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Original Article

Induced deformed Finsler metric and deformed non-linear connections

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ABSTRACT: In this paper, we study induced deformed non linear connections on Finsler sub-manifolds and prove that every non linear connection on a Finslerian sub-manifold is an induced deformed non-linear connection. In addition, we provide some conditions under which Finslerian sub-manifolds are Landsbergian manifolds.

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1. Introduction

Finsler manifolds extend the concept of Riemannian manifolds by incorporating a wider variety of metrics that account for both the magnitude and direction of tangent vectors. The exploration of Finsler geometry has produced valuable insights and applications across multiple disciplines, such as physics, optimization theory, and robotics. A Finsler manifold can be defined as a manifold M in which each tangent space is endowed with a Minkowski norm, a type of norm that is not necessarily derived from an inner product [7]. In contrast to Riemannian manifolds, this norm generates canonical inner products that depend on the direction within the tangent space TM rather than being solely determined by the points on M. Consequently, one can conceptualize a Finsler manifold as a space where the inner product is influenced not only by the location but also by the orientation of observation. In contrast, every Riemannian manifold can be regarded as a specific instance of a Finsler manifold where the inner product remains constant across all tangent directions. Finsler geometry signifies a substantial departure from the Riemannian framework. Nevertheless, Finsler geometry retains analogs for many fundamental constructs found in

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Riemannian geometry, including the length of curves, geodesics, curvature, connections, and covariant derivatives, although normal coordinates do not have a direct counterpart [13, 14].

In Finsler manifolds, geometric objects exhibit a dual nature, characterized by both forward and backward properties. This duality is a consequence of the fundamental asymmetry present in the distance function inherent to Finsler geometry. As a result, Klingenberg's notable findings regarding closed geodesics in Riemannian geometry do not extend directly to the Finsler context. Instead, it is necessary to consider reversible Finsler metrics, which possess a symmetric quality. The study of geodesics that originate from submanifolds and terminate within them, potentially at a location distinct from the origin, has been explored by Omori in [6].

In Finsler geometry and its various generalizations, nonlinear connections are of significant importance. These connections facilitate the manipulation of geometric objects, enabling us to maintain and verify the geometric interpretation of calculations within local coordinates [9].

A C^∞ distribution on TM is designated as a non-linear connection on TM if $T_vTM = H_v \oplus V_v$ for $v \in TM$, where $V \equiv \ker(d\pi)$ represents the vertical distribution. Additionally, a non-linear connection on TM is commonly known as a horizontal distribution on TM, with serving as the associated horizontal bundle. A non-linear connection is guaranteed to exist in the context of a paracompact manifold M. Nevertheless, the non-linear connections that are either derived from or related to a generalized Lagrange metric prove to be significantly more advantageous. Currently, it remains unfeasible to directly obtain a non-linear connection from an arbitrary generalized Lagrange metric. However, certain classes of generalized Lagrange metrics allow for the canonical association of a nonlinear connection with the generalized Lagrange metric. This discussion draws upon the works of M. Anastasiei and H. Shimada as referenced in [1] and [2].

In this work, we investigate induced deformed non-linear connections on Finsler sub-manifolds, demonstrating that any non-linear connection present on a Finslerian sub-manifold qualifies as an induced deformed non-linear connection. Furthermore, we outline specific conditions that characterize Finslerian sub-manifolds as Landsbergian manifolds. The structure of this paper is outlined as follows. The second section presents the essential prerequisites and core concepts pertinent to this study. The third section discusses the induced non-linear connection. In section 4, the induced Landsberg connection is illustrated. Lastly, section 5 details the induced deformed Finsler metric and the associated results.

2. Preliminaries

Suppose $\widetilde{\mathbb{F}}^{m+n} = (\widetilde{M}, \widetilde{F})$ is a smooth Finsler manifold of $\dim \widetilde{M} = m+n$, and $\widetilde{M}' := T\widetilde{M}\backslash\{0\}$ with the canonical projection map $\widetilde{\pi} : \widetilde{M}' \to \widetilde{M}$. The vertical sub-bundle $V\widetilde{M}' = ker\widetilde{\pi}_*$, is a Riemannian bundle and denoted by $(V\widetilde{M}', \widetilde{M}, \widetilde{g})$ (cf. [3, 5]), where

$$\widetilde{g}_{ij}(x,y) = \frac{1}{2}\Big(\frac{\partial^2 \widetilde{F}^2}{\partial y^i \partial y^j}\Big)(x,y), \quad 1 \le i,j \le m+n.$$

Let M be an embedded m-dimensional submanifold of $\widetilde{M}$ and $M' := TM\backslash\{0\}$. Then M' can be embedded in $\widetilde{M}'$ and for each coordinate system (x^i, y^i) on $\widetilde{M}'$, there exists a coordinate system (u^α, v^α) on M' such that the function $F(u,v) := \widetilde{F}(x(u), y(u,v))$ is a Finsler structure for M,denoted by $\mathbb{F}^m = (M, F)$. Therefore, (VM', M', g) is a Riemannian bundle, and the natural local frame and local co-frame of M and $\widetilde{M}$ are related by the following relationships:

$$\frac{\partial}{\partial u^\alpha} = B^i_\alpha \frac{\partial}{\partial x^i} + B^i_{\alpha 0}\frac{\partial}{\partial y^i}, \qquad \frac{\partial}{\partial v^\alpha} = B^i_\alpha \frac{\partial}{\partial y^i}, \tag{1}$$

$$dx^i = B^i_\alpha du^\alpha, \qquad dy^i = B^i_{\alpha 0} du^\alpha + B^i_\alpha dv^\alpha. \tag{2}$$

The components of g and g^{-1}, are given by

$$g_{\alpha\beta} = \widetilde{g}_{ij} B^{ij}_{\alpha\beta}, \qquad g^{\alpha\beta} = \widetilde{g}^{ij} B^{\alpha\beta}_{ij}, \tag{3}$$

where

$$B^i_\alpha = \frac{\partial x^i}{\partial u^\alpha}, \qquad B^{ij}_{\alpha\beta} = B^i_\alpha B^j_\beta, \quad \ldots, \quad B^i_{\alpha 0} = B^i_{\alpha\beta} v^\beta, \qquad B^i_{\alpha\beta} = \frac{\partial^2 x^i}{\partial u^\alpha \partial v^\beta}. \tag{4}$$

Throughout this paper, Latin indices $i, j, k, \ldots$ run from 1 to $m+n$, and Greek indices $\alpha, \beta, \gamma, \ldots$ run from 1 to m, and indices $a, b, c, \ldots$ run from $m+1$ to $m+n$.

