



Original Article

Some trace functional inequalities for operator (p, h) -convex functions

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ABSTRACT: In this paper, we present a theorem pertinent to singular value inequalities for positive and compact operators on a Hilbert space. Moreover, we obtain several trace inequalities for operator (p, h) -convex functions.

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1. Introduction

It is known that there are many important inequalities with miscellaneous applications in many areas of mathematics such as nonlinear analysis. One of them is the celebrated Hermite-Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a)+f(b)}{2},$$

where f is a convex function on $[a, b]$; for more information and details, we refer to [7], [9], [10] and [11]. In what follows, we state some historical notes, presented in the literature about some generalizations, modifications, refinements and improvements of the Hermite-Hadamard inequality used in this work.

Let $s \in (0, 1)$. Recall from [5] that a real valued function f on an interval $I \subseteq [0, \infty)$ is said to be s -convex in the second sense if $f(rx + ty) \leq r^s f(x) + t^s f(y)$, for all $x, y \in I$ and $r, t \geq 0$ with $r + t = 1$. In [5, Theorem 2.1],

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Dragomir and Fitzpatrick proved the following version of Hermite-Hadamard inequality for s -convex functions in the second sense: let $f : I \subseteq [0, \infty) \rightarrow \mathbb{R}$ be a s -convex function, where $s \in (0, 1]$ and $a, b \in I$ with $a < b$. If $f \in L^1(I)$, then

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x) dx \leq \frac{f(a) + f(b)}{s+1}.$$

Next, Zabandan et al. [14] presented a refinement of the Hermite-Hadamard inequality for s -convex functions in the case that $s \in [0, 1]$. In addition, they studied the Hermite-Hadamard inequality for the product of a r -convex function f and a s -convex function g .

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a complex separable Hilbert space. We denote the C^* -algebra of all bounded linear operators on $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. In the following, we provide some definitions and notations.

- An operator $A \in \mathcal{B}(\mathcal{H})$ is *positive* if $\langle Ax, x \rangle \geq 0$ (denoted by $A \geq 0$) for all $x \in \mathcal{H}$. Moreover, a positive invertible operator A is naturally denoted by $A > 0$ and the set of all positive operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})^+$.

- For self-adjoint operators $A, B \in \mathcal{B}(\mathcal{H})$ we write $B \geq A$ if $B - A \geq 0$.

- A linear map $\Phi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$ and Φ is said to be unital if $\Phi(I) = I$.

For a self-adjoint operator A in $\mathcal{B}(\mathcal{H})$, it is known that there is a $*$ -isometric isomorphism Φ between the set $C(Sp(A))$ of all continuous functions defined on $Sp(A)$, the spectrum of A , and $C^*(A)$, the C^* -algebra generated by A and the identity operator $1_{\mathcal{B}(\mathcal{H})}$ on $\mathcal{H}$. This map is called the Gelfand map; see page 3 of [13]. Suppose that f is a continuous complex-valued function on $Sp(A)$. We denote each element $\Phi(f)$ of $C^*(A)$ by $f(A)$ and call it the *continuous functional calculus* for a bounded self-adjoint operator A . Given A as a bounded self-adjoint operator, if f is a real-valued continuous function on $Sp(A)$ such that $f(t) \geq 0$ for all $t \in Sp(A)$, then $f(A) \geq 0$ and this means that $f(A)$ is a positive operator on $\mathcal{H}$. Furthermore, $f(A) \leq g(A)$ in the operator order in $\mathcal{B}(\mathcal{H})$ provided that both f and g are real-valued functions on $Sp(A)$ such that $f(t) \leq g(t)$ for any $t \in sp(A)$.

