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Original Article

Characterization and the stability of a system of multi-radical mappings related to the additive mapping

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ABSTRACT: In the current investigation, we define s-multi-radical mappings, characterize the structure of such mappings and then obtain an equation for describing them. In fact, we find a necessary and sufficient condition for a multiple mapping to be s-multi-radical. We also deal with the Hyers-Ulam stability in the spirit of Găvruţa for an s-multi-radical equation by applying the so-called direct (Hyers) method in the setting of 2-Banach spaces. For a typical case, by means of a norm, induced from a 2-norm of $\mathbb{R}^m$, we investigate the stability of a mapping $f : \mathbb{R}^{mn} \longrightarrow \mathbb{R}^m$ by a known fixed point method.

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1. Introduction and preliminaries

At first, we interpret the story of the stability of functional equations which is a interesting field in nonlinear analysis and much more authors and mathematicians have worked on it. The stability problem of functional equations began from a question of Ulam [32], regarding the stability of group homomorphisms. Note that a functional equation $\mathfrak{F}$ is said to be *stable* if any function ψ satisfying the equation $\mathfrak{F}$ approximately, must be near to an exact solution. Hyers [21] gave an answer to the question of Ulam in the context of Banach spaces in the case of additive mappings, that was an important step toward more solutions in this field. The method provided by Hyers which produces the additive function will be called a direct method. This way is the most important and useful tool to study the stability of varied functional equations. In fact, he proved a theorem as follows: Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be Banach spaces. Assume that $f : \mathbb{B}_1 \longrightarrow \mathbb{B}_2$ with

$$\|f(b_1 + b_2) - f(b_1) - f(b_2)\| \leq \delta,$$

for all $b_1, b_2 \in \mathbb{B}_1$ and for some $\epsilon \geq 0$. Then, there exists a unique additive mapping $\mathcal{A} : \mathbb{B}_1 \longrightarrow \mathbb{B}_2$ such that $\|f(b) - \mathcal{A}(b)\| \leq \delta$ for all $b \in \mathbb{B}_1$, and if $f(tb)$ is continuous in t for each fixed b, then $\mathcal{A}$ is a linear mapping.

Later, the result of Hyers was significantly generalized by Aoki [3], Rassias [27] (stability incorporated with sum of powers of norms) and Găvruţa [19] (stability controlled by a general control function).

It is well-known there is two classical methods for establishing the stability of functional equation; Hyers' (direct) [21] and fixed point ways. It is worth mentioning that the fixed point theorems have been considered for varied

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mappings and functional equations for example in [13] and [10]. Indeed, the study on fixed Point theory provides essential tools for solving and proving stability problems of functional equations. Similar investigations have been carried out in stability of general and linear recurrence; see [8] and [26]. Moreover, the fixed point theorems were applied to obtain similar stability results in many articles and books.

Here, we review a historical notes of known cubic, quintic and septic functional equations. Jun and Kim [22] defined the following cubic equations

$$C(2x+y)+C(2x-y)=2C(x+y)+2C(x-y)+12C(x). \tag{1}$$

Additionally, they studied the stability of the functional equations (1) in the setting of Banach spaces. Then, Xu et al. [35] obtained the general solution the quintic functional equation

$$Q(x+3y)-5Q(x+2y)+10Q(x+y)-10Q(x)+5Q(x-y)-Q(x-2y)=120Q(y), \tag{2}$$

for the first time. They also investigated the Ulam stability problem for it in quasi-β-normed spaces via the fixed point method. As for a septic functional equation, Shen and Chen [30] investigated the general solutions of the septic functional equation

$$\begin{aligned} S(x+4y)-7S(x+3y)+21S(x+2y)-35S(x+y)+35S(x)-21S(x-y)\\ +7S(x-2y)-S(x-3y)=5040S(y), \end{aligned} \tag{3}$$

on commutative groups; see also [34]. Note that the function $f(x)=\alpha x^t$ for $t=3,5,7$ satisfies (1),(2) and (3), respectively.

We consider the radical equation

$$R\left(\sqrt[s]{x^s+y^s}\right)=R(x)+R(y), \tag{4}$$

for $s=3,5,7$, where R is a mapping from $\mathbb{R}$ into a normed vector space V. In the case that $s=3,5,7$, we call it radical cubic, quintic and septic functional equation, respectively. We have the following observation:

Let V be a linear space over $\mathbb{R}$. Consider a mapping $\lambda:\mathbb{R}\longrightarrow V$.

(i) ([1, Theorem 2.1]) If λ fulfills (4) in the case $s=3$, then it is a cubic function and so (1) is valid for λ;
(ii) ([15, Theorem 2.1]) If λ satisfies (4) in the case $s=5$, then it is a quintic function and hence λ fulfills (2);
(iii) ([18, Theorem 1]) If λ satisfies (4) in the case $s=7$, then it is a septic function and therefore (3) holds for λ.

In the last decades, some authors have extended the results on general solutions and stability of one variable mappings and functions to multiple variables. Indeed, the definitions, characterizations of multi-additive (quadratic, cubic and quartic) of mappings were proposed in [5, 6, 7, 9] and [36] for the first time. In other words, representation of such mappings (defined as a system of functional equations) as a single equation are given in the mentioned papers. Lately, the first author [4] introduced and characterized d-dimensional multi-additive mappings and moreover established their Ulam's stability in the setting of neutrosophic Banach spaces. Recently, Alsahli and Bodaghi [2] defined the multi-radical quadratic and multi-radical quartic mappings as two systems of radical functional equations and represented them as two equations. Furthermore, the stability problem has been studied for multiple variable mappings can be found in literatures.

The definition of a 2-normed space has been defined and studied by Gähler [16, 17] and then White [33] introduced the concept of 2-Banach spaces. Next, Lewandowska defined and investigated of 2-normed sets and generalized 2-normed spaces [24, 25].

