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Original Article

Construction of an iterative method for solving a class of complex symmetric generalized Lyapunov matrix equation and application to Helmholtz equation

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ABSTRACT: The Lyapunov matrix equations occur in many branches of control theory, such as stability analysis and optimal control. In this work, we introduce a novel iterative approach to address the generalized Lyapunov matrix equation within the framework of complex matrices. At each iteration, the procedure involves solving two conventional Lyapunov equations with real-valued coefficient matrices. The scheme incorporates two positive parameters, for which we establish sufficient conditions to guarantee the convergence of the method under certain assumptions. Then we solve the Lyapunov equation arising by applying a finite difference procedure to Helmholtz equation by proposed method.

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1. Introduction

Matrix equations of linear type are fundamental tools utilized in a wide range of scientific and engineering applications. There are several iterative and direct schemes to solving linear matrix equations [11, 15, 16, 17, 18, 19, 20, 24, 25, 32]. In [35], the authors provided a comprehensive, closed-form, and explicit solution for the generalized Sylvester matrix equation

$$\mathcal{A}\mathcal{Q} - \mathcal{Q}F = \mathcal{B}\mathcal{W},$$

where F is a given matrix with arbitrary eigenvalues. In [34], a broad parametric solution to a set of generalized Sylvester matrix equations, commonly encountered in linear system theory, is provided through the use of the generalized Sylvester mapping, which exhibits several elegant characteristics.

[23] represented the general solution $\mathcal{X}$ to the pair of linear matrix equations

$$\mathcal{A}_1\mathcal{X}\mathcal{B}_1 = \mathcal{C}_1, \quad \mathcal{A}_2\mathcal{X}\mathcal{B}_2 = \mathcal{C}_2.$$

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The linear matrix equation

$$\mathcal{A}_1\mathcal{X}_1\mathcal{B}_1 + \mathcal{A}_2\mathcal{X}_2\mathcal{B}_2 = \mathcal{C},$$

has been explored by Xu et al. [33] and Chu [7], with unknown matrices $\mathcal{X}_1$ and $\mathcal{X}_2$ being real or complex.

The main purpose of this paper is to propose an efficient method for the solution of the generalized Lyapunov matrix equation

$$\mathcal{M}\mathcal{X} + \mathcal{X}\mathcal{M}^T + \sum_{j=1}^{m} \mathcal{N}_j\mathcal{X}\mathcal{N}_j^T = \mathcal{C}. \tag{1}$$

Then given matrices $\mathcal{M}$, $\mathcal{N}_j$ $(j = 1, \dots, m)$, $\mathcal{C} \in \mathbb{C}^{n\times n}$, the problem is to find the possible matrices $\mathcal{X} \in \mathbb{C}^{n\times n}$ that obey this equation.

Equation (1) simplifies to the classical Lyapunov equation when $\mathcal{N}_j = 0$ $(j = 1, \dots, m)$. This matrix equation plays several important roles in various scientific fields, as seen in works like [8, 21]. Furthermore, equation (1) is equivalent to the following set of linear system equations:

$$(\mathcal{I} \otimes \mathcal{M} + \mathcal{M} \otimes \mathcal{I} + \sum_{j=1}^{m} \mathcal{N}_j \otimes \mathcal{N}_j)vec(\mathcal{X}) = vec(\mathcal{C}). \tag{2}$$

Here, $\otimes$ denotes the Kronecker product, and vec($\mathcal{X}$) represents an operator defined as:

$$vec: \; \mathcal{X} = [x_1, \dots, x_n] \mapsto [x_1^T, \dots, x_n^T]^T, \tag{3}$$

where x_k refers to the k-th column of $\mathcal{X}$. The linear system in (2) can be solved through various approaches, such as Krylov subspace methods. However, this transformation significantly raises computational complexity, especially when the sizes of the coefficient matrices in (1) are large. Therefore, this method isn't directly applicable for large-scale problems. For example, see [6] for numerical Krylov subspace methods used to solve (1). Additionally, Dehghan and Shirilord [13], by parameterizing the MHSS method, proposed a generalized modified HSS (GMHSS) iteration method (refer to [2] for the HSS method) to solve a specific case of (1). Authors of [26] introduced two-parameter generalized Hermitian and skew-Hermitian splitting (TGHSS) iteration method for numerical solution of the Sylvester equation $\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{B} = \mathcal{C}$. In recent years, several approaches for solving linear matrix equations have been introduced in the literature [1, 4, 9, 10, 14, 27, 28, 29, 30, 31].

2. Main results

In this section, we introduce a new method for solving Eq. (1). Let $\mathcal{M} = \mathcal{S} + i\mathcal{Q}$, where $\mathcal{S}$ and $\mathcal{Q}$ are symmetric positive definite (S.P.D.) matrices. It is important to observe that we can express it as follows:

$$(\mathcal{S} + i\mathcal{Q})\mathcal{X} + \mathcal{X}(\mathcal{S} + i\mathcal{Q})^T = \mathcal{C} - \sum_{j=1}^{m} \mathcal{N}_j\mathcal{X}\mathcal{N}_j^T. \tag{4}$$

Now suppose that α and β are positive real constants. Multiplying $\alpha - i$ in (4) gives:

$$(\alpha\mathcal{S} + \mathcal{Q})\mathcal{X} + \mathcal{X}(\alpha\mathcal{S} + \mathcal{Q}) = i(\mathcal{S} - \alpha\mathcal{Q})\mathcal{X} + i\mathcal{X}(\mathcal{S} - \alpha\mathcal{Q}) + (\alpha - i)\mathcal{C} - (\alpha - i)\sum_{j=1}^{m} \mathcal{N}_j\mathcal{X}\mathcal{N}_j^T, \tag{5}$$

and multiplying $1 - i\beta$ in (4) concludes

$$(\beta\mathcal{Q} + \mathcal{S})\mathcal{X} + \mathcal{X}(\beta\mathcal{Q} + \mathcal{S}) = i(\beta\mathcal{S} - \mathcal{Q})\mathcal{X} + i\mathcal{X}(\beta\mathcal{S} - \mathcal{Q}) + (1 - i\beta)\mathcal{C} - (1 - i\beta)\sum_{j=1}^{m} \mathcal{N}_j\mathcal{X}\mathcal{N}_j^T. \tag{6}$$

Now by considering relations (5) and (6) we can obtain the following iterative method for solving matrix equation (1).

2.1. New method for solving matrix equation (1)

Compute $\mathcal{X}^{(k+1)} \in \mathbb{C}^{n\times n}$ for $k \in \{0, 1, 2, \dots\}$ by using the following iterative method:

$$\begin{cases} (\alpha\mathcal{S} + \mathcal{Q})\mathcal{X}^{(k+\frac{1}{2})} + \mathcal{X}^{(k+\frac{1}{2})}(\alpha\mathcal{S} + \mathcal{Q}) = i(\mathcal{S} - \alpha\mathcal{Q})\mathcal{X}^{(k)} + i\mathcal{X}^{(k)}(\mathcal{S} - \alpha\mathcal{Q}) \\ \qquad\qquad + (\alpha - i)\mathcal{C} - (\alpha - i)\sum_{j=1}^{m} \mathcal{N}_j\mathcal{X}^{(k)}\mathcal{N}_j^T, \\ (\beta\mathcal{Q} + \mathcal{S})\mathcal{X}^{(k+1)} + \mathcal{X}^{(k+1)}(\beta\mathcal{Q} + \mathcal{S}) = i(\beta\mathcal{S} - \mathcal{Q})\mathcal{X}^{(k+\frac{1}{2})} + i\mathcal{X}^{(k+\frac{1}{2})}(\beta\mathcal{S} - \mathcal{Q}) \\ \qquad\qquad + (1 - i\beta)\mathcal{C} - (1 - i\beta)\sum_{j=1}^{m} \mathcal{N}_j\mathcal{X}^{(k)}\mathcal{N}_j^T, \end{cases} \tag{7}$$

where $\alpha, \beta > 0$ are parameters and $\mathcal{X}^{(0)} \in \mathbb{C}^{n\times n}$ is an initial guess.
To derive convergence theorem for iterative method (7) the following definitions are required. Let $\mathcal{M} = \mathcal{S} + i\mathcal{Q}$, where $\mathcal{S}$ and $\mathcal{Q}$ be real S.P.D matrices. Define $\hat{\mathcal{H}} := \mathcal{G}^{-\frac{1}{2}}\mathcal{H}\mathcal{G}^{-\frac{1}{2}}$, where

$$\mathcal{G} = \mathcal{I}\otimes\mathcal{S} + \mathcal{S}\otimes\mathcal{I}, \qquad \text{and} \qquad \mathcal{H} = \mathcal{I}\otimes\mathcal{Q} + \mathcal{Q}\otimes\mathcal{I}. \tag{8}$$

Before proceeding, let introduce some notation:

$$\gamma_1 := \lambda_{\min}(\hat{\mathcal{H}}), \qquad \gamma_2 := \lambda_{\max}(\hat{\mathcal{H}}),$$

$$\delta_1 := \kappa_2(\mathcal{G}^{\frac{1}{2}}) = \|\mathcal{G}^{\frac{1}{2}}\|_2\|\mathcal{G}^{-\frac{1}{2}}\|_2,$$

$$\delta_2 := 2\|\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j\|_2\|\mathcal{G}^{-\frac{1}{2}}\|_2^2\|\mathcal{I} - i\hat{\mathcal{H}}\|_2 = 2\|\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j\|_2\|\mathcal{G}^{-\frac{1}{2}}\|_2^2\sqrt{\gamma_2^2+1}.$$