Suppose $VM'^{\perp}$ is the orthogonal complementary vector bundle of VM' in $V\widetilde{M}'$ with respect to $\widetilde{g}$, and $\{B_a\}$ is an orthogonal frame for $VM'^{\perp}$, where $B_a := B^i_a \frac{\partial}{\partial y^i}$. Let $[B^i \alpha B^i_a]$ be the transition matrix from $\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^{m+n}}\}$ to $\{\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^m}, B_{m+1}, \ldots, B_{m+n}\}$, and $[B^\alpha_i B^a_i]$ be its inverse. It is easy to show that (cf. [5])

$$B^\alpha_i B^i_\beta = \delta^\alpha_\beta, \qquad B^\alpha_i B^i_a = 0, \qquad B^a_i B^i_\alpha = 0, \qquad B^a_i B^i_b = \delta^a_b, \tag{5}$$

$$B^\alpha_i = g^{\alpha\beta} B^j_\beta \widetilde{g}_{ij}. \tag{6}$$

Let $\widetilde{G}^i_j$ and G^α_β be the canonical non-linear connections on $\widetilde{M}'$ and M', respectively, where

$$\widetilde{G}^i_j = \frac{\partial \widetilde{G}^i}{\partial y^j}, \qquad G^\alpha_\beta = \frac{\partial G^\alpha}{\partial v^\beta}, \tag{7}$$

and

$$\widetilde{G}^i = \frac{1}{4}\widetilde{g}^{ij}\Big(\frac{\partial^2 \widetilde{F}^2}{\partial y^j \partial x^k} y^k - \frac{\partial \widetilde{F}^2}{\partial x^j}\Big), \qquad G^\alpha = \frac{1}{4} g^{\alpha\beta}\Big(\frac{\partial^2 F^2}{\partial v^\beta \partial u^\gamma} v^\gamma - \frac{\partial F^2}{\partial u^\beta}\Big). \tag{8}$$

It is well-known that every arbitrary non-linear connection on $\widetilde{M}'$, say $\widetilde{N}^i_j$, defines a *Sasaki-Finsler* mertic on $\widetilde{M}'$, denoted by $\widetilde{G}$ and given by

$$\widetilde{G} = \widetilde{g}_{ij} dx^i dx^j + \widetilde{g}_{ij} \delta y^i \delta y^j, \tag{9}$$

where $\{dx^i, \delta y^i\}$ is the co-frame of the local frame $\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\}$, and we have

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \widetilde{N}^j_i \frac{\partial}{\partial y^j}, \qquad \delta y^i = dy^i + \widetilde{N}^i_j dx^j. \tag{10}$$

The sub-bundle $H\widetilde{M}' = span\{\frac{\delta}{\delta x^i}\}$ is called the *horizontal distribution* of $\widetilde{N}^i_j$.

We note that $V\widetilde{M}' = span\{\frac{\partial}{\partial y^i}\}$ is an integrable distribution on $\widetilde{M}'$, while $H\widetilde{M}'$ is not integrable in general case. It is important to note that we have

$$T\widetilde{M}' = V\widetilde{M}' \oplus H\widetilde{M}'. \tag{11}$$

A. Bejancu, and H. R. Farran show that (cf. [5]) a non-linear connection $\widetilde{N}^i_j$ on $\widetilde{M}'$ induces a non-linear connection N^α_β on M' given by

$$N^\alpha_\beta = g^{\alpha\gamma}\widetilde{G}\Big(\frac{\partial}{\partial u^\beta}, \frac{\partial}{\partial v^\gamma}\Big). \tag{12}$$

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Let $\{\frac{\delta}{\delta u^\alpha}\}$ be a local frame of the horizontal distribution of N^α_β, denoted by HM', where $\frac{\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - N^\beta_\alpha \frac{\partial}{\partial v^\beta}$. Then we have the following lemma.

Lemma 3.1 ([5]). *HM' is orthogonal to VM' with respect to $\widetilde{G}$ and is the horizontal distribution of N^α_beta, where*

$$N^\alpha_\beta = B^\alpha_i (B^i_{\beta 0} + B^j_\beta N^i_j). \tag{13}$$

Using Lemma 3.1, we obtain

$$\delta v^\alpha = B^\alpha_j \delta y^j, \qquad \frac{\delta}{\delta u^\alpha} = B^i_\alpha \frac{\delta}{\delta x^i} + H^a_\alpha B_a, \tag{14}$$

where

$$H^a_\alpha := B^a_i (B^i_{\alpha 0} + B^j_\alpha N^i_j). \tag{15}$$

Definition 3.2. *Let N^i_j be a non-linear connection, and A^i_j be a d-tensor of type $(1,1)$. Then the non-linear connection $\dot{N}^i_j = N^i_j + A^i_j$ is called the deformation of N^i_j by A^i_j.*

Proposition 3.3. *Let $\dot{\widetilde{N}}^i_j$ be the deformation of $\widetilde{N}^i_j$ by $\widetilde{A}^i_j$. Also, let $\dot{N}^\alpha_\beta$ and N^α_β be induced non-linear connection of $\dot{\widetilde{N}}^i_j$ and $\widetilde{N}^i_j$, respectively. Then $\dot{N}^\alpha_\beta$ is the deformation of N^α_β by $A^\alpha_\beta = \widetilde{A}^i_j B^j_\beta B^\alpha_i$.*

Proof. Let $\dot{H}\widetilde{M}' = span\{\frac{\dot{\delta}}{\delta x^i}\}$ be the horizontal distribution of $\dot{\widetilde{N}}^i_j$, and

$$\dot{\widetilde{G}} = \widetilde{g}_{ij} dx^i dx^j + \widetilde{g}_{ij} \dot{\delta} y^i \dot{\delta} y^j,$$

be the Sasaki-Finsler metric on $\widetilde{M}'$, where

$$\frac{\dot{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - \dot{\widetilde{N}}^j_i \frac{\partial}{\partial y^j}, \qquad \dot{\delta} y^i = dy^i + \dot{\widetilde{N}}^i_j dx^j.$$

Hence according to (12) and Lemma (3.1) we have

$$\dot{N}^\alpha_\beta = B^\alpha_i (B^i_{\beta 0} + B^j_\alpha \dot{\widetilde{N}}^i_j) = B^\alpha_i (B^i_{\beta 0} + B^j_\beta \widetilde{N}^i_j + B^j_\beta \widetilde{A}^i_j) = N^\alpha_\beta + B^\alpha_i B^j_\beta \widetilde{A}^i_j.$$