An extension of the previously mentioned class of functions to operators was proposed by Ghazanfari [6] as follows: Let $s \in (0, 1]$ and let I be an interval in $[0, \infty)$. A continuous function $f : I \rightarrow \mathbb{R}$ is called *operator s -convex* on I provided that $f((1-\lambda)A + \lambda B) \leq (1-\lambda)^s f(A) + \lambda^s f(B)$ for all $\lambda \in [0, 1]$ and all A, B belonging to $\mathcal{B}(\mathcal{H})^+$ whose spectra are contained in I . Under these assumptions, he presented an inequality for operator s -convex functions as follows:

$$2^{s-1} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f(\lambda A + (1-\lambda)B) d\lambda \leq \frac{f(A) + f(B)}{s+1}.$$

Let $I, J \subseteq \mathbb{R}^+$, $(0, 1) \subseteq J$ and $h : J \rightarrow \mathbb{R}$ be a nonnegative function that is not identically zero. Then

(1) (See [4].) A real-valued continuous function f on $K \subseteq [0, \infty)$ is called *operator convex* (*operator concave*) if

$$f(\lambda A + (1-\lambda)B) \leq (\geq) \lambda f(A) + (1-\lambda)f(B),$$

in the operator order in $\mathcal{B}(\mathcal{H})$, for all $\lambda \in [0, 1]$ and all bounded self-adjoint operators A and B in $\mathcal{B}(\mathcal{H})$ whose spectra are contained in K ;

(2) (See [2] and [3].) We say that a continuous function $f : I \rightarrow \mathbb{R}^+$ is an *operator h -convex function* (resp. *operator (p, h) -convex*) on I if

$$f(\lambda A + (1-\lambda)B) \leq h(\lambda)f(A) + h(1-\lambda)f(B)$$

$$(\text{resp. } f((\lambda A^p + (1-\lambda)B^p)^{\frac{1}{p}}) \leq h(\lambda)f(A) + h(1-\lambda)f(B)),$$

for all $A, B \in \mathcal{B}(\mathcal{H})^+$ whose spectra are contained in I , and for all $\lambda \in (0, 1)$.

The next inequalities, due to authors [2], provides a version of the Hermite-Hadamard inequalities for operator h -convex functions.

Let f be an operator h -convex function. Then

$$\frac{1}{2h(\frac{1}{2})} f\left(\frac{A+B}{2}\right) \leq \int_0^1 f(tA + (1-t)B) dt \leq (f(A) + f(B)) \int_0^1 h(t) dt,$$

for any self-adjoint operators A and B whose spectra lie in K .

Similarly, the following inequalities, establish the Hermite-Hadamard-type inequalities for operator (p, h) -convex functions [12].

Let $f : K \longrightarrow \mathbb{R}^+$ be a continuous operator (p, h) -convex function. Then

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})} f\left(\left(\frac{A^p + B^p}{2}\right)^{\frac{1}{p}}\right) &\leq \int_0^1 f((\lambda A^p + (1 - \lambda)B^p)^{\frac{1}{p}}) d\lambda \\ &\leq (f(A) + f(B)) \int_0^1 h(\lambda) d\lambda, \end{aligned} \quad (1)$$

for any $A, B \in \mathcal{B}(\mathcal{H})^+$ with spectra in K and $h : J \longrightarrow \mathbb{R}^+$ be a continuous non-zero function.

An interval $K \subseteq \mathbb{R}^+$ is called a p -convex set if

$$Sp((\lambda A^p + (1 - \lambda)B^p)^{\frac{1}{p}}) \subseteq K,$$

for all positive operators $A, B \in \mathcal{B}(\mathcal{H})^+$ with spectra contained in K , for all $\lambda \in (0, 1)$ and $p > 0$.

In this paper, we present some inequalities related to positive operators in $K(\mathcal{H})$, the C^* -algebra of compact operators on $\mathcal{H}$. As a consequence, we will show that if f is an operator (p, h) -convex function, then

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})} Tr\left(f\left(\left(\frac{A^p + B^p}{2}\right)^{\frac{1}{p}}\right)\right) &\leq \int_0^1 Tr(f((tA^p + (1 - t)B^p)^{\frac{1}{p}})) dt \\ &\leq (Tr(f(A)) + Tr(f(B))) \int_0^1 h(t) dt, \end{aligned}$$

for any self-adjoint operators A and B on $\mathcal{H}$. Furthermore, we establish the singular value inequalities for positive operators in $K(\mathcal{H})$. Lastly, we present several trace inequalities for positive operators on $\mathcal{B}(\mathcal{H})$.