Now we define the following two-variables functions $\psi_i(\alpha,\beta)$, $i = 1,\dots,9$, i.e,

$$\psi_1(\alpha,\beta) = \frac{-\alpha\beta\gamma_1 + \alpha\gamma_1^2 + \beta - \gamma_1}{\alpha\beta\gamma_1 + \alpha\gamma_1^2 + \beta + \gamma_1}, \qquad \psi_2(\alpha,\beta) = -\theta_1\frac{\alpha\beta\gamma_1 + \alpha\gamma_1^2 + \beta + \gamma_1}{\alpha\beta\gamma_1 - \alpha\gamma_1^2 - \beta + \gamma_1},$$

$$\psi_3(\alpha,\beta) = \theta_2\frac{\alpha\beta\gamma_1 + \alpha + \beta\gamma_1^2 + \gamma_1}{\alpha+\beta}, \qquad \psi_4(\alpha,\beta) = \frac{\alpha\beta\gamma_1 - \alpha\gamma_1\gamma_2 - \beta + \gamma_2}{\alpha\beta\gamma_1 + \alpha\gamma_1\gamma_2 + \beta + \gamma_2},$$

$$\psi_5(\alpha,\beta) = \theta_1\frac{\alpha\beta\gamma_1 + \alpha\gamma_1\gamma_2 + \beta + \gamma_2}{\alpha\beta\gamma_1 - \alpha\gamma_1\gamma_2 - \beta + \gamma_2}, \qquad \psi_6(\alpha,\beta) = \theta_1\frac{\alpha\beta\gamma_2 + \alpha\gamma_1\gamma_2 + \beta + \gamma_1}{\alpha\beta\gamma_2 - \alpha\gamma_1\gamma_2 - \beta + \gamma_1},$$

$$\psi_7(\alpha,\beta) = \frac{\alpha\beta\gamma_2 - \alpha\gamma_1\gamma_2 - \beta + \gamma_1}{\alpha\beta\gamma_2 + \alpha\gamma_1\gamma_2 + \beta + \gamma_1}, \qquad \psi_8(\alpha,\beta) = \frac{-\alpha\beta\gamma_2 + \alpha\gamma_2^2 + \beta - \gamma_2}{\alpha\beta\gamma_2 + \alpha\gamma_2^2 + \beta + \gamma_2},$$

$$\psi_9(\alpha,\beta) = -\theta_1\frac{\alpha\beta\gamma_2 + \alpha\gamma_2^2 + \beta + \gamma_2}{\alpha\beta\gamma_2 - \alpha\gamma_2^2 - \beta + \gamma_2},$$

where $\theta_1 > 0$, $\theta_2 > 0$ and $\theta_1 + \theta_2 < 1$. Moreover consider region Ω_i', $i = 1,\dots,4$, as

$$\Omega_1' = \left\{(\alpha,\beta)|\sqrt{\gamma_1\gamma_2} < \beta \le \sqrt{\gamma_1/\alpha} \wedge \alpha < \gamma_2^{-1} \wedge \psi_1(\alpha,\beta) < \theta_1 \wedge \delta_1 < \psi_2(\alpha,\beta)\right\},$$

$$\Omega_2' = \left\{(\alpha,\beta)|\gamma_1 > \beta \wedge \alpha < (\gamma_1\gamma_2)^{-1/2} \wedge \psi_4(\alpha,\beta) < \theta_1 \wedge \delta_1 < \psi_5(\alpha,\beta)\right\},$$

$$\Omega_3' = \left\{(\alpha,\beta)|\sqrt{\gamma_1\gamma_2} < \beta \le 1 \wedge \gamma_1^{-1} \ge \alpha > (\gamma_1\gamma_2)^{-1/2} \wedge \psi_7(\alpha,\beta) < \theta_1 \wedge \delta_1 < \psi_6(\alpha,\beta)\right\},$$

$$\Omega_4' = \left\{(\alpha,\beta)|\sqrt{\gamma_1\gamma_2} > \beta \wedge \beta^{-1} \ge \alpha > (\gamma_1\gamma_2)^{-1/2} \wedge \psi_8(\alpha,\beta) < \theta_1 \wedge \delta_1 < \psi_9(\alpha,\beta)\right\}.$$

Then consider region Ω_i as

$$\Omega_i = \Omega_i' \cup \{(\alpha,\beta)|\delta_2 < \psi_3(\alpha,\beta)\}, \qquad i = 1,2,3,4. \tag{9}$$

The following theorem presents the convergence result of the algorithm in (7).

Theorem 2.1. *Let $\mathcal{M} = \mathcal{S} + i\mathcal{Q}$, where $\mathcal{S}$ and $\mathcal{Q}$ are real S.P.D matrices. The iteration matrix of method (7) is of the form*

$$\Theta(\alpha,\beta) = (\beta\mathcal{H}+\mathcal{G})^{-1}(\beta\mathcal{G}-\mathcal{H})(\alpha\mathcal{G}+\mathcal{H})^{-1}(\alpha\mathcal{H}-\mathcal{G}) - (\alpha+\beta)(\beta\mathcal{H}+\mathcal{G})^{-1}(\mathcal{G}-i\mathcal{H})(\alpha\mathcal{G}+\mathcal{H})^{-1}\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j, \tag{10}$$

where $\mathcal{H}$ and $\mathcal{G}$ are defined in (8).
Then under

$$\textit{Case (1): } (\alpha,\beta)\in\Omega_1, \tag{11}$$
$$\textit{Case (2): } (\alpha,\beta)\in\Omega_2, \tag{12}$$
$$\textit{Case (3): } (\alpha,\beta)\in\Omega_3, \tag{13}$$
$$\textit{Case (4): } (\alpha,\beta)\in\Omega_4, \tag{14}$$

method (7) converges unconditionally to the unique exact solution $\mathcal{X}^\star \in \mathbb{C}^{n\times n}$ of Eq. (1), where Ω_i, $i = 1,2,3,4$ are defined at (9).

Proof. We prove this theorem at two steps.
Step 1: First we show that the iteration matrix of iterative method (7) is of the form (10). Using Kronecker product, scheme (7) will have the following structure:

$$\begin{cases} \Big(\mathcal{I}\otimes(\alpha\mathcal{S}+\mathcal{Q})+(\alpha\mathcal{S}+\mathcal{Q})\otimes\mathcal{I}\Big)vec\left(\mathcal{X}^{(k+\frac{1}{2})}\right)=(\mathcal{I}\otimes(i\mathcal{S}-i\alpha\mathcal{Q})+(i\mathcal{S}-i\alpha\mathcal{Q})\otimes\mathcal{I})vec\left(\mathcal{X}^{(k)}\right) \\ \qquad +(\alpha-i)vec(\mathcal{C})-(\alpha-i)\left(\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j\right)vec\left(\mathcal{X}^{(k)}\right), \\ \Big(\mathcal{I}\otimes(\beta\mathcal{Q}+\mathcal{S})+(\beta\mathcal{Q}+\mathcal{S})\otimes\mathcal{I}\Big)vec\left(\mathcal{X}^{(k+1)}\right)=(\mathcal{I}\otimes(i\beta\mathcal{S}-i\mathcal{Q})+(i\beta\mathcal{S}-i\mathcal{Q})\otimes\mathcal{I})vec\left(\mathcal{X}^{(k+\frac{1}{2})}\right) \\ \qquad +(1-i\beta)vec(\mathcal{C})-(1-i\beta)\left(\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j\right)vec\left(\mathcal{X}^{(k)}\right), \end{cases} \tag{15}$$

or

$$\begin{cases} (\alpha\mathcal{I}\otimes\mathcal{S}+\alpha\mathcal{S}\otimes\mathcal{I}+\mathcal{I}\otimes\mathcal{Q}+\mathcal{Q}\otimes\mathcal{I})vec\left(\mathcal{X}^{(k+\frac{1}{2})}\right)=i(\mathcal{I}\otimes\mathcal{S}+\mathcal{S}\otimes\mathcal{I}-\alpha\mathcal{I}\otimes\mathcal{Q}-\alpha\mathcal{Q}\otimes\mathcal{I})vec\left(\mathcal{X}^{(k)}\right) \\ \qquad +(\alpha-i)vec(\mathcal{C})-(\alpha-i)\left(\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j\right)vec\left(\mathcal{X}^{(k)}\right), \\ (\mathcal{I}\otimes\mathcal{S}+\mathcal{S}\otimes\mathcal{I}+\beta\mathcal{I}\otimes\mathcal{Q}+\beta\mathcal{Q}\otimes\mathcal{I})vec\left(\mathcal{X}^{(k+1)}\right)=i(\beta\mathcal{I}\otimes\mathcal{S}+\beta\mathcal{S}\otimes\mathcal{I}-\mathcal{I}\otimes\mathcal{Q}-\mathcal{Q}\otimes\mathcal{I})vec\left(\mathcal{X}^{(k+\frac{1}{2})}\right) \\ \qquad +(1-i\beta)vec(\mathcal{C})-(1-i\beta)\left(\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j\right)vec\left(\mathcal{X}^{(k)}\right). \end{cases} \tag{16}$$