Let V be an at least two-dimensional real linear space over $\mathbb{R}$. Suppose that $\|\cdot,\cdot\|:V\times V\longrightarrow\mathbb{R}$ is a function such that for each $u,v,w\in V$ and $\alpha\in\mathbb{R}$ the following properties hold:

(i) $\|u,v\|=0$ if and only if u and v are linearly dependent;
(ii) $\|u,v\|=\|v,u\|$;
(iii) $\|\alpha u,v\|=|\alpha|\|u,v\|$;
(iv) $\|u,v+w\|\leq\|u,v\|+\|u,w\|$.

Then, the function $\|\cdot,\cdot\|$ is called a 2-*norm* on V and $(V,\|\cdot,\cdot\|)$ is called a 2-*normed space*.

There are some known definitions and facts in each normed spaces that we reminder them as follows and will be used throughout this paper. Consider a sequence $(u_m)_m$ of elements of a 2-normed space $(V,\|\cdot,\cdot\|)$. This mentioned sequence is said to be 2-*Cauchy* if there exist linearly independent $v,w\in V$ for which

$$\lim_{k,j\to\infty}\|u_k-u_j,v\|=\lim_{k,j\to\infty}\|u_k-u_j,w\|=0.$$

Moreover, $(u_m)_m$ is called 2-*convergent* provided there is a $v \in V$ with $\lim_{m\to\infty} \|u_m - v, w\| = 0$. A 2-normed space $(V, \|\cdot,\cdot\|)$ is called a 2-*Banach space* if every 2-Cauchy sequence in X is 2-convergent.

In the sequel, the element v is called the limit of the sequence $(u_m)_m$ and we denote it by $\lim_{m\to\infty} u_m$. Obviously, every convergent sequence possesses a unique limit. Moreover, the standard properties of the limit of a sum and a scalar product hold. Now, we state the following results as lemma.

Lemma 1.1 ([33]). *Let $(V, \|\cdot,\cdot\|)$ be a 2-normed space. Then, the following assertions hold:*

(i) $|\|u, v\| - \|w, v\|| \le \|u - w, v\|$ *for all* $u, v, w \in V$*;*

(ii) *If* $\|u, v\| = 0$ *for all* $v \in V$*, then* $u = 0$*;*

(iii) *For any 2-convergent sequence* $(u_m)_m$ *in* $(V, \|\cdot,\cdot\|)$*, we have*

$$\lim_{m\to\infty} \|u_m, v\| = \| \lim_{m\to\infty} u_m, v\|,$$

for all $v \in V$*.*

Here, we recall that many Hyers-Ulam stability problems for various functional equations and mappings in the setting of 2-Banach spaces were studied by authors; see for instance [9], [10], [11], [29] and other resources.

Motivated by the mentioned discussion earlier, in this study, by using functional equation (4), we define s-multi-radical mappings when $s \in \{3, 5, 7\}$. We also characterize the structure of such mappings. In other words, we reduce the system of n equations defining the s-multi-radical mappings to obtain a single functional equation. The benefit of presentation of such systems as a single equation allows us to prove the Hyers-Ulam stability of s-multi-radical mappings by means of only one functional inequality, not a number of functional equations like what are done in [9] and [36]. Eventually, we establish the Găvruţa and Hyers-Ulam stability for such functional equations (s-multi-radical mappings) in a general 2-normed space by using the direct method. For a typical case, by means of a norm, induced from a 2-norm of $\mathbb{R}^m$, we established the stability of a functional equation by a known fixed point technique, whose an exact solution is a mapping $f : \mathbb{R}^{mn} \longrightarrow \mathbb{R}^m$. Note that the Hyers' method for proving the stability of a functional equation is that finding the Cauchy sequences and their convergency by a recurrence way (which is sometimes long and tedious) not using a fixed point theorem.

2. Characterization of s-multi-radical mappings

In this section, we define s-multi-radical mappings and present them as a single equation.

Definition 2.1. *Let V be a vector spaces over $\mathbb{R}$ and $s \in \{3, 5, 7\}$. A multivariable mapping $f : \mathbb{R}^n \longrightarrow V$ is said to be n-radical cubic ($s = 3$), n-radical quintic ($s = 5$), n-radical septic ($s = 7$) if it fulfills the general system of radical equations*

$$f\left(u_1, \ldots, u_{i-1}, \sqrt[s]{u_i^s + v_i^s}, u_{i+1}, \ldots, u_n\right) = f(u_1, \ldots, u_{i-1}, u_i, u_{i+1}, \ldots, u_n) + f(u_1, \ldots, u_{i-1}, v_i, u_{i+1}, \ldots, u_n),$$

for all $i \in \{1, \ldots, n\}$*.*

Hereafter, we call the above mappings as *s-multi-radical* when $s \in \{3, 5, 7\}$. It is easily checked that the mapping $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ defined via $f(r_1, \ldots, r_n) = \prod_{j=1}^n r_j^s$ satisfies the system of the multi-radical functional equations introduced in Definition 2.1.

Throughout this article, $n \in \mathbb{N}$ with $n \ge 2$ and $\mathbb{R}_+ = [0, \infty)$. Moreover, $x_i^{[n]} = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n$, where $i \in \{1, 2\}$. In the next result, we describe the mapping defined in the above as an equation.

Theorem 2.2. *Given $s \in \{3, 5, 7\}$. A mapping $f : \mathbb{R}^n \longrightarrow V$ is s-multi-radical if and only if it satisfies the equation*

$$f\left(\sqrt[s]{x_{11}^s + x_{21}^s}, \sqrt[s]{x_{12}^s + x_{22}^s}, \ldots, \sqrt[s]{x_{1n}^s + x_{2n}^s}\right) = \sum_{j_1,\ldots,j_n \in \{1,2\}} f(x_{j_1 1}, \ldots, x_{j_n n}), \tag{5}$$

for all $x_i^{[n]} = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n$*, where* $i \in \{1, 2\}$*.*