Let $A^\alpha_\beta := B^\alpha_i B^j_\beta \widetilde{A}^i_j$, so we have

$$\dot{N}^\alpha_\beta = N^\alpha_\beta + A^\alpha_\beta. \tag{16}$$

It suffices to show that A^α_β is the d-tensor of type $(1,1)$ on M'. Suppose $(\bar{x},\bar{y})$ and $(\bar{u},\bar{v})$ are another coordinate systems on $\widetilde{M}'$ and M', respectively, and

$$U^\alpha_\beta := \frac{\partial u^\alpha}{\partial \bar{u}^\beta}. \tag{17}$$

Then, it is easy to show that the following two relations hold (c.f., [12]):

$$\dot{N}^\alpha_\beta(u,v) U^\gamma_\alpha(u) = \dot{\bar{N}}^\gamma_\eta(\bar{u},\bar{v}) U^\eta_\beta(u) + U^\gamma_{\beta\lambda}(u) v^\lambda, \tag{18}$$

$$N^\alpha_\beta(u,v) U^\gamma_\alpha(u) = \bar{N}^\gamma_\eta(\bar{u},\bar{v}) U^\eta_\beta(u) + U^\gamma_{\beta\lambda}(u) v^\lambda, \tag{19}$$

where

$$U^\gamma_{\alpha\lambda} := \frac{\partial^2 u^\gamma}{\partial \bar{u}^\alpha \partial \bar{v}^\lambda}.$$

By (16), (18) and (19), we obtain

$$N^\alpha_\beta U^\gamma_\alpha + A^\alpha_\beta U^\gamma_\alpha = \bar{N}^\gamma_\eta U^\eta_\beta + \bar{A}^\gamma_\eta U^\eta_\beta + U^\gamma_{\beta\lambda} v^\lambda = N^\alpha_\beta U^\gamma_\alpha + \bar{A}^\gamma_\eta U^\eta_\beta \Rightarrow A^\alpha_\beta U^\gamma_\alpha = \bar{A}^\gamma_\eta U^\eta_\beta \Rightarrow \bar{A}^\gamma_\eta = A^\alpha_\beta U^\gamma_\alpha \bar{U}^\beta_\eta,$$

where $\bar{U}^\alpha_\eta := \frac{\partial \bar{u}^\alpha}{\partial u^\eta}$. This means that A^α_β is the d-tensor of type $(1,1)$ on M', and the proof is complete. □

Corollary 3.4. *Let $\overset{1}{\widetilde{N}}{}^i_j$ and $\overset{2}{\widetilde{N}}{}^i_j$ be two non-linear connections on $\widetilde{M}'$. Then $\widetilde{A}^i_j = \overset{1}{\widetilde{N}}{}^i_j - \overset{2}{\widetilde{N}}{}^i_j$ is a d-tensor field of type $(1,1)$ on $\widetilde{M}'$.*

Now, we study the induced non-linear connection of canonical non-linear connection. Let

$$\widetilde{g}_{ijk} = \frac{1}{2} \frac{\partial g_{ij}}{\partial y^k}, \qquad g_{\alpha\alpha\beta} = \widetilde{g}_{ijk} B^i_a B^{jk}_{\alpha\beta}, \qquad g^\alpha_{a\beta} = g_{a\beta\gamma} g^{\alpha\gamma}. \tag{20}$$

Lemma 3.5. *Let G^{α}_{β} be the induced non-linear connection of G^{i}_{j}. Then we have*

$$\dot{G}^{\alpha}_{\beta} = G^{\alpha}_{\beta} + D^{\alpha}_{\beta}, \tag{21}$$

where

$$D^{\alpha}_{\beta} = g^{\alpha}_{a\beta} H^{a}_{\lambda} v^{\lambda}. \tag{22}$$

Lemma 3.6. *Let $A^{i}_{j} = g^{i}_{jk}(B^{h}_{a}B^{a}_{k})(B^{k}_{\gamma 0}B^{\gamma}_{l} + G^{k}_{l})y^{l}$. Then for D^{β}_{α} in Lemma 3.5, we have*

$$D^{\beta}_{\alpha} = B^{j}_{\alpha} B^{\beta}_{i} A^{i}_{j}. \tag{23}$$

Proof. From (15), (5) and (6) we obtain:

$$\begin{aligned} D^{\beta}_{\alpha} &= g^{\beta}_{a\alpha} B^{a}_{k}(B^{k}_{\gamma 0} + B^{l}_{\gamma} G^{k}_{l}) v^{\gamma} \\ &= g_{a\eta\alpha} g^{\beta\eta} B^{a}_{k}(B^{k}_{\gamma 0} + B^{l}_{\gamma} B^{k}_{l}) v^{\gamma} \\ &= g_{ijq} B^{i}_{a} B^{j}_{\eta} B^{q}_{\alpha} B^{a}_{k} g^{\beta\eta}(B^{k}_{\gamma 0} + B^{\gamma}_{l} G^{k}_{\gamma}) v^{\gamma} \\ &= g_{ijq} g^{\beta\eta}(\delta^{i}_{k} - N^{i}_{\xi} B^{\xi}_{k}) B^{j}_{\eta} B^{q}_{a}(B^{k}_{\gamma 0} B^{\gamma}_{l} + G^{k}_{l}) y^{l} \\ &= g^{i}_{kj} B^{\beta}_{i} B^{j}_{\alpha}(B^{h}_{a} B^{a}_{k})(B^{k}_{\gamma 0} B^{\gamma}_{l} + G^{k}_{l}) y^{l} \\ &= B^{j}_{\alpha} B^{\beta}_{i} A^{i}_{j}. \end{aligned}$$

□

Theorem 3.7. *There exists a non-linear connection (N^{i}_{j}) on $\widetilde{M}'$ such that the induced non-linear connection of (BN^{i}_{j}) is a canonical non-linear connection (G^{α}_{β}) on M'.*

Proof. Let $N^{j}_{i} = G^{j}_{i} + A^{j}_{i}$, where A^{j}_{i} be defined by Lemma 3.6, and let $\dot{G}^{\beta}_{\alpha}$ be the induced non-linear connection of G^{j}_{i} on M'. Using Proposition 3.3, we have

$$\dot{N}^{\alpha}_{\beta} = \dot{G}^{\alpha}_{\beta} + D^{\alpha}_{\beta}.$$

On the other hand, using Lemma 3.5, we obtain

$$G^{\alpha}_{\beta} = \dot{G}^{\alpha}_{\beta} + D^{\alpha}_{\beta}.$$

Comparing (22) with (23), we conclude that $G^{\alpha}_{\beta} = \dot{N}^{\alpha}_{\beta}$. □

Definition 3.8. *The non-linear connection $N^{i}_{j} = G^{i}_{j} + A^{i}_{j}$ is called a lift of the canonical non-linear connection G^{α}_{β}.*

Corollary 3.9. *Every non-linear connection on M' is an induced non-linear connection of a non-linear on $\widetilde{M}'$.*

Proof. Let $N^{\alpha}_{\beta} = G^{\alpha}_{\beta} + A^{\alpha}_{\beta}$ be a non-linear connection on M', where A^{α}_{β} be a d-tensor field of type $(1,1)$ on M', and let $N^{i}_{j} = G^{i}_{j} + A^{i}_{j} + E^{i}_{j}$, where $E^{i}_{j} = B^{\alpha}_{i} B^{j}_{\alpha} A^{\alpha}_{\beta}$. Then by Theorem 3.7 and Lemma 3.1 we have

$$\dot{N}^{\alpha}_{\beta} = B^{\alpha}_{i}(B^{i}_{\beta 0} + B^{i}_{\alpha}(G^{j}_{i} + A^{j}_{i})) + B^{\alpha}_{i} B^{j}_{\beta} E^{j}_{i} = G^{\alpha}_{\beta} + A^{\alpha}_{\beta}.$$