Now (16) becomes

$$\begin{cases} (\alpha\mathcal{G}+\mathcal{H})vec\left(\mathcal{X}^{(k+\frac{1}{2})}\right)=i(\mathcal{G}-\alpha\mathcal{H})vec\left(\mathcal{X}^{(k)}\right)+(\alpha-i)vec(\mathcal{C})-(\alpha-i)\left(\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j\right)vec\left(\mathcal{X}^{(k)}\right) \\ (\beta\mathcal{H}+\mathcal{G})vec\left(\mathcal{X}^{(k+1)}\right)=i(\beta\mathcal{G}-\mathcal{H})vec\left(\mathcal{X}^{(k+\frac{1}{2})}\right)+(1-i\beta)vec(\mathcal{C})-(1-i\beta)\left(\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j\right)vec\left(\mathcal{X}^{(k)}\right) \end{cases} \tag{17}$$

After straightforward derivations we can reformulate (17) into the standard form

$$vec\left(\mathcal{X}^{(k+1)}\right)=\Theta(\alpha,\beta)vec\left(\mathcal{X}^{(k)}\right)+\Upsilon(\alpha,\beta)vec(\mathcal{C}), \tag{18}$$

where

$$\Theta(\alpha,\beta)=\Psi(\alpha,\beta)-\Upsilon(\alpha,\beta)\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j, \tag{19}$$

$$\Psi(\alpha,\beta)=(\beta\mathcal{H}+\mathcal{G})^{-1}(\beta\mathcal{G}-\mathcal{H})(\alpha\mathcal{G}+\mathcal{H})^{-1}(\alpha\mathcal{H}-\mathcal{G}), \tag{20}$$

and

$$\Upsilon(\alpha,\beta)=(\alpha+\beta)(\beta\mathcal{H}+\mathcal{G})^{-1}(\mathcal{G}-i\mathcal{H})(\alpha\mathcal{G}+\mathcal{H})^{-1}. \tag{21}$$

Equation (18) shows that the matrix $\Theta(\alpha,\beta)$ is iteration matrix of method (7).
Step 2: Now for convergence of method (7) we will show that $\|\Theta(\alpha,\beta)\|_2<1$. For 2-norm of iteration matrix $\Theta(\alpha,\beta)$ we can write

$$\begin{aligned} \|\Theta(\alpha,\beta)\|_2 &= \Big\|\Psi(\alpha,\beta)-\Upsilon(\alpha,\beta)\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j\Big\|_2 \\ &\le \|\Psi(\alpha,\beta)\|_2+\|\Upsilon(\alpha,\beta)\|_2\Big\|\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j\Big\|_2 \\ &= \|(\beta\mathcal{G}-\mathcal{H})(\beta\mathcal{G}+\mathcal{H})^{-1}(\alpha\mathcal{H}-\mathcal{G})(\alpha\mathcal{H}+\mathcal{G})^{-1}\|_2 \\ &\quad +(\alpha+\beta)\Big\|\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j\Big\|_2\|(\beta\mathcal{H}+\mathcal{G})^{-1}(\mathcal{G}-i\mathcal{H})(\alpha\mathcal{G}+\mathcal{H})^{-1}\|_2. \end{aligned}$$

Then we obtain:

$$\begin{aligned}\|\Theta(\alpha,\beta)\|_2 \le\; & \|(\beta\mathcal{G}-\mathcal{G}^{\frac{1}{2}}\hat{\mathcal{H}}\mathcal{G}^{\frac{1}{2}})(\beta\mathcal{G}+\mathcal{G}^{\frac{1}{2}}\hat{\mathcal{H}}\mathcal{G}^{\frac{1}{2}})^{-1}(\alpha\mathcal{G}^{\frac{1}{2}}\hat{\mathcal{H}}\mathcal{G}^{\frac{1}{2}}-\mathcal{G})(\alpha\mathcal{G}^{\frac{1}{2}}\hat{\mathcal{H}}\mathcal{G}^{\frac{1}{2}}+\mathcal{G})^{-1}\|_2\\
&+(\alpha+\beta)\|\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j\|_2\|(\beta\mathcal{G}^{\frac{1}{2}}\hat{\mathcal{H}}\mathcal{G}^{\frac{1}{2}}+\mathcal{G})^{-1}(\mathcal{G}-i\mathcal{G}^{\frac{1}{2}}\hat{\mathcal{H}}\mathcal{G}^{\frac{1}{2}})(\alpha\mathcal{G}+\mathcal{G}^{\frac{1}{2}}\hat{\mathcal{H}}\mathcal{G}^{\frac{1}{2}})^{-1}\|_2\\
\le\; & \|(\beta\mathcal{I}-\hat{\mathcal{H}})(\beta\mathcal{I}+\hat{\mathcal{H}})^{-1}(\alpha\hat{\mathcal{H}}-\mathcal{I})(\alpha\hat{\mathcal{H}}+\mathcal{I})^{-1}\|_2\|\mathcal{G}^{\frac{1}{2}}\|_2\|\mathcal{G}^{-\frac{1}{2}}\|_2\\
&+(\alpha+\beta)\|\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j\|_2\|(\beta\hat{\mathcal{H}}+\mathcal{I})^{-1}(\mathcal{I}-i\hat{\mathcal{H}})(\alpha\mathcal{I}+\hat{\mathcal{H}})^{-1}\|_2\|\mathcal{G}^{-\frac{1}{2}}\|_2^2\\
\le\; & \max_{\lambda\in sp(\hat{\mathcal{H}})}\left|\frac{(\beta-\lambda)(\alpha\lambda-1)}{(\beta+\lambda)(\alpha\lambda+1)}\right|\|\mathcal{G}^{\frac{1}{2}}\|_2\|\mathcal{G}^{-\frac{1}{2}}\|_2\\
&+2\,(\alpha+\beta)\,\|\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j\|_2\|\mathcal{G}^{-\frac{1}{2}}\|_2^2\|\mathcal{I}-i\hat{\mathcal{H}}\|_2\|(\alpha\mathcal{I}+\hat{\mathcal{H}})^{-1}\|_2\|(\beta\hat{\mathcal{H}}+\mathcal{I})^{-1}\|_2\\
\le\; & \max_{\lambda\in sp(\hat{\mathcal{H}})}\left|\frac{\alpha\lambda-1}{\alpha\lambda+1}\right|\max_{\lambda\in sp(\hat{\mathcal{H}})}\left|\frac{\beta-\lambda}{\beta+\lambda}\right|\|\mathcal{G}^{\frac{1}{2}}\|_2\|\mathcal{G}^{-\frac{1}{2}}\|_2\\
&+2\,(\alpha+\beta)\,\|\sum_{j=1}^{m}\mathcal{N}_j\otimes\mathcal{N}_j\|_2\|\mathcal{G}^{-\frac{1}{2}}\|_2^2\|\mathcal{I}-i\hat{\mathcal{H}}\|_2\|(\alpha\mathcal{I}+\hat{\mathcal{H}})^{-1}\|_2\|(\beta\hat{\mathcal{H}}+\mathcal{I})^{-1}\|_2\\
=\; & \delta_1\max_{\lambda\in sp(\hat{\mathcal{H}})}\left|\frac{\alpha\lambda-1}{\alpha\lambda+1}\right|\max_{\lambda\in sp(\hat{\mathcal{H}})}\left|\frac{\beta-\lambda}{\beta+\lambda}\right|+\delta_2\,(\alpha+\beta)\,\|(\alpha\mathcal{I}+\hat{\mathcal{H}})^{-1}\|_2\|(\beta\hat{\mathcal{H}}+\mathcal{I})^{-1}\|_2,\end{aligned}$$

where $sp(\hat{\mathcal{H}})$ denotes the spectrum of matrix $\hat{\mathcal{H}}$. Then since $\lambda>0,\ \beta+\lambda>0,\ \alpha\lambda+1>0$ we have

$$\|\Theta(\alpha,\beta)\|_2\le\delta_1\max_{\lambda\in sp(\hat{\mathcal{H}})}\frac{|\alpha\lambda-1|}{\alpha\lambda+1}\max_{\lambda\in sp(\hat{\mathcal{H}})}\frac{|\beta-\lambda|}{\beta+\lambda}+\delta_2\,(\alpha+\beta)\,\|(\alpha\mathcal{I}+\hat{\mathcal{H}})^{-1}\|_2\|(\beta\hat{\mathcal{H}}+\mathcal{I})^{-1}\|_2. \tag{22}$$

On the other hand

$$\|(\alpha\mathcal{I}+\hat{\mathcal{H}})^{-1}\|_2=\frac{1}{\gamma_1+\alpha}, \tag{23}$$

$$\|(\beta\hat{\mathcal{H}}+\mathcal{I})^{-1}\|_2=\frac{1}{\beta\gamma_1+1}. \tag{24}$$

Substituting equations (23) and (24) into equation (22), demonstrates:

$$\|\Theta(\alpha,\beta)\|_2\le\delta_1\max_{\lambda\in sp(\hat{\mathcal{H}})}\frac{|\alpha\lambda-1|}{\alpha\lambda+1}\max_{\lambda\in sp(\hat{\mathcal{H}})}\frac{|\beta-\lambda|}{\beta+\lambda}+\frac{\delta_2\,(\alpha+\beta)}{(\gamma_1+\alpha)(\beta\gamma_1+1)}. \tag{25}$$