Proof. A routine computation and by using induction (maybe a direct extension), one can show that every multi-radical quadratic mapping fulfills (5). For other implication, we denote each element of $\mathbb{R}^n$ by $x^{[n]}$ and firstly prove that $f\left(x^{[n]}\right) = 0$ for any $x^{[n]} \in \mathbb{R}^n$ with at least one component which is equal to zero. Putting $x_1^{[n]} = x_2^{[n]} = (0, \ldots, 0)$ in (5), we get

$$f(0, \ldots, 0) = 2^n f(0, \ldots, 0),$$

and thus $f(0,\dots,0)=0$. Letting $x_{1k}=0$ for all $k\in\{1,\dots,n\}\backslash\{j\}$ and $x_{2k}=0$ for $1\leq k\leq n$ in (5) and using $f(0,\dots,0)=0$, we obtain

$$f(0,\dots,0,x_{1j},0,\dots,0)=2^{n-1}f(0,\dots,0,x_{1j},0,\dots,0).$$

Consequently, $f(0,\dots,0,x_{1j},0,\dots,0)=0$. We continue in this fashion in obtaining $f\left(x^{[n]}\right)=0$ for any $x^{[n]}\in\mathbb{R}^n$ with at least one component which is equal to zero. Lastly, fixing $j\in\{1,\dots,n\}$ and putting $x_{2k}=0$ for all $k\in\{1,\dots,n\}\backslash\{j\}$ in (5), we have

$$\begin{aligned}f(x_{11},\dots,x_{1,j-1},\sqrt[s]{x_{1j}^s+x_{2j}^s},x_{1,j+1},\dots,x_{1n})=\;& f(x_{11},\dots,x_{1,j-1},x_{1j},x_{1,j+1},\dots,x_{1n})\\ &+f(x_{11},\dots,x_{1,j-1},x_{2j},x_{1,j+1},\dots,x_{1n}),\end{aligned}$$

which proves that f is an s-multi-radical mapping. □

3. Stability of s-multi-radical mappings in 2-normed spaces

In this section, we present the Găvruţa stability of (5) from $\mathbb{R}^n$ to a 2-Banach space by the direct and fixed point methods.

3.1. Stability results: the direct method

We establish the Găvruţa and Hyers-Ulam stability of (5) from $\mathbb{R}^n$ to a 2-Banach space by the direct method which is one of the main results in this subsection.

Theorem 3.1. *Given $\beta\in\{-1,1\}$. Let V be a 2-Banach space. Suppose that $f:\mathbb{R}^n\longrightarrow V$ is a mapping for which there exists a function $\psi:\mathbb{R}^n\times\mathbb{R}^n\times V\longrightarrow\mathbb{R}_+$ such that*

$$\widetilde{\psi}\left(x_1^{[n]},x_2^{[n]},v\right):=\sum_{j=0}^{\infty}\frac{1}{2^{n\beta j}}\psi\left(\sqrt[s]{2}^{\frac{\beta-1}{2}+\beta j}x_1^{[n]},\sqrt[s]{2}^{\frac{\beta-1}{2}+\beta j}x_2^{[n]},v\right)<\infty, \tag{6}$$

and

$$\left\|f\left(\sqrt[s]{x_{11}^s+x_{21}^s},\dots,\sqrt[s]{x_{1n}^s+x_{2n}^s}\right)-\sum_{j_1,\dots,j_n\in\{1,2\}}f(x_{j_11},\dots,x_{j_nn}),v\right\|\leq\psi\left(x_1^{[n]},x_2^{[n]},v\right), \tag{7}$$

for all $x_i^{[n]}=(x_{i1},\dots,x_{in})\in\mathbb{R}^n$ and $v\in V$, where $i\in\{1,2\}$. Then, there exists a unique s-multi-radical mapping $\mathcal{R}:\mathbb{R}^n\longrightarrow V$ such that

$$\left\|f\left(x^{[n]}\right)-\mathcal{R}\left(x^{[n]}\right),v\right\|\leq\frac{1}{2^{\frac{\beta+1}{2}n}}\widetilde{\psi}\left(x^{[n]},x^{[n]},v\right), \tag{8}$$

for all $x^{[n]}\in\mathbb{R}^n$ and $v\in V$.

Proof. Putting $x_1^{[n]}=x_2^{[n]}$ in (7), we have

$$\left\|f\left(\sqrt[s]{2}x_1^{[n]}\right)-2^nf\left(x_1^{[n]}\right),v\right\|\leq\psi\left(x_1^{[n]},x_1^{[n]},v\right), \tag{9}$$

for all $x_1^{[n]}\in\mathbb{R}^n$ and $v\in V$. For the rest of proof, we set $x_1^{[n]}$ by $x^{[n]}$ unless otherwise stated explicitly. It follows from (9) that

$$\left\|\frac{f\left(\sqrt[s]{2}^{\beta}x^{[n]}\right)}{2^{n\beta}}-f\left(x^{[n]}\right),v\right\|\leq\frac{1}{2^{\frac{\beta+1}{2}n}}\psi\left(\sqrt[s]{2}^{\frac{\beta-1}{2}}x^{[n]},\sqrt[s]{2}^{\frac{\beta-1}{2}}x^{[n]},v\right), \tag{10}$$

for all $x^{[n]}\in\mathbb{R}^n$ and $v\in V$. Substituting $x^{[n]}$ into $\sqrt[s]{2}^{\beta}x^{[n]}$ in (10) and continuing this way, we find

$$\left\|\frac{f\left(\sqrt[s]{2}^{m\beta}x^{[n]}\right)}{2^{mn\beta}}-f\left(x^{[n]}\right),v\right\|\leq\frac{1}{2^{\frac{\beta+1}{2}n}}\sum_{j=0}^{m-1}\frac{1}{2^{n\beta j}}\psi\left(\sqrt[s]{2}^{\frac{\beta-1}{2}+j\beta}x^{[n]},\sqrt[s]{2}^{\frac{\beta-1}{2}+j\beta}x^{[n]},v\right), \tag{11}$$

for all $x^{[n]} \in \mathbb{R}^n$ and $v \in V$. On the other hand, by applying induction, we obtain

$$\left\| \frac{f\left(\sqrt[s]{2}^{m\beta} x^{[n]}\right)}{2^{mn\beta}} - \frac{f\left(\sqrt[s]{2}^{l\beta} x^{[n]}\right)}{2^{ln\beta}}, v \right\| \leq \frac{1}{2^{\frac{\beta+1}{2}n}} \sum_{j=l}^{m-1} \frac{1}{2^{n\beta j}} \psi\left(\sqrt[s]{2}^{\frac{\beta-1}{2}+j\beta} x^{[n]}, \sqrt[s]{2}^{\frac{\beta-1}{2}+j\beta} x^{[n]}, v\right), \tag{12}$$