2. Main Results

We begin by restating some definitions, notations, commonly used terminologies and conventions from operator theory, as presented in the literature.

Let $\mathcal{H}$ be a Hilbert space and let $K(\mathcal{H})$ be the two sided ideal of compact operators in $\mathcal{B}(\mathcal{H})$. For any $A \in \mathcal{B}(\mathcal{H})$, the operator norm is defined by $\|A\| = \sup \{\|Ax\| : \|x\| = 1\}$. Given $A, B \in \mathcal{B}(\mathcal{H})$, the direct sum $A \oplus B$ denotes the block diagonal matrix $\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$ defined on $\mathcal{H} \oplus \mathcal{H}$; see [15] for more details. It is clear that $\|A \oplus B\| = \max \{\|A\|, \|B\|\}$. It is known that the operator A^*A is always positive and it has a unique positive square root, denoted via $|A|$. The eigenvalues of $|A|$ counted with multiplicities are called the *singular values* of A . We will always enumerate these in descending order, and denoted through $s_1(A) \geq s_2(A) \geq \dots \geq s_n(A)$. Note that $\|A\| = s_1(A) = \|A^*A\|^{\frac{1}{2}}$.

Let $\{e_i\}_{i \in I}$ be an orthonormal basis of $\mathcal{H}$. We say that $A \in \mathcal{B}(\mathcal{H})$ is *trace class* if

$$\|A\|_1 = \sum_{i \in I} \langle |A|e_i, e_i \rangle < \infty.$$

It is important to note that the definition of $\|A\|_1$ does not depend on the choice of the orthonormal basis $\{e_i\}_{i \in I}$. We denote the set of all trace class operators in $\mathcal{B}(\mathcal{H})$ by $\mathcal{B}_1(\mathcal{H})$. In addition, the trace of a trace class operator $A \in \mathcal{B}_1(\mathcal{H})$ is defined via

$$Tr(A) := \sum_{i \in I} \langle Ae_i, e_i \rangle,$$

where $\{e_i\}_{i \in I}$ is an orthonormal basis of $\mathcal{H}$. Note that $Tr(\cdot)$ is a bounded linear functional on $\mathcal{B}_1(\mathcal{H})$ with $\|Tr\| = 1$. A useful result from [8] asserts that if $A, B \in K(\mathcal{H})$, then $s_j(\frac{A+B}{2}) \leq s_j(A \oplus B)$ for all $j \in \mathbb{N}$.

Lemma 2.1. [[1], P. 75] *Let $A, B \in \mathcal{B}(\mathcal{H})$ such that A is a compact operator. Then*

$$s_j(AB) \leq \|B\|s_j(A),$$

for all $j \in \mathbb{N}$.

From now on, we set

$$\mathcal{M}(\mathcal{H}) := \{(A, B) \in \mathcal{B}(\mathcal{H})^+ \times \mathcal{B}(\mathcal{H})^+ | AB + BA \geq 0\}.$$

In the upcoming result, we state the singular value inequalities for positive operators in $K(\mathcal{H})$.

Theorem 2.2. *Let A, B be positive operators in $K(\mathcal{H})$ such that $AB + BA \geq 0$ and $\frac{1}{2} \leq s \leq p$. If $\mathcal{T}$ is an arbitrary operator in $\mathcal{B}(\mathcal{H})$, then*

$$\frac{1}{2} s_j((A+B)^{\frac{1}{2}} \mathcal{T})^{\frac{2s}{p}} \leq s_j\left(\int_0^1 (\mathcal{T}^*(\lambda A + (1-\lambda)B)\mathcal{T})^{\frac{s}{p}} d\lambda\right) \leq \frac{2p}{s+p} \|\mathcal{T}\|^{\frac{2s}{p}} s_j(A \oplus B)^{\frac{2s}{p}},$$

for all $\lambda \in (0, 1)$ and all $j \in \mathbb{N}$.