It is not difficult to see that

$$\max_{\lambda\in sp(\hat{\mathcal{H}})}\frac{|\alpha\lambda-1|}{\alpha\lambda+1}=\max\left\{\frac{1-\alpha\gamma_1}{\alpha\gamma_1+1},\frac{\alpha\gamma_2-1}{\alpha\gamma_2+1}\right\},\quad\text{and}\quad\max_{\lambda\in sp(\hat{\mathcal{H}})}\frac{|\beta-\lambda|}{\beta+\lambda}=\max\left\{\frac{\beta-\gamma_1}{\beta+\gamma_1},\frac{\gamma_2-\beta}{\beta+\gamma_2}\right\}.$$

This implies

$$\|\Theta(\alpha,\beta)\|_2\le\delta_1\max\left\{\frac{1-\alpha\gamma_1}{\alpha\gamma_1+1},\frac{\alpha\gamma_2-1}{\alpha\gamma_2+1}\right\}\max\left\{\frac{\beta-\gamma_1}{\beta+\gamma_1},\frac{\gamma_2-\beta}{\beta+\gamma_2}\right\}+\frac{\delta_2\,(\alpha+\beta)}{(\gamma_1+\alpha)(\beta\gamma_1+1)}.$$

Now for convergence of method (7) it is enough to show that

$$\delta_1\max\left\{\frac{1-\alpha\gamma_1}{\alpha\gamma_1+1},\frac{\alpha\gamma_2-1}{\alpha\gamma_2+1}\right\}\max\left\{\frac{\beta-\gamma_1}{\beta+\gamma_1},\frac{\gamma_2-\beta}{\beta+\gamma_2}\right\}+\frac{\delta_2\,(\alpha+\beta)}{(\gamma_1+\alpha)(\beta\gamma_1+1)}<1.$$

Now, let us consider the following cases:

Case (1): Let

$$\max_{\lambda\in sp(\hat{\mathcal{H}})}\frac{|\alpha\lambda-1|}{\alpha\lambda+1}=\frac{1-\alpha\gamma_1}{\alpha\gamma_1+1},\quad\text{and}\quad\max_{\lambda\in sp(\hat{\mathcal{H}})}\frac{|\beta-\lambda|}{\beta+\lambda}=\frac{\beta-\gamma_1}{\beta+\gamma_1}.$$

Then $(\alpha,\beta)\in\Omega_1$ yields

$$\|\Theta(\alpha,\beta)\|_2 \leq \frac{\delta_1(1-\alpha\gamma_1)(\beta-\gamma_1)}{(\alpha\gamma_1+1)(\beta+\gamma_1)} + \frac{\delta_2\,(\alpha+\beta)}{(\gamma_1+\alpha)(\beta\gamma_1+1)} < 1.$$

Case (2): Let

$$\max_{\lambda\in sp(\hat{\mathcal{H}})} \frac{|\alpha\lambda-1|}{\alpha\lambda+1} = \frac{1-\alpha\gamma_1}{\alpha\gamma_1+1}, \quad \text{and} \quad \max_{\lambda\in sp(\hat{\mathcal{H}})} \frac{|\beta-\lambda|}{\beta+\lambda} = \frac{\gamma_2-\beta}{\beta+\gamma_2}.$$

Then $(\alpha,\beta)\in\Omega_2$ concludes

$$\|\Theta(\alpha,\beta)\|_2 \leq \frac{\delta_1(1-\alpha\gamma_1)(\gamma_2-\beta)}{(\alpha\gamma_1+1)(\beta+\gamma_2)} + \frac{\delta_2\,(\alpha+\beta)}{(\gamma_1+\alpha)(\beta\gamma_1+1)} < 1.$$

Case (3): Let

$$\max_{\lambda\in sp(\hat{\mathcal{H}})} \frac{|\alpha\lambda-1|}{\alpha\lambda+1} = \frac{\alpha\gamma_2-1}{\alpha\gamma_2+1}, \quad \text{and} \quad \max_{\lambda\in sp(\hat{\mathcal{H}})} \frac{|\beta-\lambda|}{\beta+\lambda} = \frac{\beta-\gamma_1}{\beta+\gamma_1}.$$

Then $(\alpha,\beta)\in\Omega_3$ gives

$$\|\Theta(\alpha,\beta)\|_2 \leq \frac{\delta_1(\alpha\gamma_2-1)(\beta-\gamma_1)}{(\alpha\gamma_2+1)(\beta+\gamma_1)} + \frac{\delta_2\,(\alpha+\beta)}{(\gamma_1+\alpha)(\beta\gamma_1+1)} < 1.$$

Case (4): Let

$$\max_{\lambda\in sp(\hat{\mathcal{H}})} \frac{|\alpha\lambda-1|}{\alpha\lambda+1} = \frac{\alpha\gamma_2-1}{\alpha\gamma_2+1}, \quad \text{and} \quad \max_{\lambda\in sp(\hat{\mathcal{H}})} \frac{|\beta-\lambda|}{\beta+\lambda} = \frac{\gamma_2-\beta}{\beta+\gamma_2}.$$

Then $(\alpha,\beta)\in\Omega_4$ yields

$$\|\Theta(\alpha,\beta)\|_2 \leq \frac{\delta_1(\alpha\gamma_2-1)(\gamma_2-\beta)}{(\alpha\gamma_2+1)(\beta+\gamma_2)} + \frac{\delta_2\,(\alpha+\beta)}{(\gamma_1+\alpha)(\beta\gamma_1+1)} < 1.$$

The proof is finished. □

Remark 2.2. *Although Theorem 2.1 theoretically guarantees the existence of regions in which the proposed method converges for any parameter choices within those regions, determining the optimal values of the parameters in practice is a very challenging task. A comprehensive study devoted to the practical determination of these parameters can be conducted in future work. One possible idea is to minimize the upper bound of the iteration matrix norm, namely,*

$$\delta_1 \max\left\{\frac{1-\alpha\gamma_1}{\alpha\gamma_1+1}, \frac{\alpha\gamma_2-1}{\alpha\gamma_2+1}\right\} \max\left\{\frac{\beta-\gamma_1}{\beta+\gamma_1}, \frac{\gamma_2-\beta}{\beta+\gamma_2}\right\} + \frac{\delta_2\,(\alpha+\beta)}{(\gamma_1+\alpha)(\beta\gamma_1+1)}.$$

In the current version of the paper, these parameters have been experimentally selected to achieve the minimum number of iterations. A more precise investigation of the optimal selection strategy, particularly for the parameters α,β, will be the subject of our future research.

For small n, one may compute $\hat{\mathcal{H}}$ directly to obtain the precise values of γ_1 and γ_2, but for large n the following theorem allows us to efficiently estimate γ_1, γ_2, and δ_1 while avoiding the formation of large Kronecker matrices.

Theorem 2.3. *Let $\mathcal{S},\mathcal{Q}\in\mathbb{R}^{n\times n}$ be symmetric positive definite matrices, and define $\hat{\mathcal{H}} := \mathcal{G}^{-1/2}\mathcal{H}\mathcal{G}^{-1/2}$, where*

$$\mathcal{G} = \mathcal{I}\otimes\mathcal{S} + \mathcal{S}\otimes\mathcal{I}, \qquad \textit{and} \qquad \mathcal{H} = \mathcal{I}\otimes\mathcal{Q} + \mathcal{Q}\otimes\mathcal{I}.$$

Then the eigenvalues of $\hat{\mathcal{H}}$ satisfy the following bounds:

$$\frac{\lambda_{\min}(\mathcal{Q})}{\lambda_{\max}(\mathcal{S})} \leq \lambda(\hat{\mathcal{H}}) \leq \frac{\lambda_{\max}(\mathcal{Q})}{\lambda_{\min}(\mathcal{S})}.$$

In particular, the smallest and largest eigenvalues are bounded by

$$\gamma_1 := \lambda_{\min}(\hat{\mathcal{H}}) \geq \frac{\lambda_{\min}(\mathcal{Q})}{\lambda_{\max}(\mathcal{S})}, \qquad \gamma_2 := \lambda_{\max}(\hat{\mathcal{H}}) \leq \frac{\lambda_{\max}(\mathcal{Q})}{\lambda_{\min}(\mathcal{S})}.$$

Also we have $\delta_1 := \kappa_2(\mathcal{G}^{\frac{1}{2}}) = \sqrt{\kappa_2(\mathcal{S})}$.