for all $x^{[n]} \in \mathbb{R}^n, v \in V$ and $m > l \geq 0$. Thus, the sequence $\left\{\frac{f\left(\sqrt[s]{2}^{m\beta} x^{[n]}\right)}{2^{mn\beta}}\right\}$ is 2-Cauchy by (6) and (12). Due to the completeness of V as a 2-Banach space, one can assume that there exists a mapping $\mathcal{R} : \mathbb{R}^n \longrightarrow V$ such that

$$\lim_{m\to\infty} \frac{f\left(\sqrt[s]{2}^{m\beta} x^{[n]}\right)}{2^{mn\beta}} = \mathcal{R}(x). \tag{13}$$

Taking $m \to \infty$ in (11) and using (13), we can see that the inequality (8) is valid. Now, by interchanging $\left(x_1^{[n]}, x_2^{[n]}\right)$ into $\left(\sqrt[s]{2}^{m\beta} x_1^{[n]}, \sqrt[s]{2}^{m\beta} x_2^{[n]}\right)$, respectively in (7) and dividing to $2^{nm\beta}$, we get

$$\frac{1}{2^{nm\beta}} \left\| f\left(\sqrt[s]{2^{m\beta} x_{11}^s + 2^{m\beta} x_{21}^s}, \ldots, \sqrt[s]{2^{m\beta} x_{1n}^s + 2^{m\beta} x_{2n}^s}\right) - \sum_{j_1,\ldots,j_n \in \{1,2\}} f\left(\sqrt[s]{2}^{m\beta} x_{j_1 1}, \ldots, \sqrt[s]{2}^{m\beta} x_{j_n n}\right), v \right\|$$
$$\leq \frac{\psi\left(\sqrt[s]{2}^{m\beta} x_1^{[n]}, \sqrt[s]{2}^{m\beta} x_2^{[n]}, v\right)}{2^{nm\beta}}.$$

Letting the limit as $m \to \infty$, and applying (6) and (13), we see that $\mathcal{R}$ fulfills (5) and by Theorem 2.2, it is a s-multi-radical mapping. Let now $\mathcal{R}' : \mathbb{R}^n \longrightarrow V$ be another s-multi-radical mapping satisfying (8). Then, we have

$$\begin{aligned}
\left\|\mathcal{R}\left(x^{[n]}\right) - \mathcal{R}'\left(x^{[n]}\right), v\right\| &= \frac{1}{2^{nm\beta}} \left\|\mathcal{R}\left(\sqrt[s]{2}^{\beta m} x^{[n]}\right) - \mathcal{R}'\left(\sqrt[s]{2}^{\beta m} x^{[n]}\right), v\right\| \\
&\leq \frac{1}{2^{nm\beta}} \left(\left\|\mathcal{R}\left(\sqrt[s]{2}^{\beta m} x^{[n]}\right) - f\left(\sqrt[s]{2}^{\beta m} x^{[n]}\right), v\right\| + \left\|f\left(\sqrt[s]{2}^{\beta m} x^{[n]}\right) - \mathcal{R}'\left(\sqrt[s]{2}^{\beta m} x^{[n]}\right), v\right\|\right) \\
&\leq \frac{2}{2^{nm\beta}} \frac{1}{2^{\frac{\beta+1}{2}n}} \widetilde{\psi}\left(x^{[n]}, x^{[n]}, v\right) \\
&= \frac{2}{2^{nm\beta}} \frac{1}{2^{\frac{\beta+1}{2}n}} \sum_{j=0}^{\infty} \frac{1}{2^{n\beta j}} \psi\left(\sqrt[s]{2}^{\frac{\beta-1}{2}+j\beta} x^{[n]}, \sqrt[s]{2}^{\frac{\beta-1}{2}+j\beta} x^{[n]}, v\right) \\
&= \frac{2}{2^{\frac{\beta+1}{2}n}} \sum_{j=m}^{\infty} \frac{1}{2^{n\beta j}} \psi\left(\sqrt[s]{2}^{\frac{\beta-1}{2}+(j-m)\beta} x^{[n]}, \sqrt[s]{2}^{\frac{\beta-1}{2}+(j-m)\beta} x^{[n]}, v\right),
\end{aligned}$$

for all $x^{[n]} \in \mathbb{R}^n$ and $v \in V$. Taking $m \to \infty$ in the above inequality, we have $\mathcal{R} = \mathcal{R}'$ and hence it is shown the uniqueness of solution. ☐

Remark 3.2. *Let $(V, \|\cdot,\cdot\|)$ be a 2-normed space and $\{v_1, v_2\}$ be a linearly independent set in V. It was proved in [28, Theorem 8] that the function $\|\cdot\| : V \longrightarrow [0,\infty)$ defined by $\|v\| = \|v, v_1\| + \|v, v_2\|$ is a norm on V.*

The upcoming corollary is a direct consequence of Theorem 3.1 concerning the Hyers-Ulam stability of s-multi-radical mappings.

Corollary 3.3. *Given $r, \alpha \in (0,\infty)$ with $r \neq ns$. Let V be a 2-Banach space and $\{v_1, v_2\}$ be a linearly independent set in V. Suppose that a mapping $f : \mathbb{R}^n \longrightarrow V$ satisfying the inequality*

$$\left\| f\left(\sqrt[s]{x_{11}^s + x_{21}^s}, \ldots, \sqrt[s]{x_{1n}^s + x_{2n}^s}\right) - \sum_{j_1,\ldots,j_n \in \{1,2\}} f(x_{j_1 1}, \ldots, x_{j_n n}), v \right\| \leq \|v\|^\alpha \sum_{i=1}^{2} \sum_{j=1}^{n} |x_{ij}|^r,$$

for all $x_i^{[n]} = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n$ and $v \in V$, where $i \in \{1,2\}$ and $\|v\|$ is defined in Remark 3.2. Then, there exists a unique s-multi-radical mapping $\mathcal{R} : \mathbb{R}^n \longrightarrow V$ such that

$$\left\| f\left(x^{[n]}\right) - \mathcal{R}\left(x^{[n]}\right), v \right\| \leq \frac{\|v\|^\alpha}{\left|2^n - \sqrt[s]{2}^r\right|} \sum_{j=1}^{n} |x_{1j}|^r,$$

for all $x^{[n]} := x_1^{[n]} \in \mathbb{R}^n$ and $v \in V$.

