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Original Article

Algebraic property of weighted Lebesgue spaces on a class of hypergroups

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ABSTRACT: In this paper, in the context of an important class of locally compact hypergroups, namely $\mathcal{K}_\rho$, which were introduced by Dunkl and Ramirez, we find some sufficient conditions on a weight $(w_n)_{n=0}^\infty \subseteq (0,\infty)$ such that the set

$$\left\{\big((f_k)_k,(g_k)_k\big) \in L^p(\mathcal{K}_\rho) \times L^q(\mathcal{K}_\rho) : \sum_{k=1}^{\infty} |f_k g_k| w_k^2\, \rho^{k-1} < \infty\right\}$$

is a σ-c-lower porous set.

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1. Introduction and Notations

Consider $\mathbb{N}_0 := \{0,1,2,\dots\}$ equipped with the discrete topology, and fix some prime number ρ. For any $k \in \mathbb{N}_0$ and each $m,n \in \mathbb{N}$, define

$$\delta_k * \delta_0 = \delta_0 * \delta_k := \delta_k,$$

and

$$\delta_n * \delta_n := \frac{1}{\rho^{n-1}(\rho-1)}\delta_0 + \sum_{k=1}^{n-1} \rho^{k-n}\delta_k + \frac{\rho-2}{\rho-1}\delta_n,$$

and $\delta_m * \delta_n := \delta_{\max\{m,n\}}$. Then, by [5], $\mathbb{N}_0$ equipped with the above convolution would be a Hermitian hypergroup. Note that the measure λ on $\mathbb{N}_0$ given by

$$\lambda(\{k\}) := \begin{cases} 1, & \text{if } k=0, \\ (\rho-1)\rho^{k-1}, & \text{if } k \geq 1, \end{cases}$$

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is a Haar measure for $\mathbb{N}_0$. This means that for any $n \in \mathbb{N}_0$,

$$\delta_n * \lambda = \lambda.$$

In the sequel, we refer to this hypergroup by $\mathcal{K}_\rho$.

This class of hypergroups, as the dual of a compact countable hypergroup, was introduced and studied in [5]. For more details about general locally compact hypergroups refer to the monographs [4, 7].

In this paper, we intend to discover some facts regarding porosity of weighted Lebesgue spaces on this hypergroups with respect to the above Haar measure λ. Let (X, d) be a metric space, $c \in (0, 1]$, and $P \subseteq X$. Then, P is called a c-lower porous set in X if for each $x \in P$,

$$\liminf_{r>0} \frac{\gamma(x, r, P)}{r} \geq \frac{c}{2},$$

where

$$\gamma(x, r, P) := \sup\{s \geq 0 : \text{ for some } z \in X,\, B(z; s) \subseteq B(z; r) \setminus P\},$$

where $B(z; s)$ is a ball with center $z \in X$ and radius $s > 0$. A subset P of X is called σ-c-lower porous if it can be written as a countable union of c-lower porous subsets of X. For more details, see [10, 11] and their references.

Recall that if f, g are measurable complex-valued functions on a hypergroup $(K, *, \check{\cdot}\,)$, then

$$(f * g)(x) := \int_K f(y)g(\check{y} * x)\, d\lambda(y),$$

for all $x \in K$, while this integral exists. In [3, 8] convolution properties of Orlicz spaces and weighted Orlicz spaces on hypergroups have been studied.

In [6, Theorem 1] for a locally compact non-compact group G and any $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} < 1$, it is proved that the set

$$\{(f, g) \in L^p(G) \times L^q(G) : \exists x \in B,\, (f * g)(x) \text{ is well-defined}\},$$

is σ-c-lower porous in $L^p(G) \times L^q(G)$ for some $0 < c < 1$, where B is a compact subset of G. Also, it is proved in [6, Corollary 2] that a locally compact group would be non-compact if and only if the set

$$\{(f, g) \in L^p(G) \times L^q(G) : f * g \text{ is defined a.e. on } G\},$$

is σ-c-lower porous for some $0 < c < 1$. See [2, 9] for some new results in this direction for Lebesgue spaces in the context of semigroups and hypergroups. In [1] some similar facts are given for subsets of Orlicz spaces.

In this paper, we have a try to study this topic in some new situation. More precisely, in Theorem 2.1, we find some sufficient condition on a weight $(w_n)_{n=0}^\infty \subseteq (0, \infty)$ such that the set

$$\left\{ \big((f_k)_k, (g_k)_k\big) \in L^p(\mathcal{K}_\rho) \times L^q(\mathcal{K}_\rho) : \sum_{k=1}^\infty |f_k g_k| w_k^2\, \rho^{k-1} < \infty \right\},$$

to be σ-c-lower porous.