Proof. For any $y \neq 0$, consider the Rayleigh quotient of $\hat{\mathcal{H}}$:

$$\frac{y^T \hat{\mathcal{H}} y}{y^T y} = \frac{y^T \mathcal{G}^{-1/2} \mathcal{H} \mathcal{G}^{-1/2} y}{y^T y}.$$

Set $z := \mathcal{G}^{-1/2} y$, then $y = \mathcal{G}^{1/2} z$, and

$$\frac{y^T \hat{\mathcal{H}} y}{y^T y} = \frac{z^T \mathcal{H} z}{z^T \mathcal{G} z}.$$

Using the inequality $z^T \mathcal{G} z \geq \lambda_{\min}(\mathcal{G}) z^T z$ gives

$$\frac{z^T \mathcal{H} z}{z^T \mathcal{G} z} \leq \frac{z^T \mathcal{H} z}{\lambda_{\min}(\mathcal{G}) z^T z} \leq \frac{\lambda_{\max}(\mathcal{H})}{\lambda_{\min}(\mathcal{G})} = \frac{2\lambda_{\max}(\mathcal{Q})}{2\lambda_{\min}(\mathcal{S})} = \frac{\lambda_{\max}(\mathcal{Q})}{\lambda_{\min}(\mathcal{S})}.$$

Taking the maximum over all $z \neq 0$ yields the upper bound for γ_2. Similarly, the lower bound follows a similar proof. Finally we obtain

$$\delta_1 := \kappa_2(\mathcal{G}^{\frac{1}{2}}) = \sqrt{\kappa_2(\mathcal{G})} = \sqrt{\frac{\lambda_{\max}(\mathcal{G})}{\lambda_{\min}(\mathcal{G})}} = \sqrt{\frac{\lambda_{\max}(\mathcal{S})}{\lambda_{\min}(\mathcal{S})}} = \sqrt{\kappa_2(\mathcal{S})}.$$ □

Remark 2.4. *Theorem 2.1 provides a theoretical guarantee under this specific condition. While this condition is automatically satisfied in our experiments and across a broad range of test cases, empirical evidence shows that the method converges reliably even when the theoretical prerequisite is not strictly met.*

Now we are going to perform numerical examples with a partial differential equation with a complex term. The efficiency of the introduced method is numerically tested.

3. Test problems

In this section, the provided numerical examples illustrate that the proposed iterative method is highly effective for solving generalized Sylvester equation (1) in terms of both iteration count and computing time, and the logarithm of the norm of the relative error defined by

$$\log_{10}\left(\frac{\|\mathcal{X}^\star - \mathcal{X}^{(k)}\|_F}{\|\mathcal{X}^\star\|_F}\right).$$

The iteration schemes are started from the zero matrix. The suggested method is compared with the GIGMRES algorithm [6].

Example 3.1. *Consider the following Helmholtz equation by Dirichlet boundary conditions [3, 5, 12, 22]:*

$$\begin{cases} -\dfrac{\partial^2 w}{\partial x^2} - \dfrac{\partial^2 w}{\partial y^2} + \sigma_1 w + i\sigma_2 w = f, & \text{on } \Xi \\ w = 0, & \text{on } \partial\Xi, \end{cases} \tag{26}$$

where Ξ is a distinct square region defined as $\Xi = \{(x,y) \mid 0 < x, y < 1\}$, and $\partial\Xi$ represents the boundary of this region. The functions σ_1 and σ_2 are real-valued coefficient functions, and $i = \sqrt{-1}$. Furthermore, we assume f is a continuous function defined on Ξ. Let n be a positive integer that is known in advance. We now consider the following set of points:

$$\overline{\Xi_h} := \{(jh, kh) \mid j,\ k = 0, 1, \ldots, n+1\},$$

$$\Xi_h := \{(jh, kh) \mid j,\ k = 0, 1, 2, \ldots, n\},$$

where $h = \frac{1}{n+1}$ represents the step length. The set Ξ_h refers to the interior of this region, and the boundary points are given by $\overline{\Xi_h} - \Xi_h$.

Let $w_{j,k} = w(jh, kh)$ be the exact solution of the equation at the point (j, k), and $\mathcal{W}_{j,k}$ be its approximation. To approximate the second derivative u with respect to x and y, we use the following central difference approximations:

$$\frac{\partial^2 w(x,y)}{\partial x^2}\Big|_{(x=jh, y=kh)} \simeq \frac{\mathcal{W}_{j-1,k} - 2\mathcal{W}_{jk} + \mathcal{W}_{j+1,k}}{h^2},$$

$$\frac{\partial^2 w(x,y)}{\partial y^2}\Big|_{(x=jh, y=kh)} \simeq \frac{\mathcal{W}_{j,k-1} - 2\mathcal{W}_{jk} + \mathcal{W}_{j,k+1}}{h^2}.$$

Therefore, the given problem is discretized as follows

$$-\frac{\mathcal{W}_{j-1,k}-2\mathcal{W}_{jk}+\mathcal{W}_{j+1,k}}{h^2}-\frac{\mathcal{W}_{j,k-1}-2\mathcal{W}_{jk}+\mathcal{W}_{j,k+1}}{h^2}+\sigma_1\mathcal{W}_{jk}+i\sigma_2\mathcal{W}_{jk}=f_{jk},\quad j,\ k=1,2,\ldots,n.$$

This implies

$$(4+h^2(\sigma_1+i\sigma_2))\mathcal{W}_{jk}-\mathcal{W}_{j-1,k}-\mathcal{W}_{j+1,k}-\mathcal{W}_{j,k-1}-\mathcal{W}_{j,k+1}=h^2f_{jk},\quad j,\ k=1,2,\ldots,n.$$

These equations, using the boundary conditions, can be expressed as follows:

$$j=1,\quad\begin{cases}k=1:(4+h^2(\sigma_1+i\sigma_2))\mathcal{W}_{11}-0-\mathcal{W}_{21}-0-\mathcal{W}_{12}=h^2f_{11},\\ k=2:(4+h^2(\sigma_1+i\sigma_2))\mathcal{W}_{12}-0-\mathcal{W}_{22}-\mathcal{W}_{11}-\mathcal{W}_{13}=h^2f_{12},\\ \quad\vdots\\ k=n:(4+h^2(\sigma_1+i\sigma_2))\mathcal{W}_{1n}-0-\mathcal{W}_{2n}-\mathcal{W}_{1,k-1}-0=h^2f_{1n},\end{cases}$$

$$j=2,\quad\begin{cases}k=1:(4+h^2(\sigma_1+i\sigma_2))\mathcal{W}_{21}-\mathcal{W}_{11}-\mathcal{W}_{31}-0-\mathcal{W}_{22}=h^2f_{21},\\ k=2:(4+h^2(\sigma_1+i\sigma_2))\mathcal{W}_{22}-\mathcal{W}_{12}-\mathcal{W}_{32}-\mathcal{W}_{21}-\mathcal{W}_{23}=h^2f_{22},\\ \quad\vdots\\ k=n:(4+h^2(\sigma_1+i\sigma_2))\mathcal{W}_{2n}-\mathcal{W}_{1n}-\mathcal{W}_{3n}-\mathcal{W}_{2,n-1}-0=h^2f_{2n},\end{cases}$$

$$\vdots$$

$$j=n,\quad\begin{cases}k=1:(4+h^2(\sigma_1+i\sigma_2))\mathcal{W}_{n1}-\mathcal{W}_{n-1,1}-0-0-\mathcal{W}_{n2}=h^2f_{n1},\\ k=2:(4+h^2(\sigma_1+i\sigma_2))\mathcal{W}_{n2}-\mathcal{W}_{n-1,2}-0-\mathcal{W}_{n1}-\mathcal{W}_{n3}=h^2f_{n2},\\ \quad\vdots\\ k=n:(4+h^2(\sigma_1+i\sigma_2))\mathcal{W}_{nn}-\mathcal{W}_{n-1,n}-0-\mathcal{W}_{n,n-1}-0=h^2f_{nn},\end{cases}$$

At this point, the equations can be written in the following matrix representation:

$$\begin{bmatrix}(2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{11}-\mathcal{W}_{21} & (2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{12}-\mathcal{W}_{22} & \cdots & (2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{1n}-\mathcal{W}_{2n}\\ -\mathcal{W}_{11}+(2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{21}-\mathcal{W}_{31} & -\mathcal{W}_{12}+(2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{22}-\mathcal{W}_{32} & \cdots & -\mathcal{W}_{1n}+(2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{2n}-\mathcal{W}_{3n}\\ \vdots & \vdots & \ddots & \vdots\\ -\mathcal{W}_{n-1,1}+(2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{n1} & -\mathcal{W}_{n-1,2}+(2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{n2} & \cdots & -\mathcal{W}_{n-1,n}+(2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{nn}\end{bmatrix}$$

$$+\begin{bmatrix}(2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{11}-\mathcal{W}_{12} & -\mathcal{W}_{11}+(2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{12}-\mathcal{W}_{13} & \cdots & -\mathcal{W}_{1,n-1}+(2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{1n}\\ (2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{21}-\mathcal{W}_{22} & -\mathcal{W}_{21}+(2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{22}-\mathcal{W}_{23} & \cdots & -\mathcal{W}_{2,n-1}+(2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ (2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{n1}-\mathcal{W}_{n2} & -\mathcal{W}_{n1}+(2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{n2}-\mathcal{W}_{n3} & \cdots & -\mathcal{W}_{n,n-1}+(2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2))\mathcal{W}_{nn}\end{bmatrix}$$

$$=h^2\begin{bmatrix}f_{11} & f_{12} & \cdots & f_{1n}\\ f_{21} & f_{22} & \cdots & f_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ f_{n1} & f_{n2} & \cdots & f_{nn}\end{bmatrix},$$

or

$$\begin{bmatrix}2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2) & -1 & 0 & \ldots & 0\\ -1 & 2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2) & -1 & \ldots & 0\\ 0 & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & -1\\ 0 & \ldots & \ldots & -1 & 2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2)\end{bmatrix}\begin{bmatrix}\mathcal{W}_{11} & \mathcal{W}_{12} & \ldots & \mathcal{W}_{1n}\\ \mathcal{W}_{21} & \mathcal{W}_{22} & \ldots & \mathcal{W}_{2n}\\ \vdots & \ddots & & \vdots\\ \mathcal{W}_{n1} & \mathcal{W}_{n2} & \ldots & \mathcal{W}_{nn}\end{bmatrix}$$