Proof. Replacing (A, B) by $(A^{\frac{1}{p}}, B^{\frac{1}{p}})$ in inequality (1), we obtain

$$\frac{1}{2h(\frac{1}{2})} f\left(\left(\frac{A+B}{2}\right)^{\frac{1}{p}}\right) \leq \int_0^1 f((\lambda A + (1-\lambda)B)^{\frac{1}{p}}) d\lambda \leq (f(A^{\frac{1}{p}}) + f(B^{\frac{1}{p}})) \int_0^1 h(\lambda) d\lambda.$$

Switching A, B with $\Phi(A), \Phi(B)$, respectively, where Φ is a positive unital operator in $\mathcal{B}(\mathcal{H})$, we obtain

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})} f\left(\left(\frac{\Phi(A) + \Phi(B)}{2}\right)^{\frac{1}{p}}\right) &\leq \int_0^1 f((\lambda \Phi(A) + (1-\lambda)\Phi(B))^{\frac{1}{p}}) d\lambda \\ &\leq (f(\Phi^{\frac{1}{p}}(A)) + f(\Phi^{\frac{1}{p}}(B))) \int_0^1 h(\lambda) d\lambda. \end{aligned} \quad (2)$$

Put $f(x) = x^s$, $h(\lambda) = \lambda^{\frac{s}{p}}$ for $\frac{1}{2} \leq s \leq p$ and $\Phi(A) = \mathcal{T}^* A \mathcal{T}$ in (2). Then, we get

$$\begin{aligned} \frac{1}{2} (\mathcal{T}^* A \mathcal{T} + \mathcal{T}^* B \mathcal{T})^{\frac{s}{p}} &\leq \int_0^1 (\lambda \mathcal{T}^* A \mathcal{T} + (1-\lambda) \mathcal{T}^* B \mathcal{T})^{\frac{s}{p}} d\lambda \\ &\leq \frac{p}{s+p} ((\mathcal{T}^* A \mathcal{T})^{\frac{s}{p}} + (\mathcal{T}^* B \mathcal{T})^{\frac{s}{p}}). \end{aligned} \quad (3)$$

We compute the first part of inequality (3) as follows:

$$\begin{aligned} \frac{1}{2} (\mathcal{T}^* A \mathcal{T} + \mathcal{T}^* B \mathcal{T})^{\frac{s}{p}} &= \frac{1}{2} (\mathcal{T}^* (A+B) \mathcal{T})^{\frac{s}{p}} \\ &= \frac{1}{2} (\mathcal{T}^* (A+B)^{\frac{1}{2}} (A+B)^{\frac{1}{2}} \mathcal{T})^{\frac{s}{p}} \\ &= \frac{1}{2} (|(A+B)^{\frac{1}{2}} \mathcal{T}|^2)^{\frac{s}{p}} = \frac{1}{2} (|(A+B)^{\frac{1}{2}} \mathcal{T}|)^{\frac{2s}{p}}, \end{aligned}$$

and so

$$\frac{1}{2} (\mathcal{T}^* A \mathcal{T} + \mathcal{T}^* B \mathcal{T})^{\frac{s}{p}} = \frac{1}{2} (|(A+B)^{\frac{1}{2}} \mathcal{T}|)^{\frac{2s}{p}}. \quad (4)$$

For the third part of inequality (3), we find

$$\frac{p}{s+p} ((\mathcal{T}^* A \mathcal{T})^{\frac{s}{p}} + (\mathcal{T}^* B \mathcal{T})^{\frac{s}{p}}) = \frac{p}{s+p} (|A^{\frac{1}{2}} \mathcal{T}|^{\frac{2s}{p}} + |B^{\frac{1}{2}} \mathcal{T}|^{\frac{2s}{p}}). \quad (5)$$