2. Main Results

In the next result, for each $p, q > 1$ we consider the norm

$$\|(f, g)\| := \max\{\|f\|_p, \|g\|_q\}$$

for all $(f, g) \in L^p(\mathcal{K}_\rho) \times L^q(\mathcal{K}_\rho)$.

Theorem 2.1. *Let ρ be a prime number and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} < 1$. Assume that $w := (w_n)_{n=0}^\infty \subseteq (0, \infty)$ such that for some sequence $n_1 < n_2 < \cdots$,*

$$0 < \inf_{k \in \mathbb{N}} w_{n_k} \leq \sup_{k \in \mathbb{N}} w_{n_k} < \infty.$$

Then,

$$F := \left\{ \big((f_k)_k, (g_k)_k\big) \in L^p(\mathcal{K}_\rho) \times L^q(\mathcal{K}_\rho) : \sum_{k=1}^\infty |f_k g_k| w_k^2\, \rho^{k-1} < \infty \right\}, \tag{1}$$

is σ-c-lower porous in $L^p(\mathcal{K}_\rho) \times L^q(\mathcal{K}_\rho)$ for some $c > 0$.

Proof. For each $N \in \mathbb{N}$ denote

$$F_N := \left\{ \big((f_k)_k, (g_k)_k\big) \in L^p(\mathcal{K}_\rho) \times L^q(\mathcal{K}_\rho) : \sum_{k=1}^{\infty} |f_k g_k| w_k^2 \, \rho^{k-1} < N \right\}.$$

Then, there are $\delta, c \in (0,1)$ and $\eta \in (0, 1-\delta)$ such that

$$\theta := \big(\inf_{k\in\mathbb{N}} w_{n_k}\big)^2 - \left(\frac{c}{(1-\delta)\,\eta}\right)^p \big(\sup_{k\in\mathbb{N}} w_{n_k}\big)^2 - \left(\frac{c}{(1-\delta)\,\eta}\right)^q \big(\sup_{k\in\mathbb{N}} w_{n_k}\big)^2 > 0.$$

Let $r > 0$, $f := (f_k)_k \in L^p(\mathcal{K}_\rho)$ and $g := (g_k)_k \in L^q(\mathcal{K}_\rho)$. Without loss of generality, assume that $w_0 = 1$. Note that

$$\begin{aligned}
\sum_{k=1}^{\infty} \|\chi_{\{n_k\}} \, f\|_p^p &= \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} \left|\chi_{\{n_k\}}(n) \, f_n\right|^p \, (\rho-1)\,\rho^{n-1} \\
&= \sum_{k=1}^{\infty} |f_{n_k}|^p \, (\rho-1)\,\rho^{n_k-1} \\
&\leq \sum_{n=0}^{\infty} |f_n|^p \, (\rho-1)\,\rho^{n-1} \\
&= \|f\|_p^p < \infty.
\end{aligned}$$

Similarly,

$$\sum_{k=1}^{\infty} \|\chi_{\{n_k\}} \, g\|_q^q \leq \|g\|_q^q < \infty.$$

So, there is some $\ell \in \mathbb{N}$ such that

$$\sum_{k=\ell}^{\infty} \|\chi_{\{n_k\}} \, f\|_p^p < [(1-c-\eta)r]^p$$

and

$$\sum_{k=\ell}^{\infty} \|\chi_{\{n_k\}} \, g\|_q^q < [(1-c-\eta)r]^q.$$

Since $\frac{1}{p} + \frac{1}{q} < 1$, one can pick a natural number $\kappa > \ell$ such that

$$\left(\sum_{k=\ell}^{\kappa} \rho^{n_k-1}\right)^{1-\frac{1}{p}-\frac{1}{q}} > \frac{N(\rho-1)^{\frac{1}{p}+\frac{1}{q}}}{(\delta\eta\, r)^2\, \theta}.$$

Put

$$A := \{n_\ell, n_{\ell+1}, \ldots, n_\kappa\}.$$

So,

$$\|\chi_A \, f\|_p^p = \left\|\sum_{k=\ell}^{\kappa} \chi_{\{n_k\}} \, f\right\|_p^p = \sum_{k=\ell}^{\kappa} \left\|\chi_{\{n_k\}} \, f\right\|_p^p \leq \sum_{k=\ell}^{\infty} \left\|\chi_{\{n_k\}} \, f\right\|_p^p < (1-c-\eta)^p r^p.$$