Proof. Setting $\psi\left(x_1^{[n]}, x_2^{[n]}, v\right) = \|v\|^\alpha \sum_{i=1}^{2}\sum_{j=1}^{n} |x_{ij}|^r$ in Theorem 3.1, one can obtain the desired results. $\square$

Suppose that $(V, \|\cdot,\cdot\|)$ is a 2-normed space. Moreover, it is assumed that V has dimension d, where $2 \le d < \infty$. Fix $\mathbf{B} = \{\mathbf{v}_1, \ldots, \mathbf{v}_d\}$ to be a basis for V. With respect to the basis $\mathbf{B}$, one can define a norm on V, denoted by $\|\cdot\|_\infty$ as follows:

$$\|v\|_\infty := \max\{\|v, \mathbf{v}_j\| : 1 \le j \le d\}.$$

For the case that $1 \le p < \infty$, the function $\|\cdot\|_p : V \longrightarrow [0, \infty)$ defined through

$$\|v\|_p := \left(\sum_{j=1}^{d} \|v, \mathbf{v}_j\|^p\right)^{\frac{1}{p}},$$

is also a norm on V. Such norms were defined and studied in [20]. Note that all norms introduced above are equivalent and moreover the choice of the basis is not essential. In other words, by choosing another basis, the resultant norm will be equivalent to the one considered with respect to $\mathbf{B}$.

Corollary 3.4. *Given $\alpha > 0$ and $r \in (0, \infty)$ with $r \neq ns$. Let V be a 2-Banach space and $\|\cdot\|_p$ as the norm on V for $1 \le p \le \infty$. Suppose that the mapping $f : \mathbb{R}^n \longrightarrow V$ fulfilling*

$$\left\| f\left(\sqrt[s]{x_{11}^s + x_{21}^s}, \ldots, \sqrt[s]{x_{1n}^s + x_{2n}^s}\right) - \sum_{j_1,\ldots,j_n \in \{1,2\}} f(x_{j_1 1}, \ldots, x_{j_n n}), v \right\| \le \|v\|_p^\alpha \sum_{i=1}^{2}\sum_{j=1}^{n} |x_{ij}|^r,$$

for all $x_i^{[n]} = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n$ and $v \in V$, where $i \in \{1, 2\}$ and $\|v\|$ is defined in Remark 3.2. Then, there exists a unique s-multi-radical mapping $\mathcal{R} : \mathbb{R}^n \longrightarrow V$ such that

$$\left\| f\left(x^{[n]}\right) - \mathcal{R}\left(x^{[n]}\right), v \right\| \le \frac{\|v\|_p^\alpha}{\left|2^n - \sqrt[s]{2}^r\right|} \sum_{j=1}^{n} |x_{1j}|^r,$$

for all $x^{[n]} := x_1^{[n]} \in \mathbb{R}^n$ and $v \in V$.

Proof. Defining $\psi\left(x_1^{[n]}, x_2^{[n]}, v\right) = \|v\|_p^\alpha \sum_{i=1}^{2}\sum_{j=1}^{n} |x_{ij}|^r$ in Theorem 3.1, we find the result. $\square$

In the next result, the Hyers stability of s-multi-radical mapping, which is a direct consequence of Theorem 3.1, indicated.

Corollary 3.5. *Let $\delta \in [0, \infty)$. Let also V be a 2-Banach space. Suppose that the mapping $f : \mathbb{R}^n \longrightarrow V$ satisfies*

$$\left\| f\left(\sqrt[s]{x_{11}^s + x_{21}^s}, \ldots, \sqrt[s]{x_{1n}^s + x_{2n}^s}\right) - \sum_{j_1,\ldots,j_n \in \{1,2\}} f(x_{j_1 1}, \ldots, x_{j_n n}), v \right\| \le \delta,$$

for all $x_1^{[n]}, x_2^{[n]} \in \mathbb{R}^n$ and $v \in V$.for all $x_i^{[n]} = (x_{i1}, \ldots, x_{in}) \in \mathbb{R}^n$ and $v \in V$, where $i \in \{1, 2\}$. Then, there exists a unique s-multi-radical mapping $\mathcal{R} : \mathbb{R}^n \longrightarrow V$ such that

$$\left\| f\left(x^{[n]}\right) - \mathcal{R}\left(x^{[n]}\right), v \right\| \le \frac{\delta}{2^n - 1},$$

for all $x^{[n]} \in \mathbb{R}^n$ and $v \in V$.

Proof. Setting $\psi\left(x_1^{[n]}, x_2^{[n]}, v\right) = \delta$ in Theorem 3.1 in the case of $\beta = 1$, we can reach to the result. $\square$

3.2. Stability results: the fixed point method

In this subsection, we present the Găvruţa and Hyers-Ulam stability of an s-multi-radical mapping $f : \mathbb{R}^{dn} \longrightarrow \mathbb{R}^d$ by a fixed point method.

Let X be a set. A function $d : X \times X \longrightarrow [0, \infty]$ is called a *generalized metric* on X if and only if d satisfies the statements

(i) $d(x, y) = 0$ if and only if $x = y$ for all $x, y \in X$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;
(iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Note that the only substantial difference of the generalized metric from the metric is that the range of generalized metric includes the infinity.

In the next theorem, we bring a fundamental result in fixed point theory which is useful to our purpose in this subsection (an extension of the result was stated in [31]).