It follows from (3), (5) and (4) that

$$\begin{aligned} \frac{1}{2} (|(A+B)^{\frac{1}{2}} \mathcal{T}|)^{\frac{2s}{p}} &\leq \int_0^1 (\lambda \mathcal{T}^* A \mathcal{T} + (1-\lambda) \mathcal{T}^* B \mathcal{T})^{\frac{s}{p}} d\lambda \\ &\leq \frac{p}{s+p} (|A^{\frac{1}{2}} \mathcal{T}|^{\frac{2s}{p}} + |B^{\frac{1}{2}} \mathcal{T}|^{\frac{2s}{p}}). \end{aligned}$$

Since for each j , s_j is a monotone operator, we arrive at

$$\begin{aligned} \frac{1}{2}s_j\left(\left|(A+B)^{\frac{1}{2}}\mathcal{T}\right|^{\frac{2s}{p}}\right) &\leq s_j\left(\int_0^1 (\mathcal{T}^*(\lambda A + (1-\lambda)B)\mathcal{T})^{\frac{s}{p}} d\lambda\right) \\ &\leq s_j\left(\frac{p}{s+p}\left(\left|A^{\frac{1}{2}}\mathcal{T}\right|^{\frac{2s}{p}} + \left|B^{\frac{1}{2}}\mathcal{T}\right|^{\frac{2s}{p}}\right)\right) \\ &\leq \frac{2p}{s+p}s_j\left(\left|A^{\frac{1}{2}}\mathcal{T}\right|^{\frac{2s}{p}} \oplus \left|B^{\frac{1}{2}}\mathcal{T}\right|^{\frac{2s}{p}}\right) \\ &\leq \frac{2p}{s+p}s_j\left(\begin{pmatrix} A^{\frac{1}{2}} & 0 \\ 0 & B^{\frac{1}{2}} \end{pmatrix} \oplus \begin{pmatrix} \mathcal{T} & 0 \\ 0 & \mathcal{T} \end{pmatrix}\right)^{\frac{2s}{p}} \\ &\leq \frac{2p}{s+p}\|\mathcal{T} \oplus \mathcal{T}\|^{\frac{2s}{p}}s_j\left(A^{\frac{1}{2}} \oplus B^{\frac{1}{2}}\right)^{\frac{2s}{p}} \\ &= \frac{2p}{s+p}\|\mathcal{T}\|^{\frac{2s}{p}}s_j(A \oplus B)^{\frac{2s}{p}}. \end{aligned}$$

Therefore, the proof is now complete. $\square$

In the next corollary, we include [2, Example 3.2] as a direct consequence of Theorem 2.2.

Corollary 2.3. *Let $\mathcal{T}$ be an operator in $\mathcal{B}(\mathcal{H})$. Then, for each pair of positive compact operators A and B in $M(\mathcal{H})$, we have*

$$\begin{aligned} \frac{1}{2}s_j((A+B)^{\frac{1}{2}}\mathcal{T})^{2s} &\leq s_j\left(\int_0^1 (\mathcal{T}^*(\lambda A + (1-\lambda)B)\mathcal{T})^s d\lambda\right) \\ &\leq \frac{2}{s+1}\|\mathcal{T}\|^{2s}s_j(A \oplus B)^s, \end{aligned}$$

for all $j \in \mathbb{N}$, where $s \in [\frac{1}{2}, 1]$.