So,

$$\left\|\chi_A \, f\right\|_p < (1-c-\eta)r \tag{2}$$

and similarly

$$\left\|\chi_A \, g\right\|_q < (1-c-\eta)r. \tag{3}$$

Denote

$$M_1 := \eta\, r \left(\sum_{k=\ell}^{\kappa} (\rho-1)\rho^{n_k-1}\right)^{\frac{-1}{p}} \quad \text{and} \quad M_2 := \eta\, r \left(\sum_{k=\ell}^{\kappa} (\rho-1)\rho^{n_k-1}\right)^{\frac{-1}{q}}.$$

and define

$$\tilde{f}_n := \begin{cases} M_1, & \text{if } n = n_k \ (k = \ell, \ldots, \kappa) \\ f_n, & \text{otherwise} \end{cases}$$

and

$$\tilde{g}_n := \begin{cases} M_2, & \text{if } n = n_k \ (k = \ell, \dots, \kappa) \\ g_n, & \text{otherwise} \end{cases}$$

for all $n \in \mathcal{K}_\rho$. Put $\tilde{f} := (\tilde{f}_n)_n$ and $\tilde{g} := (\tilde{g}_n)_n$. Then, by (2) and (3),

$$\max\{\|\tilde{f} - f\|_p, \|\tilde{g} - g\|_q\} < (1-c)r,$$

and so

$$B((\tilde{f}, \tilde{g}); c\,r) \subseteq B((f,g); r).$$

Let $(h,s) \in B((\tilde{f}, \tilde{g}); c\,r)$, where $h := (h_n)_n$ and $s := (s_n)_n$. Put $A_1 := \{n \in A : \, |h_n| \le \delta\, \tilde{f}_n\}$. Then,

$$c\,r > \big(1-\delta\big)\, M_1\, \|\chi_{A_1}\|_1^{\frac{1}{p}} \ge \big(1-\delta\big)\, M_1 \, \left\| \chi_{A_1} \frac{w^2}{\big(\sup_{k\in\mathbb{N}} w_{n_k}\big)^2} \right\|_1^{\frac{1}{p}},$$

thus

$$\big\|\chi_{A_1}\, w^2\big\|_1 < \left(\frac{c\,r}{(1-\delta)\, M_1}\right)^p \big(\sup_{k\in\mathbb{N}} w_{n_k}\big)^2 \le \left(\frac{c}{(1-\delta)\,\eta}\right)^p \left(\sum_{k=\ell}^{\kappa} (\rho-1)\rho^{n_k - 1}\right) \big(\sup_{k\in\mathbb{N}} w_{n_k}\big)^2.$$

Now, put $A_2 := \{n \in A : \, |s_n| \le \delta\, \tilde{g}_n\}$. Then,

$$c\,r > \big(1-\delta\big)\, M_2\, \|\chi_{A_2}\|_1^{\frac{1}{q}} \ge \big(1-\delta\big)\, M_2 \left\| \chi_{A_2} \frac{w^2}{\big(\sup_{k\in\mathbb{N}} w_{n_k}\big)^2} \right\|_1^{\frac{1}{q}}$$

and so,

$$\big\|\chi_{A_2}\, w^2\big\|_1 < \left(\frac{c\,r}{(1-\delta)\, M_2}\right)^q \big(\sup_{k\in\mathbb{N}} w_{n_k}\big)^2 = \left(\frac{c}{(1-\delta)\,\eta}\right)^q \left(\sum_{k=\ell}^{\kappa} (\rho-1)\rho^{n_k - 1}\right) \big(\sup_{k\in\mathbb{N}} w_{n_k}\big)^2.$$

Put $A_3 := A \setminus (A_1 \cup A_2)$. Then,

$$\begin{aligned} \sum_{k=1}^{\infty} |h_k s_k| w_k^2\, \rho^{k-1} &= \sum_{k=1}^{\infty} |h_k w_k|\ |s_k w_k|\, \rho^{k-1} \\ &\ge \sum_{k=1}^{\infty} \chi_{A_3}(k)\, |h_k w_k|\ |s_k w_k|\, \rho^{k-1} \\ &\ge \delta^2 \sum_{k=1}^{\infty} \chi_{A_3}(k)\, |\tilde{f}_k w_k|\ |\tilde{g}_k w_k|\, \rho^{k-1} \\ &\ge \delta^2\, M_1\, M_2 \sum_{k=1}^{\infty} \chi_{A_3}(k)\, w_k^2\, \rho^{k-1}. \end{aligned}$$