$$+\begin{bmatrix}\mathcal{W}_{11} & \mathcal{W}_{12} & \ldots & \mathcal{W}_{1n}\\ \mathcal{W}_{21} & \mathcal{W}_{22} & \ldots & \mathcal{W}_{2n}\\ \vdots & \ddots & & \vdots\\ \mathcal{W}_{n1} & \mathcal{W}_{n2} & \ldots & \mathcal{W}_{nn}\end{bmatrix}\begin{bmatrix}2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2) & -1 & 0 & \ldots & 0\\ -1 & 2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2) & -1 & \cdots & 0\\ 0 & \ddots & \ddots & \ddots & \vdots\\ \vdots & & \ddots & \ddots & -1\\ 0 & \ldots & \ldots & -1 & 2+h^2(\frac{1}{2}\sigma_1+\frac{i}{2}\sigma_2)\end{bmatrix},$$

$$=h^2\begin{bmatrix}f_{11} & f_{12} & \cdots & f_{1n}\\ f_{21} & f_{22} & \cdots & f_{2n}\\ \vdots & \vdots & \ddots & \vdots\\ f_{n1} & f_{n2} & \cdots & f_{nn}\end{bmatrix},$$

or

$$\mathcal{M}\mathcal{X} + \mathcal{X}\mathcal{M}^T = h^2\mathcal{H}, \tag{27}$$

where

$$\mathcal{X} = \begin{bmatrix} \mathcal{W}_{11} & \mathcal{W}_{12} & \dots & \mathcal{W}_{1n} \\ \mathcal{W}_{21} & \mathcal{W}_{22} & \dots & \mathcal{W}_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ \mathcal{W}_{n1} & \mathcal{W}_{n2} & \dots & \mathcal{W}_{nn} \end{bmatrix}_{n\times n}, \quad \mathcal{H} = \begin{bmatrix} f_{11} & f_{12} & \dots & f_{1n} \\ f_{21} & f_{22} & \dots & f_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ f_{n1} & f_{n2} & \dots & f_{nn} \end{bmatrix}_{n\times n},$$

and

$$\mathcal{M} = \begin{bmatrix} 2+h^2(\frac{1}{2}\sigma_1 + \frac{i}{2}\sigma_2) & -1 & 0 & \dots & 0 \\ -1 & 2+h^2(\frac{1}{2}\sigma_1 + \frac{i}{2}\sigma_2) & -1 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \dots & \dots & -1 & 2+h^2(\frac{1}{2}\sigma_1 + \frac{i}{2}\sigma_2) \end{bmatrix}_{n\times n}.$$

It is evident that solving the Helmholtz equation using the finite difference method results in solving a Lyapunov matrix equation with a complex coefficient matrix. Furthermore, equation (27) represents a specific instance of general equation (1) examined in this study. It is important to observe that the coefficient matrix of this Lyapunov equation can be decomposed as follows:

$$\mathcal{M} = \mathcal{S} + i\mathcal{Q},$$

where

$$\mathcal{S} = \begin{bmatrix} 2+\frac{h^2}{2}\sigma_1 & -1 & 0 & \dots & 0 \\ -1 & 2+\frac{h^2}{2}\sigma_1 & -1 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & -1 \\ 0 & \dots & \dots & -1 & 2+\frac{h^2}{2}\sigma_1 \end{bmatrix}, \quad \mathcal{Q} = \begin{bmatrix} \frac{h^2}{2}\sigma_2 & 0 & 0 & \dots & 0 \\ 0 & \frac{h^2}{2}\sigma_2 & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \dots & \dots & 0 & \frac{h^2}{2}\sigma_2 \end{bmatrix}.$$

Here we solve matrix equation (27) when in Helmholtz equation (26) we have $\sigma_1 = 1,\ \sigma_2 = 5$ and

$$f(x,y) = e^{x+y}(2\pi^2 \sin(\pi x)\sin(\pi y) + 5i\sin(\pi x)\sin(\pi y) - 2\sin(\pi x)\pi\cos(\pi y) - 2\pi\cos(\pi x)\sin(\pi y) - \sin(\pi x)\sin(\pi y)).$$

In this case the exact solution is

$$w(x,y) = e^{x+y}\sin(\pi x)\sin(\pi y).$$

The numerical results about iterations, CPU time and relative error of GIGMRES(5) and new method presented in Tables 1, 2, 3 and 4. As we can see from these tables that the new method for $(\alpha,\beta) = (9.3, 6)$ considerably outperforms IGMRES(5) method, because it needs less iteration counts and CPU times.

Table 1: Example 3.1: Numerical results for $(\alpha,\beta) = (9.3, 6)$ and $n = 500$.

$Iteration(=k)$	5	10	15	20	25
$\log_{10}\left(\Vert\mathcal{X}^\star - \mathcal{X}^{(k)}\Vert_F / \Vert\mathcal{X}^\star\Vert_F\right)$	-2.5771	-4.7053	-5.9117	-7.0224	-8.0968
$CPU - time(seconds)$	8.2466	8.8138	9.3620	9.4167	9.5527

Table 2: Example 3.1: Numerical results for $(\alpha,\beta) = (9.3, 6)$ and $n = 500$.

$Iteration(=k)$	30	35	40	45	50
$\log_{10}\left(\Vert\mathcal{X}^\star - \mathcal{X}^{(k)}\Vert_F / \Vert\mathcal{X}^\star\Vert_F\right)$	-9.1454	-9.9676	-10.0657	-10.0669	-10.0669
$CPU - time(seconds)$	9.5949	9.6330	9.6854	9.7334	9.9474

Moreover, for $n = 100$ with different parameters (α,β), we plot the logarithm of relative error as a function of iterations in Figures 1 and 2. From these figures the parameters $(\alpha,\beta) = (9.3, 6)$ are optimum.

For this value for parameters α and β in Fig. 3 we plot the approximate solutions for imaginary and real parts; after 2, 5, 10 and 50 iterations.

Table 3: Example 3.1: Numerical results for GlGMRES(10) method and $n = 500$.

$Iteration(= k)$	5	10	15	20	25
$\log_{10}\left(\|\mathcal{X}^\star - \mathcal{X}^{(k)}\|_F/\|\mathcal{X}^\star\|_F\right)$	-0.0081	-0.0133	-0.0188	-0.0238	-0.0292
$CPU - time(seconds)$	17.1073	20.7798	17.2189	17.5226	17.5074

Table 4: Example 3.1: Numerical results for GlGMRES(10) method and $n = 500$.

$Iteration(= k)$	30	35	40	45	50
$\log_{10}\left(\|\mathcal{X}^\star - \mathcal{X}^{(k)}\|_F/\|\mathcal{X}^\star\|_F\right)$	-0.0341	-0.0394	-0.0443	-0.0496	-0.0545
$CPU - time(seconds)$	17.4516	17.9568	18.0315	17.9112	17.9236

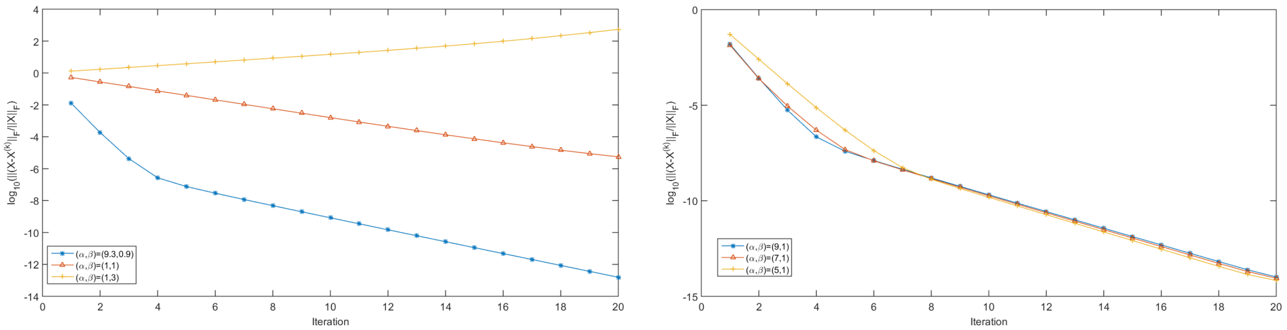

Figure 1: Relative error as a function of iterations and different parameters for Example 3.1.

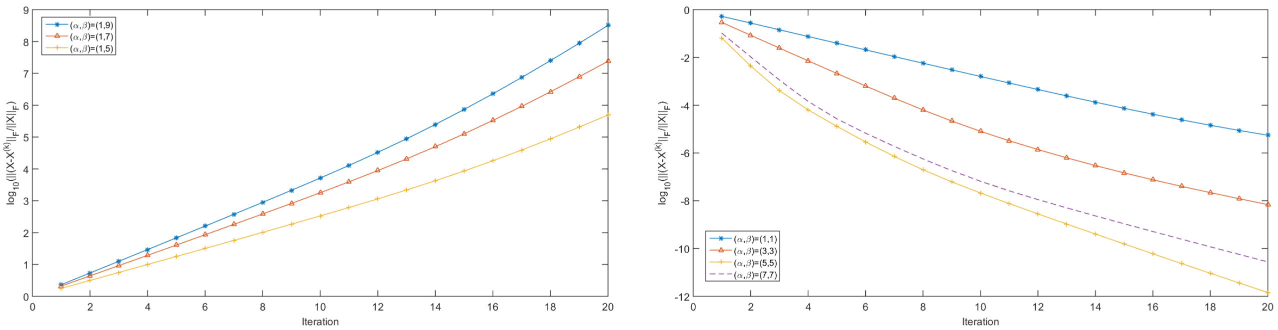

Figure 2: Relative error as a function of iterations and different parameters for Example 3.1.