Theorem 3.6 ([14]). *Let (Ω, d) be a complete generalized metric space and $\mathcal{J} : \Omega \to \Omega$ be a mapping with Lipschitz constant $L < 1$. Then, for each element $y \in \Omega$, either $d(\mathcal{J}^n y, \mathcal{J}^{n+1} y) = \infty$ for all $n \geq 0$, or there exists a natural number n_0 such that:*

(i) $d(\mathcal{J}^n y, \mathcal{J}^{n+1} y) < \infty$ for all $n \geq n_0$;
(ii) the sequence $\{\mathcal{J}^n y\}$ is convergent to a fixed point y^* of $\mathcal{J}$;
(iii) y^* is the unique fixed point of $\mathcal{J}$ in the set

$$\Lambda = \{y \in \Omega : d(\mathcal{J}^{n_0} y, y) < \infty\};$$

(iv) $d(y, y^*) \leq \frac{1}{1-L} d(y, \mathcal{J} y)$ for all $y \in \Lambda$.

For $\mathbb{R}^m$, define $\|\cdot, \cdot\| : \mathbb{R}^m \times \mathbb{R}^m \longrightarrow [0, \infty)$ via the formula

$$\|\mathbf{x}, \mathbf{y}\| = \left(\left(\sum_{j=1}^{m} x_j^2 \right) \left(\sum_{j=1}^{m} y_j^2 \right) - \left(\sum_{j=1}^{m} x_j y_j \right)^2 \right)^{\frac{1}{2}}, \tag{14}$$

where $\mathbf{x} = (x_1, \ldots, x_m), \mathbf{y} = (y_1, \ldots, y_m) \in \mathbb{R}^m$. Then, $\|\mathbf{x}, \mathbf{y}\|$, is the area of the parallelogram spanned by $\mathbf{x}$ and $\mathbf{y}$ is 2-norm on $\mathbb{R}^m$ [20]. If $\{\mathbf{e}_1, \ldots, \mathbf{e}_m\}$ is the standard basis for $\mathbb{R}^m$, then for $j = 1, \ldots, m$, we have $\|\mathbf{x}, \mathbf{e}_j\| = \left(\left(\sum_{j=1}^{m} x_j^2 \right) - x_j^2 \right)^{\frac{1}{2}}$, and therefore

$$\|\mathbf{x}\|_\infty = \max \left\{ \left(\left(\sum_{j=1}^{m} x_j^2 \right) - x_j^2 \right)^{\frac{1}{2}} : j = 1, \ldots, m \right\}. \tag{15}$$

In the following, we have a stability result for a mapping $f : \mathbb{R}^{mn} \longrightarrow \mathbb{R}^m$ when

$$\left\| f\left(\sqrt[s]{x_{11}^s + x_{21}^s}, \ldots, \sqrt[s]{x_{1n}^s + x_{2n}^s} \right) - \sum_{j_1, \ldots, j_n \in \{1,2\}} f(x_{j_1 1}, \ldots, x_{j_n n}), \mathbf{y} \right\|,$$

is controlled as pointwise.

Theorem 3.7. *Let $\sigma \in \{-1, 1\}$. Suppose also $\mathbb{R}^m$ is considered as a 2-Banach space equipped with 2-norm (14). Let $\psi : \mathbb{R}^{mn} \times \mathbb{R}^{mn} \times \mathbb{R}^m \longrightarrow \mathbb{R}_+$ be a mapping fulfilling the inequality*

$$\frac{1}{2^{n\sigma}} \psi\left(\sqrt[s]{2}^{\sigma} \mathbf{x}_1^{[n]}, \sqrt[s]{2}^{\sigma} \mathbf{x}_2^{[n]}, \mathbf{y} \right) \leq L \psi(\mathbf{x}_1^{[n]}, \mathbf{x}_2^{[n]}, \mathbf{y}), \tag{16}$$

and

$$\lim_{l \to \infty} \frac{1}{2^{nl}} \psi\left(\sqrt[s]{2}^{l} \mathbf{x}_1^{[n]}, \sqrt[s]{2}^{l} \mathbf{x}_2^{[n]}, \mathbf{y} \right) = 0, \tag{17}$$

for all $\mathbf{x}_1^{[n]}, \mathbf{x}_2^{[n]} \in \mathbb{R}^{mn}$, $\mathbf{y} \in \mathbb{R}^m$ and for some $0 < L < 1$, where $\mathbf{x}_i^{[n]} = (x_{i1}, \ldots, x_{in})$, whereas $x_{ij} \in \mathbb{R}^m$ with $i \in \{1, 2\}$ and $j \in \{1, \ldots, n\}$. Assume also $f : \mathbb{R}^{mn} \longrightarrow \mathbb{R}^m$ is a mapping satisfying

$$\left\| f\left(\sqrt[s]{x_{11}^s + x_{21}^s}, \ldots, \sqrt[s]{x_{1n}^s + x_{2n}^s} \right) - \sum_{j_1, \ldots, j_n \in \{1,2\}} f(x_{j_1 1}, \ldots, x_{j_n n}), \mathbf{y} \right\| \leq \psi(\mathbf{x}_1^{[n]}, \mathbf{x}_2^{[n]}, \mathbf{y}), \tag{18}$$

for all $\mathbf{x}_1^{[n]}, \mathbf{x}_2^{[n]} \in \mathbb{R}^{mn}$, $\mathbf{y} \in \mathbb{R}^m$. Then, there exists a unique s-multi-radical mapping $\mathcal{R} : \mathbb{R}^{mn} \longrightarrow \mathbb{R}^m$ such that

$$\left\| f\left(\mathbf{x}^{[n]} \right) - \mathcal{R}\left(\mathbf{x}^{[n]} \right), \mathbf{y} \right\| \leq \frac{L^{\frac{|\sigma-1|}{2}}}{2^n(1-L)} \psi\left(\mathbf{x}^{[n]}, \mathbf{x}^{[n]}, \mathbf{y} \right), \tag{19}$$

for all $\mathbf{x}^{[n]} \in \mathbb{R}^{mn}$, $\mathbf{y} \in \mathbb{R}^m$.