□

Definition 3.10. *The non-linear connection $N^{i}_{j} = G^{i}_{j} + A^{i}_{j} + E^{i}_{j}$ is called a lift of the non-linear connection of N^{α}_{β}.*

4. Induced Landsberg Connection

Let $\widetilde{FC}^{R} = (\widetilde{G}^{i}_{j}, \widetilde{F}^{i}_{j\ k}, 0)$ and $\widetilde{FC}^{B} = (\widetilde{G}^{i}_{j}, \widetilde{g}^{i}_{j\ k}, 0)$ be the Rund and Berwald connections on $\widetilde{F}^{m+n} = (\widetilde{M}, \widetilde{F})$, respectively, where

$$\widetilde{F}^{i}_{j\ k} = \frac{1}{2}\widetilde{g}^{ih}\Big(\frac{\delta \widetilde{g}_{jh}}{\delta x^{k}} + \frac{\delta \widetilde{g}_{hk}}{\delta x^{j}} - \frac{\delta \widetilde{g}_{jk}}{\delta x^{h}}\Big), \qquad \widetilde{g}^{i}_{j\ k} = \frac{\partial \widetilde{G}^{i}_{j}}{\partial y^{k}}.$$

We consider $FC^{B} = (G^{\alpha}_{\beta}, g^{\alpha}_{\beta\ \gamma}, 0)$ as a Berwald Finsler linear connection on F^{m}, where $\widetilde{G}^{i}_{j}$ and G^{α}_{β} are canonical non-linear connection on $\widetilde{M}'$ and M', respectively.

Definition 4.1. *Let $\widetilde{N}^i_j = \widetilde{G}^i_j + \widetilde{A}^i_j$ be the deformation of $\widetilde{G}^i_j$ by $\widetilde{A}^i_j$ (as defined in Theorem 3.7), and $H\widetilde{M}' = span\{\frac{\widehat{\delta}}{\delta x^i}\}$ be the horizontal distribution of $\widetilde{N}^i_j$. Then $H\widetilde{M}'$-Rund and $H\widetilde{M}'$-Berwald connections, respectively are defined by*

$$\widehat{\widetilde{RFC}} = (\widetilde{N}^i_j, \widehat{F}^i_{j\ k}, 0), \qquad \widehat{\widetilde{BFC}} = (\widetilde{N}^i_j, \widehat{N}^i_{j\ k}, 0), \tag{24}$$

where

$$\widehat{F}^i_{j\ k} = \frac{1}{2}\widetilde{g}^{ih}\Big(\frac{\widehat{\delta}\widetilde{g}_{kh}}{\delta x^j} + \frac{\widehat{\delta}\widetilde{g}_{hj}}{\delta x^k} - \frac{\widehat{\delta}\widetilde{g}_{jk}}{\delta x^h}\Big), \qquad \widehat{N}^i_{j\ k} = \frac{\partial \widetilde{N}^i_j}{\partial y^k}. \tag{25}$$

In general the induced non-linear connection of the Berwald connection is not Berwald connection, that is, being Berwald connection is not a hereditary property. Here we show that being $H\widetilde{M}'$-Berwald connection is a hereditary property.

Theorem 4.2. *The induced non-linear connection of $H\widetilde{M}'$-Berwald connection $\widehat{\widetilde{RFC}} = (\widetilde{N}^i_j, \widehat{F}^i_{j\ k}, 0)$ of $\widetilde{F}^{m+n}$ is an HM'-Berwald connection of F^m.*

Proof. Let $FC = (N^\alpha_\beta, N^\alpha_{\beta\ \gamma}, C^\alpha_{\beta\ \gamma})$ be the induced non-linear connection of $\widehat{\widetilde{BFC}}$ on F^m. Therefore (cf. [5], Theorem 4.1 p. 103)

$$N^\alpha_{\beta\ \gamma} = B^\gamma_k(B^k_{\beta\gamma} + B^{ij}_{\beta\gamma}N^k_{i\ j}), \qquad C^\alpha_{\beta\ \gamma} = 0. \tag{26}$$

It suffices to show that $N^\alpha_{\beta\ \gamma} = \frac{\partial N^\alpha_\beta}{\partial y^\gamma}$. By derivation with respect to v^γ of (13) we have

$$\begin{aligned} \frac{\partial N^\alpha_\beta}{\partial v^\gamma} &= \frac{\partial}{\partial v^\gamma}\Big(B^\alpha_i(B^i_{\beta 0} + B^j_\beta N^i_j)\Big) \\ &= B^\alpha_i B^i_{\beta\gamma} + B^j_\beta B^k_\gamma B^\alpha_i N^i_{jk} \\ &= B^\alpha_k(B^k_{\beta\gamma} + B^{ij}_{\beta\gamma}N^k_{ij}). \end{aligned} \tag{27}$$

Using (24) and (26) we deduce that $N^\alpha_{\beta\ \gamma} = \frac{\partial N^\alpha_\beta}{\partial v^\gamma}$. □

Theorem 4.3. *If F^m is a totally geodesic submanifold of $\widetilde{F}^{m+n}$, then the induced Berwald-Finsler connection $\widetilde{FC} = (\widetilde{G}^i_j, \widetilde{g}^i_{j\ k}, 0)$ of $\widetilde{F}^{m+n}$ on Finsler submanifold F^m is Berwald- Finsler connection $FC = (G^\alpha_\beta, g^\alpha_{\beta\ \gamma}, 0)$.*

Proof. We know that F^m is a totally geodesic submanifold of $\widetilde{F}^{m+n}$ if and only if the normal curvature is vanishing (see [5] Theorem 1.2, p. 131), i.e.

$$\forall a, \alpha \quad H^a_\alpha = 0.$$

By Lemma 3.5, if F^m is a totally geodesic submanifold of $\widetilde{F}^{m+n}$, then $D^\beta_\alpha = 0$. Thus by Theorem 3.7, the proof is complete. □

We say that the Finsler manifold is a *Landsberg manifold (Landsbergian)*, if the Berwald connection coincides with the Run connection, i.e.,

$$\widetilde{g}^i_{j\ k} = \widetilde{F}^i_{j\ k},$$

and $\widetilde{FC}^L = (\widetilde{G}^i_j, \widetilde{F}^i_{j\ k} = \widetilde{g}^i_{j\ k}, 0)$ is called a *Landsberg connection.*

Theorem 4.4. *If F^m is a totally geodesic submanifold of $\widetilde{F}^{m+n}$, then the induced Landsberg manifold is Landsbergian.*