Note that

$$\begin{aligned} \sum_{k=1}^{\infty} \chi_{A_3}(k)\, w_k^2\, \rho^{k-1} &= \sum_{k=1}^{\infty} \chi_{A\setminus A_1}(k)\, w_k^2\, \rho^{k-1} - \sum_{k=1}^{\infty} \chi_{(A\setminus A_1)\setminus(A\setminus A_2)}(k)\, w_k^2\, \rho^{k-1} \\ &\ge \sum_{k=1}^{\infty} \chi_{A\setminus A_1}(k)\, w_k^2\, \rho^{k-1} - \sum_{k=0}^{\infty} \chi_{A\setminus(A\setminus A_2)}(k)\, w_k^2\, \rho^{k-1} \\ &= \sum_{k=1}^{\infty} \chi_{A\setminus A_1}(k)\, w_k^2\, \rho^{k-1} - \sum_{k=0}^{\infty} \chi_{A_2}(k)\, w_k^2\, \rho^{k-1} \\ &= \sum_{k=1}^{\infty} \chi_{A}(k)\, w_k^2\, \rho^{k-1} - \sum_{k=1}^{\infty} \chi_{A_1}(k)\, w_k^2\, \rho^{k-1} - \sum_{k=1}^{\infty} \chi_{A_2}(k)\, w_k^2\, \rho^{k-1}, \end{aligned}$$

Thus, setting

$$P := \left(\frac{c}{(1-\delta)\,\eta}\right)^p - \left(\frac{c}{(1-\delta)\,\eta}\right)^q$$

we have

$$\begin{aligned}\sum_{k=1}^{\infty} \chi_{A_3}(k)\, w_k^2\, \rho^{k-1} &\geq \left(\big(\inf_{k\in\mathbb{N}} w_{n_k}\big)^2 - \big(\sup_{k\in\mathbb{N}} w_{n_k}\big)^2 P\right) \left(\sum_{k=\ell}^{\kappa} \rho^{n_k-1}\right) \\ &= \theta \left(\sum_{k=\ell}^{\kappa} \rho^{n_k-1}\right).\end{aligned}$$

Finally, by the above calculation,

$$\begin{aligned}\sum_{k=1}^{\infty} |h_k s_k| w_k^2\, \rho^{k-1} &\geq \delta^2 M_1 M_2\, \theta \left(\sum_{k=\ell}^{\kappa} \rho^{n_k-1}\right) \\ &= \delta^2 \eta\, r \left(\sum_{k=\ell}^{\kappa} (\rho-1)\, \rho^{n_k-1}\right)^{\frac{-1}{p}} \eta\, r \left(\sum_{k=\ell}^{\kappa} (\rho-1)\, \rho^{n_k-1}\right)^{\frac{-1}{q}} \theta \left(\sum_{k=\ell}^{\kappa} \rho^{n_k-1}\right) \\ &= \delta^2 \eta^2\, r^2\, \theta \left(\sum_{k=\ell}^{\kappa} \rho^{n_k-1}\right)^{1-\frac{1}{p}-\frac{1}{q}} (\rho-1)^{-\frac{1}{p}-\frac{1}{q}} \\ &> N,\end{aligned}$$

therefore, $(h,s) \notin F_N$. This shows that any F_N is c-lower porous, and so the proof is complete. □

By some calculation, we can conclude an improvement of the above result as below.

Theorem 2.2. *Let ρ be a prime number and $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} < 1$. Assume that $w := (w_n)_{n=0}^{\infty} \subseteq (0,\infty)$ such that for some sequence $n_1 < n_2 < \cdots$, we have $0 < \inf_{k\in\mathbb{N}} w_{n_k}$. Then,*

$$F := \left\{ \big((f_k)_k, (g_k)_k\big) \in L^p(\mathcal{K}_\rho) \times L^q(\mathcal{K}_\rho) : \sum_{k=1}^{\infty} |f_k g_k| w_k^2\, \rho^{k-1} < \infty \right\}$$

is σ-c-lower porous in $L^p(\mathcal{K}_\rho) \times L^q(\mathcal{K}_\rho)$ for some $c > 0$.

Note that, the set F given by (1) in the above theorem equals to

$$\left\{ \big((f_k)_k, (g_k)_k\big) \in L^p(\mathcal{K}_\rho) \times L^q(\mathcal{K}_\rho) : \; ((f_k)_k *_w (g_k)_k)_0 < \infty \right\}.$$

So, in the case that $p = q > 2$, if $(L^p(\mathcal{K}_\rho), *_w)$ is an algebra, then we have $F = L^p(\mathcal{K}_\rho) \times L^p(\mathcal{K}_\rho)$. Hence, thanks to Theorem 2.1, the next fact will be obtained.

Corollary 2.3. *Let $p > 2$, and $w := (w_k)_k \subseteq (0,\infty)$. If $(L^p(\mathcal{K}_\rho), *_w)$ is an algebra, then for every sequence $n_1 < n_2 < \cdots$, $\inf_{k\in\mathbb{N}} w_{n_k} = 0$. In particular, $L^p(\mathcal{K}_\rho)$ is not a convolution algebra for all $p > 2$.*

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