Proof. We establish the result for the case $\sigma = 1$. Other case can be obtained similarly. Let us define a generalized metric space (Ω, d), where $\Omega := (\mathbb{R}^m)^{\mathbb{R}^{mn}}$ (the set of all mappings from $\mathbb{R}^{mn}$ to $\mathbb{R}^m$) and

$$d(\lambda,\mu) := \inf\left\{C \in [0,\infty) : \left\|\lambda\left(\mathbf{x}^{[n]}\right) - \mu\left(\mathbf{x}^{[n]}\right), \mathbf{y}\right\| \le C\psi\left(\mathbf{x}^{[n]}, \mathbf{x}^{[n]}, \mathbf{y}\right), \quad \forall \mathbf{x}^{[n]} \in \mathbb{R}^{mn}, \mathbf{y} \in \mathbb{R}^m\right\},$$

where, as usual, $\inf\emptyset = +\infty$, for which $g, h \in \Omega$. Similar to the proof of [23, Theorem 3.1] (see also [12]), one can show that (Ω, d) is a complete generalized metric space. Consider a mapping $\mathcal{J} : \Omega \longrightarrow \Omega$ defined through

$$\mathcal{J} f\left(\sqrt[s]{2}\mathbf{x}^{[n]}\right) := \frac{1}{2^n} f\left(\mathbf{x}^{[n]}\right).$$

for all $\mathbf{x}^{[n]} \in \mathbb{R}^{mn}$. We claim that $\mathcal{J}$ is a strictly contractive operator with the Lipschitz constant L. Take $\lambda, \mu \in \Omega$, $\mathbf{x} \in \mathbb{R}^{mn}$ and $C \in [0, \infty]$ with $d(\lambda, \mu) \le C$. Then, $\left\|\lambda\left(\mathbf{x}^{[n]}\right) - \mu\left(\mathbf{x}^{[n]}\right), \mathbf{y}\right\| \le C\psi\left(\mathbf{x}^{[n]}, \mathbf{x}^{[n]}, \mathbf{y}\right)$ for all $\mathbf{x}^{[n]} \in \mathbb{R}^{mn}$ and $\mathbf{y} \in \mathbb{R}^m$. We have

$$\begin{aligned}\|\mathcal{J}\lambda\left(\mathbf{x}^{[n]}\right) - \mathcal{J}\mu\left(\mathbf{x}^{[n]}\right), \mathbf{y}\| &\le \frac{1}{2^n}\left\|\lambda\left(\sqrt[s]{2}\mathbf{x}\right) - \mu\left(\sqrt[s]{2}\mathbf{x}\right), \mathbf{y}\right\| \\ &\le \frac{1}{2^n} C\psi\left(\sqrt[s]{2}\mathbf{x}, \sqrt[s]{2}\mathbf{x}, \mathbf{y}\right) \le CL\psi\left(\mathbf{x}^{[n]}, \mathbf{x}^{[n]}, \mathbf{y}\right),\end{aligned}$$

for all $\mathbf{x}^{[n]} \in \mathbb{R}^{mn}$ and $\mathbf{y} \in \mathbb{R}^m$. Therefore, $d(\mathcal{J}\lambda, \mathcal{J}\mu) \le CL$. This shows that $d(\mathcal{J}\lambda, \mathcal{J}\mu) \le Ld(\lambda, \mu)$, as claimed. Putting $\mathbf{x}_2^{[n]} = \mathbf{x}_1^{[n]}$ in (18), similar to the proof of Theorem 3.1 (relation (9)), we have

$$\left\|\frac{f\left(\sqrt[s]{2}\mathbf{x}_1^{[n]}\right)}{2^n} - f\left(\mathbf{x}_1^{[n]}\right), \mathbf{y}\right\| \le \frac{1}{2^n}\psi\left(\mathbf{x}_1^{[n]}, \mathbf{x}_1^{[n]}, \mathbf{y}\right),$$

for all $\mathbf{x}_1^{[n]} \in \mathbb{R}^{mn}$ and $\mathbf{y} \in \mathbb{R}^m$. For the rest of proof, we set $\mathbf{x}_1^{[n]}$ by $\mathbf{x}^{[n]}$ unless otherwise stated explicitly. The last inequality implies that

$$\left\|\mathcal{J} f\left(\mathbf{x}^{[n]}\right) - f\left(\mathbf{x}^{[n]}\right), \mathbf{y}\right\| \le \frac{1}{2^n}\psi\left(\mathbf{x}^{[n]}, \mathbf{x}^{[n]}, \mathbf{y}\right),$$

for all $\mathbf{x} \in \mathbb{R}^{mn}$ and $\mathbf{y} \in \mathbb{R}^m$. This means that

$$d(\mathcal{J} f, f) \le \frac{1}{2^n}. \tag{20}$$

We can now apply Theorem 3.6 for the space (Ω, d), the operator $\mathcal{J}$, $n_0 = 0$ and $y = f$ to deduce that the sequence $(\mathcal{J}^l f)_{l\in\mathbb{N}}$ is convergent in (Ω, d) and its limit, $\mathcal{R}$ is a fixed point of $\mathcal{J}$. In fact,

$$\mathcal{R}\left(\mathbf{x}^{[n]}\right) = \frac{1}{2^n}\mathcal{R}\left(\sqrt[s]{2}\mathbf{x}^{[n]}\right) = \lim_{l\to\infty}\mathcal{J}^l f\left(\mathbf{x}^{[n]}\right), \tag{21}$$

for all $\mathbf{x}^{[n]} \in \mathbb{R}^{mn}$. Indeed, by induction on l, one can easily show for all $\mathbf{x}^{[n]} \in \mathbb{R}^{mn}$ that

$$\mathcal{J}^l f\left(\mathbf{x}^{[n]}\right) := \left(\frac{1}{2^n}\right)^l f\left(\sqrt[s]{2}^l\mathbf{x}^{[n]}\right), \tag{22}$$

for all $\mathbf{x}^{[n]} \in \mathbb{R}^{mn}$. Since $f \in \Lambda$, part (iv) of Theorem 3.6 and (20) necessitate that

$$d(f, \mathcal{R}) \le \frac{1}{1-L} d(\mathcal{J} f, f) \le \frac{1}{2^n(1-L)},$$