Proof. Let $\widetilde{FC}^B = (\widetilde{G}^i_j, \widetilde{g}^i_{j\ k}, 0)$ and $\widetilde{FC}^R = (\widetilde{G}^i_j, \widetilde{F}^i_{j\ k}, 0)$ be Berwald and Rund connections on $\widetilde{F}^{m+n}$, and let $FC^B = (G^\alpha_\beta, g^\alpha_{\beta\ \gamma}, 0)$ and $FC^R = (G^\alpha_\beta, F^\alpha_{\beta\ \gamma}, 0)$ be the intrinsic Berwald and Rund connections on F^m, respectively. Also, let $\overline{FC}^B = (N^\alpha_\beta, \overline{g}^\alpha_{\beta\ \gamma}, \overline{C}^\alpha_{\beta\ \gamma})$ and $\overline{FC}^R = (N^\alpha_\beta, \overline{F}^\alpha_{\beta\ \gamma}, \overline{C}^\alpha_{\beta\ \gamma})$ be induced Berwald and Rund connections on F^m, respectively. Then we have (cf.[5])

$$\overline{F}^\alpha_{\beta\ \gamma} = F^\alpha_{\beta\ \gamma}, \qquad \overline{C}^\alpha_{\beta\ \gamma} = C^\alpha_{\beta\gamma}. \tag{28}$$

By hypothesis, we have $\widetilde{g}^i_{j\ k} = \widetilde{F}^i_{j\ k}$ and

$$\overline{F}^\alpha_{\beta\ \gamma} = (B^k_{\beta\gamma} + B^{ij}_{\beta\gamma}\widetilde{F}^k_{i\ j})B^\alpha_k, \qquad \overline{g}^\alpha_{\beta\ \gamma} = (B^k_{\beta\gamma} + B^{ij}_{\beta\gamma}\widetilde{g}^k_{i\ j})B^\alpha_k. \tag{29}$$

Thus by (27), (28), (29) and Theorem 4.3 we obtain

$$g^\alpha_{\beta\ \gamma} = \overline{g}^\alpha_{\beta\ \gamma} = \overline{F}^\alpha_{\beta\ \gamma} = F^\alpha_{\beta\ \gamma},$$

i.e. F^m is Landsbergian. □

5. Induced Deformed Finsler Metric

In the second part of the paper we attend to the deformed Finsler metrics. These facts led R. Miron ([8, 10, 11]) to propose the study of generalized Lagrange metrics, GL-metrics for brevity, whose definition is tailored after the basic properties of Finsler metrics.

A *GL-metric* is a $(0,2)$ d-tensor field $\widetilde{g}_{ij}(x,y)$ on $T\widetilde{M}$ for which the matrix $(\widetilde{g}(x,y))$ is positive definite on $\widetilde{M}'$. For example, every Finsler metric with Finslerian energy function F induces is a GL-metric. For the GL-metric $\widetilde{g}_{ij}(x,y)$, the $(0,3)$ d-tensor field C_{ijk} is defined as follows

$$C_{ijk} = \frac{1}{2}\Big(\frac{\partial \widetilde{g}_{ik}}{\partial y^j} + \frac{\partial \widetilde{g}_{kj}}{\partial y^i} - \frac{\partial \widetilde{g}_{ij}}{\partial y^k}\Big), \tag{30}$$

which is symmetric in j,k. Suppose $\widetilde{F}^{m+n} = (\widetilde{M}, \widetilde{F})$ is a Finslerian manifold and $F^m = (M,F)$ is a Finslerian submanifold of $\widetilde{F}^{m+n}$, and that assumptions of previous sections are satisfied. In this case, the *deformation meter* $\widetilde{g}_{ij}(x,y)$ is defined as follows

$$*\widetilde{g}_{ij} := \widetilde{g}_{ij}(x,y) + \sigma(x,y)B_{ij}(x,y),$$

where $\sigma(x,y)$ is a scalar smooth function on $T\widetilde{M}$, and $B_{ij}(x,y)$ a symmetric d-tensor field of type $(0,2)$. Obviously $*\widetilde{g}_{ij}(x,y)$ is a GL-metric. For example, if $\sigma > 0$, $B^i(x,y) = y^i$, $B_j(x,y) = \widetilde{g}_{ij}(x,y)y^i$ and $B_{ij}(x,y) = y^iy^j$, then

$$*\widetilde{g}_{ij}(x,y) = \widetilde{g}_{ij}(x,y) + \sigma(x,y)y_iy_j,$$

is GL-metric. For applications of deformed Finsler metric you can see [8, 10, 11, 12]. In this paper, given d-tensor field $B^i(x,y)$ of type $(1,0)$, we are going to study

$$*\widetilde{g}_{ij}(x,y) = \widetilde{g}_{ij}(x,y) + \sigma(x,y)B_i(x,y)B_j(x,y), \tag{31}$$

where $B_i(x,y) = \widetilde{g}_{ij}(x,y)B^j(x,y)$. By straightforward calculations it is easy to show that the components of the inverse of $(*\widetilde{g}_{ij})$ is given by (cf. [10, 8])

$$*\widetilde{g}^{ij} = \widetilde{g}^{ij} - *\sigma(x,y)B^iB^j, \tag{32}$$

where $*\sigma(x,y) = \frac{\sigma(x,y)}{1+\sigma(x,y)B^2}$ and $B^2 = \widetilde{g}_{ij}B^iB^j$. The GL-metric has been used by R. Beil [4] for pseudo Riemann-Finsler manifolds. For the Riemannian metrics $*\widetilde{g} = (\widetilde{g}_{ij})$ on $V\widetilde{M}'$, we consider $*VM'^{\perp}$ the orthogonal complementary of $V\widetilde{M}'$ in $T\widetilde{M}'$. Let $\{*B_a := *B^i_a\frac{\partial}{\partial y^i}\}$ be the orthogonal local frames for $*VM'^{\perp}$. Denote by $[B^i_\alpha, *B^i_a]$ the transition matrix from the frames $\{\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^{m+n}}\}$ to $\{\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^m}, *B^i_{m+1}, \cdots, *B^i_{m+n}\}$, and its inverse by $[*B^\alpha_i, *B^a_i]$. Thus

$$*B^\alpha_iB^i_\beta = \delta^\alpha_\beta, \quad (*B^\alpha_i)(*B^i_a) = 0, \quad *B^a_iB^i_\alpha = 0, \tag{33}$$

$$(*B^a_i)(*B^i_b) = \delta^a_b, \qquad B^i_a(*B^\alpha_j) + (*B^i_a)(*B^a_j) = \delta^i_j, \tag{34}$$

$$*B^\alpha_j = *\widetilde{g}_{jk}B^k_\beta * \widetilde{g}^{\alpha\beta}. \tag{35}$$