Example 3.2. *Consider the equation* $\mathcal{M}\mathcal{X} + \mathcal{X}\mathcal{M}^T + \mathcal{N}_1\mathcal{X}\mathcal{N}_1^T = \mathcal{C}$, *where*

$$\mathcal{S} = \begin{bmatrix} 25 & 2 & -2 & -4 & 6 & 0 & \dots & \dots & \dots & 0 \\ 2 & 25 & 2 & -2 & -4 & 6 & & & & \vdots \\ -2 & 2 & 25 & 2 & -2 & -4 & 6 & & & \vdots \\ -4 & -2 & 2 & 25 & 2 & -2 & -4 & 6 & & \vdots \\ 6 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 6 \\ \vdots & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -4 \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & -2 \\ \vdots & & & & 6 & -4 & -2 & 2 & 25 & 2 \\ 0 & \dots & \dots & \dots & 0 & 6 & -4 & -2 & 2 & 25 \end{bmatrix}_{n\times n},$$

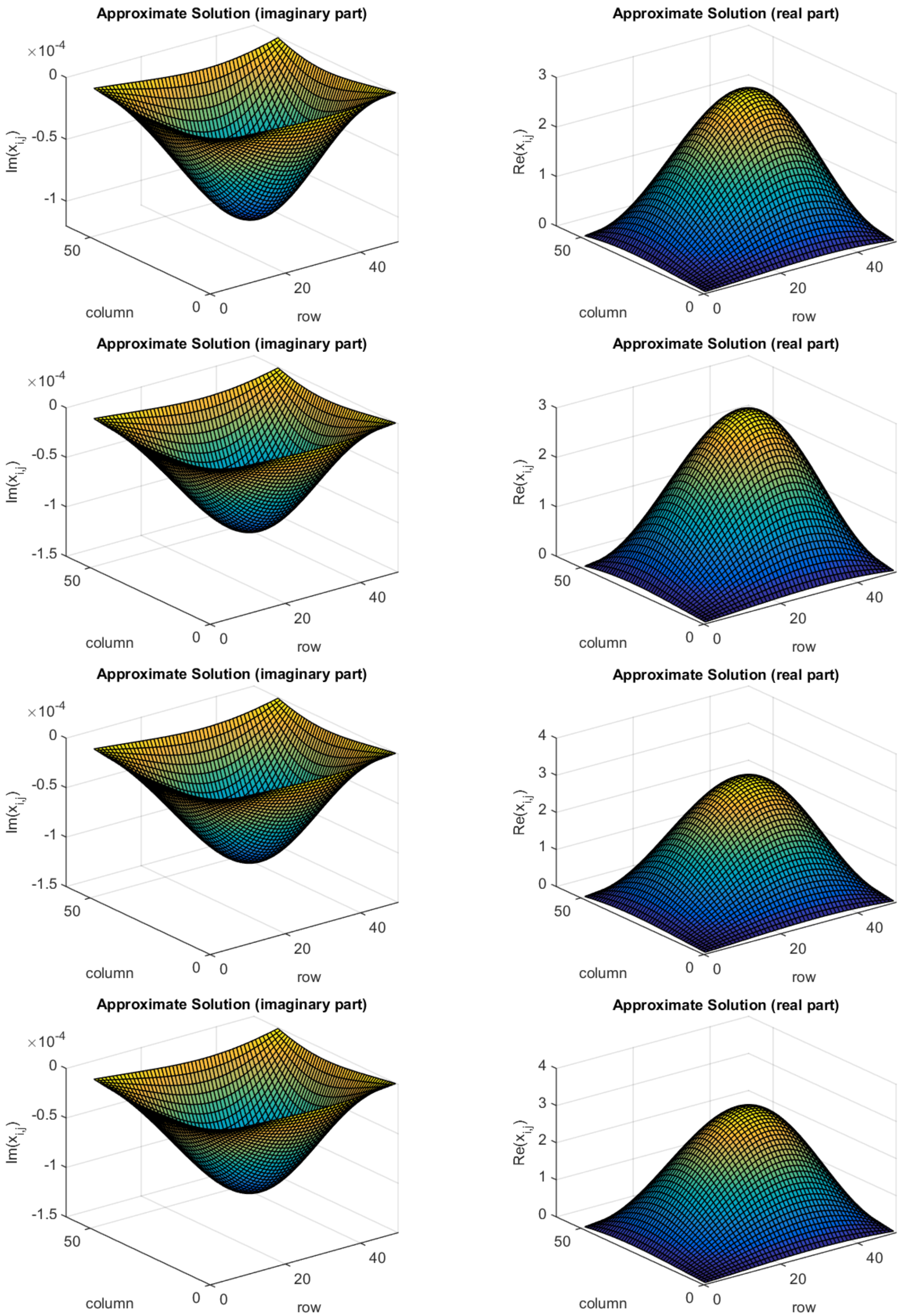

Figure 3: Approximate solutions for imaginary and real parts; after 2, 5, 10 and 50 iterations for Example 3.1.

$$\mathcal{Q}=\begin{bmatrix} 5 & -\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & 0 & \dots & \dots & \dots & 0 \\ -\frac{1}{2} & 5 & -\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & & & & \vdots \\ \frac{1}{3} & -\frac{1}{2} & 5 & -\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & & & \vdots \\ \frac{1}{4} & \frac{1}{3} & -\frac{1}{2} & 5 & -\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} & & \vdots \\ \frac{1}{5} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & \frac{1}{5} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{1}{5} \\ \vdots & & \frac{1}{5} & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{1}{4} \\ \vdots & & & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \frac{1}{3} \\ \vdots & & & & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & -\frac{1}{2} & 5 & -\frac{1}{2} \\ 0 & \dots & \dots & \dots & 0 & \frac{1}{5} & \frac{1}{4} & \frac{1}{3} & -\frac{1}{2} & 5 \end{bmatrix}_{n\times n},$$

$$\mathcal{N}_1=\begin{bmatrix} 2 & -\frac{1}{3} & -\frac{1}{2} & 0 & \dots & \dots & \dots & \dots & \dots & 0 \\ \frac{1}{4} & 2 & -\frac{1}{3} & -\frac{1}{2} & & & & & & \vdots \\ \frac{1}{5} & \frac{1}{4} & 2 & -\frac{1}{3} & -\frac{1}{2} & & & & & \vdots \\ 0 & \frac{1}{5} & \frac{1}{4} & 2 & -\frac{1}{3} & -\frac{1}{2} & & & & \vdots \\ \vdots & & \frac{1}{5} & \frac{1}{4} & 2 & -\frac{1}{3} & -\frac{1}{2} & & & \vdots \\ \vdots & & & \frac{1}{5} & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & & & & \ddots & \ddots & \ddots & \ddots & -\frac{1}{2} \\ \vdots & & & & & & \ddots & \ddots & \ddots & -\frac{1}{3} \\ 0 & \dots & \dots & \dots & \dots & \dots & 0 & \frac{1}{5} & \frac{1}{4} & 2 \end{bmatrix}_{n\times n},$$

and the matrix on the right-hand side, $\mathcal{C}$, is defined such that $\mathcal{X}^\star = (x_{i,j})$, where

$$x_{i,j} = \exp\left[-\left(\{-1+2(i-1)/(n-1)\}^2 + \{-1+2(j-1)/(n-1)\}^2\right)\right], \quad i,j = 1,2,\dots,n,$$

is the exact solution.

For this test problem we have reported the numerical results in Tables 5, 6, 7 and 8. Based on the results from these tables we conclude the new method for $(\alpha,\beta) = (9.3, 0.9)$ considerably outperforms IGMRES(5) method, because it needs less iteration counts and CPU time.

Table 5: Example 3.2: Numerical results for $(\alpha,\beta) = (9.3, 0.9)$ and $n = 500$.

Iteration $(= k)$	5	10	15	20	25
$\log_{10}\left(\Vert\mathcal{X}^\star - \mathcal{X}^{(k)}\Vert_F/\Vert\mathcal{X}^\star\Vert_F\right)$	−8.1476	−10.1805	−11.9505	−13.6138	−14.3546
CPU-time (seconds)	10.1649	10.2052	10.2291	10.2497	10.3071

Table 6: Example 3.2: Numerical results for $(\alpha,\beta) = (9.3, 0.9)$ and $n = 500$.

Iteration $(= k)$	30	35	40	45	50
$\log_{10}\left(\Vert\mathcal{X}^\star - \mathcal{X}^{(k)}\Vert_F/\Vert\mathcal{X}^\star\Vert_F\right)$	−14.3580	−14.3582	−14.3586	−14.3583	−14.3589
CPU-time (seconds)	10.4254	10.5264	10.5605	10.8451	10.9819

On the other hand we plot the logarithm of relative error as a function of iterations for $n = 100$ and different parameters (α,β), see Figures 4 and 5. According to these figure we found that the parameters $(\alpha,\beta) = (9.3, 0.9)$ are optimum.