which proves (19). It follows from (17), (18) and (22) that

$$\begin{aligned}&\left\|f\left(\sqrt[s]{x_{11}^s + x_{21}^s}, \ldots, \sqrt[s]{x_{1n}^s + x_{2n}^s}\right) - \sum_{j_1,\ldots,j_n\in\{1,2\}} f(x_{j_11}, \ldots, x_{j_nn}), \mathbf{y}\right\| \\ &\le \frac{1}{2^{nm\beta}}\left\|f\left(\sqrt[s]{2^{m\beta}x_{11}^s + 2^{m\beta}x_{21}^s}, \ldots, \sqrt[s]{2^{m\beta}x_{1n}^s + 2^{m\beta}x_{2n}^s}\right) - \sum_{j_1,\ldots,j_n\in\{1,2\}} f\left(\sqrt[s]{2}^{m\beta}x_{j_11}, \ldots, \sqrt[s]{2}^{m\beta}x_{j_nn}\right), \mathbf{y}\right\| \\ &\le \frac{\psi\left(\sqrt[s]{2}^{m\beta}x_1^{[n]}, \sqrt[s]{2}^{m\beta}x_2^{[n]}, \mathbf{y}\right)}{2^{nm\beta}}.\end{aligned}$$

for all $\mathbf{x}_1^{[n]}, \mathbf{x}_2^{[n]} \in \mathbb{R}^{mn}$. The relation above and part (ii) of Lemma 1.1 implies that

$$\mathcal{R}\left(\sqrt[s]{x_{11}^s + x_{21}^s}, \sqrt[s]{x_{12}^s + x_{22}^s}, \dots, \sqrt[s]{x_{1n}^s + x_{2n}^s}\right) = \sum_{j_1,\dots,j_n \in \{1,2\}} \mathcal{R}(x_{j_1 1}, \dots, x_{j_n n}),$$

for all $\mathbf{x}_1^{[n]}, \mathbf{x}_2^{[n]} \in \mathbb{R}^{mn}$. This means that the mapping $\mathcal{R}$ is s-multi-radical. Let us lastly assume that $\mathfrak{R} : \mathbb{R}^{mn} \longrightarrow \mathbb{R}^m$ is an s-multi-radical mapping such that inequality (19) is valid. Then, $\mathfrak{R}$ fulfills (21), and thus it is a fixed point of the operator $\mathcal{J}$. Furthermore, by (19), we find

$$d(f, \mathfrak{R}) \leq \frac{1}{2^n(1-L)} < \infty,$$

and consequently $\mathfrak{R} \in \Lambda$. Now, part (ii) of Theorem 3.6 implies that $\mathfrak{R} = \mathcal{R}$. $\square$

The upcoming stability result is deduced from Theorem 3.7. We bring only the sketch of proofs.

Corollary 3.8. *Fix $\delta \in [0, \infty)$ and $\alpha > 0$. Given $r \in (0, \infty)$ with $r \neq ns$. Let $\mathbb{R}^m$ be a 2-Banach space equipped with 2-norm (14). Assume also $f : \mathbb{R}^{mn} \longrightarrow \mathbb{R}^m$ is a mapping fulfills*

$$\left\| f\left(\sqrt[s]{x_{11}^s + x_{21}^s}, \dots, \sqrt[s]{x_{1n}^s + x_{2n}^s}\right) - \sum_{j_1,\dots,j_n \in \{1,2\}} f(x_{j_1 1}, \dots, x_{j_n n}), \mathbf{y} \right\| \\ \leq \delta \|\mathbf{y}\|_\infty^\alpha \sum_{i=1}^{2} \sum_{j=1}^{n} |\mathbf{x}_{ij}|^r,$$

for all $\mathbf{x}_1^{[n]}, \mathbf{x}_2^{[n]} \in \mathbb{R}^{mn}$, $\mathbf{y} \in \mathbb{R}^m$, where $\|\mathbf{y}\|_\infty$ is defined in (15). Then, there exists a unique s-multi-radical mapping $\mathcal{R} : \mathbb{R}^{mn} \longrightarrow \mathbb{R}^m$ such that

$$\left\| f\left(\mathbf{x}^{[n]}\right) - \mathcal{R}\left(\mathbf{x}^{[n]}\right), \mathbf{y} \right\| \leq \frac{\|\mathbf{y}\|_\infty^\alpha}{\left|2^n - \sqrt[s]{2}^r\right|} \sum_{j=1}^{n} |x_{1j}|^r,$$

for all $\mathbf{x}^{[n]} := \mathbf{x}_1^{[n]} \in \mathbb{R}^{mn}$, $\mathbf{y} \in \mathbb{R}^m$.

Proof. Set $\psi(\mathbf{x}_1^{[n]}, \mathbf{x}_2^{[n]}, \mathbf{y}) = \delta \|\mathbf{y}\|_\infty^\alpha \sum_{i=1}^{2} \sum_{j=1}^{n} |\mathbf{x}_{ij}|^r$ in Theorem 3.7. Then, in the case $\sigma = 1$ (resp. $\sigma = -1$), we have $L = \frac{\sqrt[s]{2}^r}{2^n}$ (resp. $L = \frac{2^n}{\sqrt[s]{2}^r}$). Therefore, one can reach to the result by the former replacements. $\square$

4. Conclusion and future works

We have introduced s-multi-radical mappings as a system of symmetric radical equations and expressed it as a single equation. The benefit of presentation of such systems as a single equation allows us to prove the Hyers-Ulam stability of s-multi-radical mappings by means of only one functional inequality, not a number of functional equations. Moreover, we established the Hyers-Ulam stability in the spirit of Găvruţa for an s-multi-radical equation by using the Hyers method in the setting of 2-Banach spaces. For a typical case, by means of a norm, induced from a 2-norm of $\mathbb{R}^m$, we have investigated the stability of a mapping $f : \mathbb{R}^{mn} \longrightarrow \mathbb{R}^m$ by applying a fixed point technique. In fact, we have extended some results in the former works like [1], [15] and [18], which are considered on functional equations whose solutions are one variable mappings.

It is known there are various versions of radical functional equations which are available in the literature. As a future work, by applying them, one can generalize an one dimensional radical functional equation to multiple variable and study their stability on miscellaneous spaces.

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