We put

$$*g_{\alpha\beta}(u,v) = *g_{ij}(x(u), y(u,v))B^{ij}_{\alpha\beta}, \tag{36}$$

where $*g_{\alpha\beta}(u,v)$ is a GL-metric on M'. Note that

$$*g_{\alpha\beta}(u,v)v^\alpha v^\beta = *g_{ij}(x(u),y(u,v))B^i_\alpha B^j_\beta v^\alpha v^\beta = *g_{ij}y^iy^j \geq 0,$$

then we have

$$*g^{\alpha\beta} = *g^{ij}(*B^\alpha_i)(*B^\beta_j).$$

Let $\widehat{\widetilde{CFC}} = (N^i_j, \widehat{F}^i_{j\ k}, \widehat{C}^i_{j\ k})$ and $\widehat{\widetilde{BFC}} = (N^i_j, N^i_{j\ k}, 0)$ be the Cartan and Berwald connections on $\widetilde{F}^{m+n} = (\widetilde{M}, \widetilde{M}', \widetilde{F})$, respectively. Let $H\widehat{\widetilde{M}'} = span\{\frac{\widehat{\delta}}{\delta x^i}\}$ and $H\widetilde{M}' = span\{\frac{\delta}{\delta x^i}\}$, where

$$\frac{\widehat{\delta}}{\delta x^i} = \frac{\partial}{\partial x^i} - N^j_i\frac{\partial}{\partial y^j}, \qquad \frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G^j_i\frac{\partial}{\partial y^j}. \tag{37}$$

Here we consider $N^i_j = G^i_j + A^i_j$, where the induced non-linear connection of (N^i_j) on M' coincides to canonical non-linear connection (G^α_β). We define

$$\widehat{\delta}y^i = dy^i + N^i_jdx^j, \qquad \delta y^i = dy^i + G^i_jdx^j. \tag{38}$$

Now, let $\widetilde{G}$, $\widehat{\widetilde{G}}$, $*\widetilde{G}$ and $*\widehat{\widetilde{G}}$ be Sasaki-Finsler metrics on $\widetilde{M}'$, where

$$G = \widetilde{g}_{ij}dx^i \otimes dx^j + \widetilde{g}_{ij}\delta y^i \otimes \delta y^j, \tag{39}$$

$$\widehat{G} = \widetilde{g}_{ij}dx^i \otimes dx^j + \widetilde{g}_{ij}\widehat{\delta}y^i \otimes \widehat{\delta}y^j, \tag{40}$$

$$*G = *\widetilde{g}_{ij}dx^i \otimes dx^j + *\widetilde{g}_{ij}\delta y^i \otimes \delta y^j, \tag{41}$$

$$*\widehat{G} = *\widetilde{g}_{ij}dx^i \otimes dx^j + *\widetilde{g}_{ij}\widehat{\delta}y^i \otimes \widehat{\delta}y^j. \tag{42}$$

We put $*\widehat{\widetilde{CFC}} = (N^i_j, *\widehat{F}^i_{j\ k}, *\widehat{C}^i_{j\ k})$ and $*\widetilde{CFC} = (G^i_j, *F^i_{j\ k}, *C^i_{j\ k})$, where

$$*\widehat{F}^i_{j\ k} = \frac{1}{2} * \widetilde{g}^{ih}\Big(\frac{\widehat{\delta} * \widetilde{g}_{jh}}{\delta x^k} + \frac{\widehat{\delta} * \widetilde{g}_{kh}}{\delta x^j} - \frac{\widehat{\delta} * \widetilde{g}_{jk}}{\delta x^h}\Big), \tag{43}$$

$$*\widehat{C}^i_{j\ k} = \frac{1}{2} * \widetilde{g}^{ih}\Big(\frac{\partial * \widetilde{g}_{jh}}{\partial y^k} + \frac{\partial * \widetilde{g}_{kh}}{\partial y^j} - \frac{\partial * \widetilde{g}_{jk}}{\partial y^h}\Big), \tag{44}$$

$$*F^i_{j\ k} = \frac{1}{2} * \widetilde{g}^{ih}\Big(\frac{\delta * \widetilde{g}_{jh}}{\delta x^k} + \frac{\delta * \widetilde{g}_{kh}}{\delta x^j} - \frac{\delta * \widetilde{g}_{jk}}{\delta x^h}\Big), \tag{45}$$

$$*C^i_{j\ k} = \frac{1}{2} * \widetilde{g}^{ih}\Big(\frac{\partial * \widetilde{g}_{jh}}{\partial y^k} + \frac{\partial * \widetilde{g}_{kh}}{\partial y^j} - \frac{\partial * \widetilde{g}_{jk}}{\partial y^h}\Big). \tag{46}$$

Corollary 5.1. *Using (43-46) we deduce that* $*\widehat{C}^i_{j\ k} = *C^i_{j\ k}$ *and*

$$*\widehat{F}^i_{j\ k} = *F^i_{j\ k} - \frac{1}{2} * \widetilde{g}^{ih}\Big(A^l_k\frac{\partial * \widetilde{g}_{jh}}{\partial y^l} + A^l_j\frac{\partial * \widetilde{g}_{kh}}{\partial y^l} - A^l_h\frac{\partial * \widetilde{g}_{jk}}{\partial y^l}\Big) \tag{47}$$

By (39-42), we consider the following four non-linear connections on M';

$$\begin{aligned} G^\alpha_\beta &= g^{\alpha\gamma}\widehat{G}\Big(\frac{\partial}{\partial u^\beta}, \frac{\partial}{\partial v^\gamma}\Big), \\ N^\alpha_\beta &= g^{\alpha\gamma}G\Big(\frac{\partial}{\partial u^\beta}, \frac{\partial}{\partial v^\gamma}\Big), \\ *G^\alpha_\beta &= *g^{\alpha\gamma} * \widehat{G}\Big(\frac{\partial}{\partial u^\beta}, \frac{\partial}{\partial v^\gamma}\Big), \\ *N^\alpha_\beta &= *g^{\alpha\gamma} * G\Big(\frac{\partial}{\partial u^\beta}, \frac{\partial}{\partial v^\gamma}\Big). \end{aligned} \tag{48}$$

The proof for $*N_\beta^\alpha$ and $*G_\beta^\alpha$ to be non-linear connections is similar for N_β^α and G_β^α. Also, corresponding to above non-linear connections, we define

$$\begin{aligned}\frac{\delta}{\delta u^\alpha} &= \frac{\partial}{\partial u^\alpha} - N_\alpha^\beta \frac{\partial}{\partial v^\beta},\\ \frac{\widehat{\delta}}{\delta u^\alpha} &= \frac{\partial}{\partial u^\alpha} - G_\alpha^\beta \frac{\partial}{\partial v^\beta},\\ \frac{*\delta}{\delta u^\alpha} &= \frac{\partial}{\partial u^\alpha} - *N_\alpha^\beta \frac{\partial}{\partial v^\beta},\\ \frac{*\widehat{\delta}}{\delta u^\alpha} &= \frac{\partial}{\partial u^\alpha} - *G_\alpha^\beta \frac{\partial}{\partial v^\beta}\end{aligned} \tag{49}$$

Theorem 5.2. *The following relations hold:*

(i) $*G_\beta^\alpha = *B_i^\alpha(B_{\alpha 0}^i + B_\alpha^j N_j^i)$,

(ii) $*N_\beta^\alpha = *B_i^\alpha(B_{\alpha 0}^i + B_\alpha^j G_j^i)$.