Proof. The desired result can be obtained by putting $p = 1$ in Theorem 2.2. $\square$

Proposition 2.4. *Let $f : K \rightarrow \mathbb{R}^+$ be an operator (p, h) -convex function and $h : J \rightarrow \mathbb{R}^+$ be a non-zero function. Then, $g(t) = \text{Tr}(f((tA^p + (1-t)B^p)^{\frac{1}{p}}))$ is h -convex on $(0, 1)$ for any self-adjoint operators A and B with spectra contained in K .*

Proof. Since the trace functional is convex and monotonic, for each $u, v \in (0, 1)$ and $0 < \alpha < 1$, we obtain

$$\begin{aligned} g(\alpha u + (1-\alpha)v) &= \text{Tr}\left(f((\alpha u + (1-\alpha)v)A^p + (\alpha u + (1-\alpha)v)B^p)^{\frac{1}{p}}\right) \\ &= \text{Tr}\left(f(\alpha(uA^p + (1-u)B^p) + (1-\alpha)(vA^p + (1-v)B^p))^{\frac{1}{p}}\right) \\ &= \text{Tr}\left(f(\alpha[(uA^p + (1-u)B^p)^{\frac{1}{p}}]^p + (1-\alpha)[(vA^p + (1-v)B^p)^{\frac{1}{p}}]^p)^{\frac{1}{p}}\right) \\ &\leq \text{Tr}\left(h(\alpha)f(uA^p + (1-u)B^p)^{\frac{1}{p}} + h(1-\alpha)f(vA^p + (1-v)B^p)^{\frac{1}{p}}\right) \\ &= h(\alpha)\text{Tr}\left(f(uA^p + (1-u)B^p)^{\frac{1}{p}}\right) + h(1-\alpha)\text{Tr}\left(f(vA^p + (1-v)B^p)^{\frac{1}{p}}\right) \\ &= h(\alpha)g(u) + h(1-\alpha)g(v). \end{aligned}$$

Therefore, g is h -convex. $\square$

We now state the second main result of this paper using the above proposition. In fact, the next theorem generalizes the trace Hermite-Hadamard inequality for operator (p, h) -convex functions.

Theorem 2.5. *Let $f : K \rightarrow \mathbb{R}^+$ be an operator (p, h) -convex function and $h : J \rightarrow \mathbb{R}^+$ be a non-zero function. Then, for each self-adjoint operators A and B with spectra contained in K , we have.*

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})}\text{Tr}\left(f\left(\left(\frac{A^p + B^p}{2}\right)^{\frac{1}{p}}\right)\right) &\leq \int_0^1 \text{Tr}\left(f((tA^p + (1-t)B^p)^{\frac{1}{p}})\right) dt \\ &\leq \left(\text{Tr}(f(A)) + \text{Tr}(f(B))\right) \int_0^1 h(t) dt. \end{aligned} \tag{6}$$

Proof. By Proposition 2.4, the function $g(t) = \text{Tr}(f(tA^p + (1-t)B^p)^{\frac{1}{p}})$ is h -convex on $(0, 1)$, and hence by inequality (1), we get

$$\frac{1}{2h(\frac{1}{2})}g\left(\frac{0+1}{2}\right) \leq \int_0^1 g(t) dt \leq \left(\frac{g(0)+g(1)}{2}\right) \int_0^1 h(t) dt.$$

This finishes the proof. $\square$

The incoming corollaries are direct consequences of Theorem 2.5.

Corollary 2.6. *Let $p > 0$ and $s \in [p, 2p]$. Then*

$$2^{-\frac{s}{p}} \text{Tr}((A^p + B^p)^{\frac{s}{p}}) \leq \int_0^1 \text{Tr}((\lambda A^p + (1-\lambda)B^p)^{\frac{s}{p}}) d\lambda \leq \frac{1}{2} \text{Tr}(A^s + B^s),$$

for any self-adjoint operators $A, B \in \mathcal{B}_1(\mathcal{H})^+$ with spectra contained in K .

Proof. For $h(\lambda) = \lambda$ and $s \in [p, 2p]$, the function $f(t) = t^s$ is operator (p, h) -convex. By inequality (6) the result can be deduced. $\square$

Corollary 2.7. *For $p > 0$ and $0 < s \leq p$, we have*

$$\frac{1}{2} \text{Tr}((A^p + B^p)^{\frac{s}{p}}) \leq \int_0^1 \text{Tr}((\lambda A^p + (1-\lambda)B^p)^{\frac{s}{p}}) d\lambda \leq \frac{p}{s+p} \text{Tr}(A^s + B^s), \quad (7)$$

for all positive trace class operators A and B in $M(\mathcal{H})$.