The approximate solutions for imaginary and real parts; after 2, 5, 10 and 50 iterations are plotted in Fig. 6 for optimum parameters $\alpha = 9.3$ and $\beta = 0.9$. From the results obtained in test problems we can see the proposed method is efficient.

Table 7: Example 3.2: Numerical results for GlGMRES(10) method and $n = 500$.

Iteration $(= k)$	5	10	15	20	25
$\log_{10}\left(\Vert\mathcal{X}^\star - \mathcal{X}^{(k)}\Vert_F / \Vert\mathcal{X}^\star\Vert_F\right)$	-1.7517	-1.7517	-1.7517	-1.7517	-1.7517
CPU-time (seconds)	0.0309	0.0311	0.0312	0.0316	0.0318

Table 8: Example 3.2: Numerical results for GlGMRES(10) method and $n = 500$.

Iteration $(= k)$	30	35	40	45	50
$\log_{10}\left(\Vert\mathcal{X}^\star - \mathcal{X}^{(k)}\Vert_F / \Vert\mathcal{X}^\star\Vert_F\right)$	-1.7517	-1.7517	-1.7517	-1.7517	-1.7517
CPU-time (seconds)	0.0319	0.0320	0.0322	0.0324	0.0326

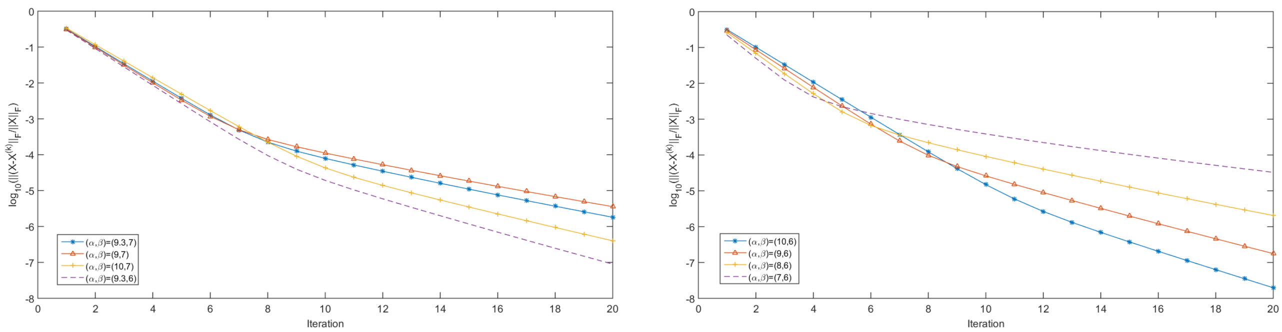

Figure 4: Relative error as a function of iterations and different parameters for Example 3.2.

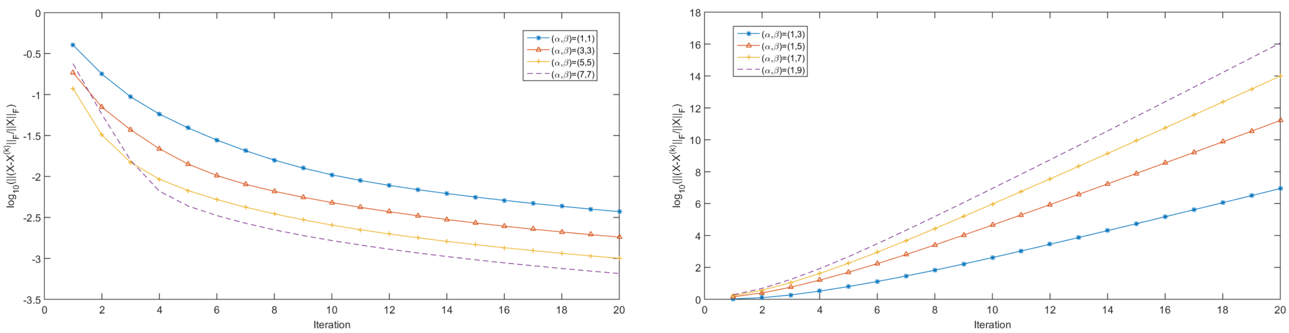

Figure 5: Relative error as a function of iterations and different parameters for Example 3.2.

4. Conclusion

In this study, we proposed an effective method for solving the generalized Lyapunov matrix equation

$$\mathcal{M}\mathcal{X} + \mathcal{X}\mathcal{M}^T + \sum_{j=1}^{m} \mathcal{N}_j \mathcal{X} \mathcal{N}_j^T = \mathcal{C},$$

where $\mathcal{M}$, $\mathcal{N}_j$ $(j = 1, \ldots, m)$, and $\mathcal{C} \in \mathbb{C}^{n\times n}$ are known matrices, and the goal is to find $\mathcal{X} \in \mathbb{C}^{n\times n}$.

The proposed approach involves solving two standard Lyapunov matrix equations at each iteration step, which can be handled either by direct methods or numerical solvers. Theoretical analysis confirms that the method performs well under suitable conditions. To demonstrate its practical efficiency, the method has been tested through two illustrative numerical examples. As a future research direction, it would be interesting to investigate the use of the proposed splitting method as an inexact preconditioner for (Flexible) Gl-GMRES, which may enhance its convergence while preserving computational efficiency.

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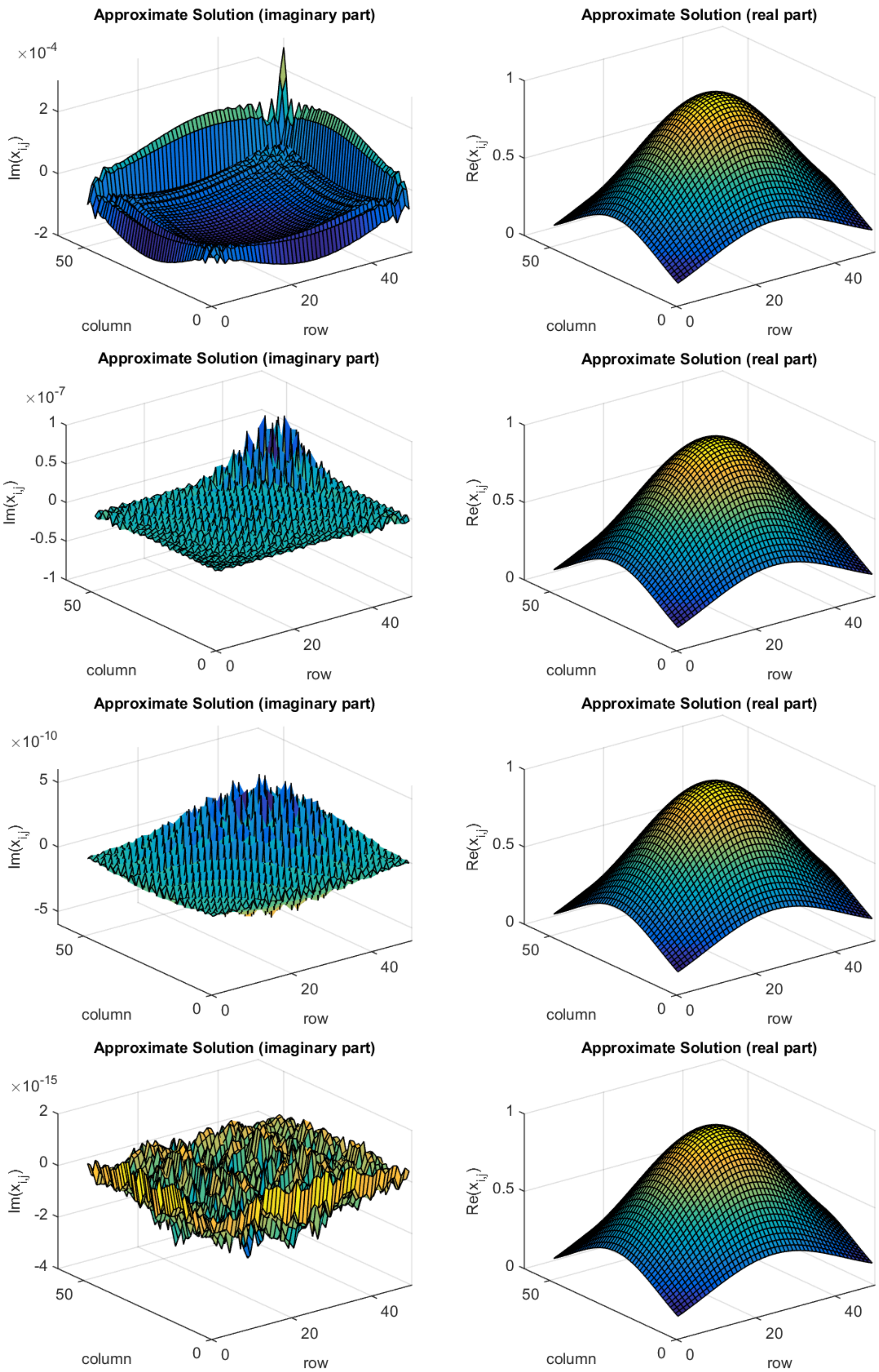

Figure 6: Approximate solutions for imaginary and real parts; after 2, 5, 10 and 50 iterations for Example 3.2.

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