Proof. We prove (i). Using (31), (32), (38) and (1) we deduce that

$$\begin{aligned}*G_\beta^\alpha &= *g^{\alpha\gamma} * \widehat{G}\Big(\frac{\partial}{\partial u^\beta}, \frac{\partial}{\partial v^\gamma}\Big)\\ &= *g^{\alpha\gamma} * \widehat{G}\Big(B_\beta^i \frac{\partial}{\partial x^i} + B_{\beta 0}^i \frac{\partial}{\partial y^i}, B_\gamma^i \frac{\partial}{\partial y^i}\Big)\\ &= *g^{\alpha\gamma} B_{\beta 0}^i * g_{ij} B_\gamma^j + g^{\alpha\gamma} B_\beta^i N_i^k \delta_j^l B_\gamma^j * g_{kl}\\ &= *g^{jk} B_{\beta 0}^i * g_{ij} * B_k^\alpha + *g^{jk} * B_k^\alpha B_\beta^i N_i^l * g_{lj}\\ &= (B_{\beta 0}^i + B_\beta^i N_i^k) * B_k^\alpha.\end{aligned}$$

Part (ii) is proved similarly. □

Theorem 5.3. *The following relations hold:*

(i) $\dfrac{*\delta}{\delta u^\alpha} = B_\alpha^i \dfrac{\widehat{\delta}}{\delta x^i} + *H_\alpha^a * B_a$,

(ii) $\dfrac{*\widehat{\delta}}{\delta u^\alpha} = B_\alpha^i \dfrac{\widehat{\delta}}{\delta x^i} + *\widehat{H}_\alpha^a * B_a$,

where $*H_\alpha^a = *B_i^a(B_{\alpha 0}^i + B_\alpha^j G_j^i)$ *and* $*\widehat{H}_\alpha^a = *B_i^a(B_{\alpha 0}^i + B_\alpha^j N_j^i)$.

Proof. Using (1), (48) and (49) we have

$$\frac{*\delta}{\delta u^\alpha} = \frac{\partial}{\partial u^\alpha} - *N_\alpha^\beta \frac{\partial}{\partial v^\beta} = B_\alpha^i \frac{\partial}{\partial x^i} - *B_j^\beta(*B_{\alpha 0}^j + B_\alpha^k * N_k^j) B_\beta^l \frac{\partial}{\partial y^l} + B_{\alpha 0}^i \frac{\partial}{\partial y^i}.$$

By (33), (34) and (35), we obtain the relation (i). Part (ii) is proved similarly. □

Now we put

$$*C_{ijk} = \frac{1}{2}\Big(\frac{\partial * \widetilde{g}_{ik}}{\partial y^j} + \frac{\partial * \widetilde{g}_{ij}}{\partial y^k} - \frac{\partial * \widetilde{g}_{jk}}{\partial y^i}\Big),$$

thus (cf. [8])

$$*\widehat{F}_{ijk} = *F_{ijk} - *C_{kil} A_j^l - *C_{ijl} A_k^l - *C_{jkl} A_i^l.$$

Therefore

$$*F_{i\ j}^{\ k} = E_l^k F_{ij}^l + \frac{1}{2}\sigma * g^{kh}\Big(\frac{\delta(B_h B_j)}{\delta x^i} + \frac{\delta(B_h B_i)}{\delta x^j} - \frac{\delta(B_i B_j)}{\delta x^h}\Big) + \frac{1}{2} * g^{kh}(\sigma_i B_h B_j + \sigma_j B_h B_i - \sigma_h B_i B_j), \tag{50}$$

and

$$*C_{i\ j}^{\ k} = E_l^k C_{ij}^l + \frac{1}{2}\sigma * g^{kh}\Big(\frac{\partial(B_h B_j)}{\partial y^i} + \frac{\partial(B_h B_i)}{\partial y^j} - \frac{\partial(B_i B_j)}{\partial x^h}\Big) + \frac{1}{2} * g^{kh}(\dot{\sigma}_i B_h B_j + \dot{\sigma}_j B_i B_h - \dot{\sigma}_h B_i B_j), \tag{51}$$

where $\sigma_i = \frac{\partial\sigma}{\partial x^i}$, $\dot{\sigma}_i = \frac{\partial\sigma}{\partial y^i}$ and $E_j^i = \delta_j^i - *\sigma B^i B_j$.

Corollary 5.4. *Using (50) and (51) we have*

$$*F_{j\ k}^{i} = F_{j\ k}^{i} + \Lambda_{jk}^{i}, \qquad *C_{j\ k}^{i} = C_{j\ k}^{i} + \overset{\circ}{\Lambda}{}_{jk}^{i},$$

where

$$\Lambda_{jk}^{i} = \frac{1}{2} * g^{ih}\Big(\frac{\delta(\sigma B_j B_h)}{\delta x^k} + \frac{\delta(\sigma B_h B_k)}{\delta x^j} - \frac{\delta(\sigma B_j B_k)}{\delta x^h}\Big) - *\sigma(B^i B^h)F_{jhk},$$

$$\overset{\circ}{\Lambda}{}_{jk}^{i} = \frac{1}{2} * g^{ih}\Big(\frac{\partial(\sigma B_j B_h)}{\partial y^k} + \frac{\partial(\sigma B_h B_k)}{\partial y^j} - \frac{\partial(\sigma B_j B_k)}{\partial y^h}\Big) - *\sigma(*B^i B^h)C_{jhk}.$$

Using (37), we can define

$$\begin{aligned} &\widetilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} = F_{i\ j}^{k}\frac{\partial}{\partial y^k}, \qquad \widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = C_{i\ j}^{k}\frac{\partial}{\partial y^k},\\ &\widehat{\widetilde{\nabla}}_{\frac{\widehat{\delta}}{\delta x^i}}\frac{\partial}{\partial y^j} = \widehat{F}_{i\ j}^{k}\frac{\partial}{\partial y^k}, \qquad \widehat{\widetilde{\nabla}}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = \widehat{C}_{i\ j}^{k}\frac{\partial}{\partial y^k}. \end{aligned} \tag{52}$$

and

$$\begin{aligned} &*\widetilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} = *F_{i\ j}^{k}\frac{\partial}{\partial y^k}, \qquad *\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = *C_{i\ j}^{k}\frac{\partial}{\partial y^k}\\ &*\widehat{\widetilde{\nabla}}_{\frac{\widehat{\delta}}{\delta x^i}}\frac{\partial}{\partial y^j} = *\widehat{F}_{i\ j}^{k}\frac{\partial}{\partial y^k}, \qquad *\widehat{\widetilde{\nabla}}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = *C_{i\ j}^{k}\frac{\partial}{\partial y^k} \end{aligned} \tag{53}$$

By applying (52) and (53), and direct calculations, the following result is obtained.