Proof. Set $h(\lambda) = \lambda^{\frac{s}{p}}$ with $0 < s \leq p$. Then, the function $f(t) = t^s$ is operator (p, h) -convex on $M(\mathcal{H})$. Applying f in inequality (6), we find the result. $\square$

Corollary 2.8. *Suppose that $0 < s \leq p$ and Φ is a unital positive linear map. For each pair of positive trace class operators A and B in $M(\mathcal{H})$, we have*

$$\frac{1}{2} \text{Tr}(\Phi((A^p + B^p)^{\frac{s}{p}})) \leq \frac{p}{s+p} \text{Tr}((\Phi(A))^s + (\Phi(B))^s).$$

Proof. Replacing $\Phi(A), \Phi(B)$ by A, B , respectively, in inequality (7), we get the result by Proposition 2.4. $\square$

Remark 2.9. *It should be note that if Φ is a unital positive trace preserving map, then by Corollary 2.8 we find*

$$\frac{1}{2} \text{Tr}((A^p + B^p)^{\frac{s}{p}}) \leq \frac{p}{s+p} \text{Tr}((\Phi(A))^s + (\Phi(B))^s).$$

Some inequalities for positive operators A and B in $\mathcal{B}_1(\mathcal{H})^+$ are presented in the upcoming proposition.

Proposition 2.10. *Let f be an operator (p, h) -convex and $h : J \rightarrow \mathbb{R}^+$ be a non-zero function. Then, we have*

$$\begin{aligned} \frac{1}{2h(\frac{1}{2})}f\left(\left(\frac{(\text{Tr}(A))^p + (\text{Tr}(B))^p}{2}\right)^{\frac{1}{p}}\right) &\leq \int_0^1 f((\lambda(\text{Tr}(A))^p + (1-\lambda)(\text{Tr}(B))^p)^{\frac{1}{p}}) d\lambda \\ &\leq (f(\text{Tr}(A)) + f(\text{Tr}(B))) \int_0^1 h(\lambda) d\lambda, \end{aligned} \quad (8)$$

for all positive operators A and B in $\mathcal{B}_1(\mathcal{H})^+$ with spectra contained in K .

Proof. Interchanging (A, B) by $(\text{Tr}(A), \text{Tr}(B))$ in inequality (1), we obtain (8). $\square$

Plugging [12, Example 1] and Proposition 2.10, we obtain a relation as follows. Since the proof is routine, we omit it.

Corollary 2.11. *Let $0 < s \leq p$ and let $A, B \in \mathcal{B}_1(\mathcal{H})^+$ such that $AB + BA \geq 0$. Then*

$$\begin{aligned} \frac{1}{2} ((\text{Tr}(A))^p + (\text{Tr}(B))^p)^{\frac{s}{p}} &\leq \int_0^1 (\lambda(\text{Tr}(A))^p + (1-\lambda)(\text{Tr}(B))^p)^{\frac{s}{p}} d\lambda \\ &\leq ((\text{Tr}(A))^s + (\text{Tr}(B))^s) \left(\frac{p}{s+p}\right). \end{aligned}$$

Corollary 2.12. Suppose that $0 < p$ and $s \in [p, 2p]$ and let $A, B \in \mathcal{B}_1(\mathcal{H})^+$. Then, we have

$$\begin{aligned} \left(\frac{(Tr(A))^p + (Tr(B))^p}{2} \right)^{\frac{s}{p}} &\leq \int_0^1 (\lambda(Tr(A))^p + (1-\lambda)(Tr(B))^p)^{\frac{s}{p}} d\lambda \\ &\leq \frac{(Tr(A))^s + (Tr(B))^s}{2}. \end{aligned}$$

Proof. The result follows from [12, Proposition 5] and Proposition 2.10. □

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