Corollary 5.5. *We have the following assertions:*

$$\begin{aligned} &*\widetilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} = \widetilde{\nabla}_{\frac{\delta}{\delta x^i}}\frac{\partial}{\partial y^j} + \Lambda_{ij}^{k}\frac{\partial}{\partial y^k}, \qquad *\widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = \widetilde{\nabla}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} + \overset{\circ}{\Lambda}{}_{ij}^{k}\frac{\partial}{\partial y^k}\\ &*\widehat{\widetilde{\nabla}}_{\frac{\widehat{\delta}}{\delta x^i}}\frac{\partial}{\partial y^j} = \widehat{\widetilde{\nabla}}_{\frac{\widehat{\delta}}{\delta x^i}}\frac{\partial}{\partial y^j} + \widehat{\Lambda}_{ij}^{k}\frac{\partial}{\partial y^k}, \qquad *\widehat{\widetilde{\nabla}}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} = \widehat{\widetilde{\nabla}}_{\frac{\partial}{\partial y^i}}\frac{\partial}{\partial y^j} + \widehat{\overset{\circ}{\Lambda}}{}_{ij}^{k}\frac{\partial}{\partial y^k}, \end{aligned}$$

where

$$\widehat{\Lambda}_{ij}^{k} = \frac{1}{2} * g^{kh}\Big(\frac{\widehat{\delta}(\sigma B_i B_h)}{\delta x^j} + \frac{\widehat{\delta}(\sigma B_h B_j)}{\delta x^i} - \frac{\widehat{\delta}(\sigma B_i B_j)}{\delta x^h}\Big) - \sigma^*(B^k B^h) * \widehat{F}_{ijh},$$

and

$$\widehat{\overset{\circ}{\Lambda}}{}_{ij}^{k} = \frac{1}{2} * g^{kh}\Big(\frac{\partial(\sigma B_i B_h)}{\partial y^j} + \frac{\partial(\sigma B_h B_j)}{\partial y^i} - \frac{\partial(\sigma B_i B_j)}{\partial y^h}\Big) - \sigma^*(B^k B^h) * C_{ijh}.$$

Now easily follows

Proposition 5.6. *The following relations hold*

$$\widehat{F}_{i\ j}^{k} = F_{i\ j}^{k} - A_i^l C_{l\ j}^{k} - \frac{1}{2}\widetilde{g}^{kh}\Big(A_j^l\frac{\partial\widetilde{g}_{hi}}{\partial y^l} + A_i^l\frac{\partial\widetilde{g}_{hj}}{\partial y^l} - A_h^l\frac{\partial\widetilde{g}_{ij}}{\partial y^l}\Big),$$

$$*\widehat{F}_{i\ j}^{k} = *F_{i\ j}^{k} - A_i^l * C_{l\ j}^{k} - \frac{1}{2}\widetilde{g}^{kh}\Big(A_j^l\frac{\partial *\widetilde{g}_{hi}}{\partial y^l} + A_i^l\frac{\partial *\widetilde{g}_{hj}}{\partial y^l} - A_h^l\frac{\partial *\widetilde{g}_{ij}}{\partial y^l}\Big).$$

Now we define for $X \in \Gamma(TM')$, $vX \in \Gamma(VM')$ is vertical part of X on VM'. Then we can consider vX as a vector field of $T\widetilde{M}'$. Also we define

$$*\widetilde{\nabla}_X vY = *\nabla_X vY + *B(X, vY), \qquad \forall X, Y \in \Gamma(TM'), \tag{54}$$

$$*\widehat{\widetilde{\nabla}}_X vY = *\widehat{\nabla}_X vY + *\widehat{B}(X, vY), \qquad \forall X, Y \in \Gamma(TM'), \tag{55}$$

where $*\widetilde{\nabla}_X vY, *\widehat{\nabla}_X vY \in \Gamma(VM')$, and $*\widehat{B}(X, vY) \in VM'^{\perp}$.
Then $*\nabla : \Gamma(TM') \times \Gamma(VM') \to \Gamma(VM')$ defines the linear connection on the vector bundle VM' on M'. We put

$$\begin{aligned} *\nabla_{\frac{*\delta}{\delta u^\alpha}} \frac{\partial}{\partial v^\beta} &= *F^{\gamma}_{\alpha\ \beta} \frac{\partial}{\partial v^\gamma}, \qquad & *\widehat{\nabla}_{\frac{*\hat{\delta}}{\delta u^\alpha}} \frac{\partial}{\partial v^\beta} &= *\widehat{F}^{\gamma}_{\alpha\ \beta} \frac{\partial}{\partial v^\gamma} \\ *\nabla_{\frac{\partial}{\partial v^\alpha}} \frac{\partial}{\partial v^\beta} &= *C^{\gamma}_{\alpha\ \beta} \frac{\partial}{\partial v^\gamma}, \qquad & *\widehat{\nabla}_{\frac{\partial}{\partial v^\alpha}} \frac{\partial}{\partial v^\beta} &= *\widehat{C}^{\gamma}_{\alpha\ \beta} \frac{\partial}{\partial v^\gamma}. \end{aligned} \tag{56}$$

Also

$$*C\widetilde{F}C = (G^j_i, *F^j_{i\ k}, *C^j_{i\ k}), \qquad *C\widehat{\widetilde{F}}C = (N^j_i, *\widehat{F}^j_{i\ k}, *C^j_{i\ k}).$$

From Theorem 5.3, Proposition 5.6 and relations (54), (55) and (56) one deduces.

Proposition 5.7. *The local coefficients of the induced* $(N^\beta_\alpha, *F^\beta_{\alpha\ \gamma}, *C^\beta_{\alpha\ \beta})$ *and* $(G^\beta_\alpha, *\widehat{F}^\beta_{\alpha\ \gamma}, *\widehat{C}^\beta_{\alpha\ \beta})$ *are given by:*

$$\begin{aligned} *F^{\gamma}_{\alpha\ \beta} &= (B^{\gamma}_{\alpha\beta} + B^{ij}_{\alpha\beta} * F^k_{ij} + B^i_\alpha * C^k_{i\ j} * B^j_a * H^a_\beta) * B^\gamma_k, \\ *\widehat{F}^{\gamma}_{\alpha\ \beta} &= (B^{\gamma}_{\alpha\beta} + B^{ij}_{\alpha\beta} * \widehat{F}^k_{ij} + B^i_\alpha * C^k_{i\ j} * B^j_a * \widehat{H}^a_\beta) * B^\gamma_k, \\ *\widehat{C}^{\gamma}_{\alpha\ \beta} &= *B^{ij}_{\alpha\beta} * C^k_{i\ j} B^\gamma_k. \end{aligned}$$

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