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Original Article

On an imprimitive maximal subgroup of $SU_7(2)$

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ABSTRACT: The special unitary group $SU_n(q)$ has a maximal imprimitive subgroup with the structure $(q+1)^{n-1}{:}S_n$. The exceptions when the subgroup is not maximal are $SU_3(5)$, $SU_4(3)$ and $SU_6(2)$. In this paper, the ordinary character table of the maximal imprimitive subgroup $\overline{G} = 3^6{:}S_7$ of $SU_7(2) = U_7(2)$ is computed by the Fischer-Clifford matrices technique. A combinatorial approach is adopted in the computation of the Fischer-Clifford matrices of $\overline{G}$.

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1. Introduction

The wreath product of $\mathbb{Z}_m$ with S_n is a semi-direct product of N by S_n called the generalized symmetric group denoted by $B(m,n)$, where $\mathbb{Z}_m$ is the cyclic group of order m, N the direct product of n copies of $\mathbb{Z}_m$ and S_n is the symmetric group of degree n. A subgroup $B_S(m,n)$ is a group of order $m^{n-1}n!$ and index m in $B(m,n)$. A combinatorial method in [1] was adapted in [22] to construct the Fischer-Clifford matrices [6] of some generalized group extensions of the form $B(m,n)$ and their associated groups of the form $B_S(m,n)$. Using Theorem 6.1 and Proposition 6.3 both in [1] together with Theorem 6.2 in [11] (see Section 6), Programme 5.2.4 was developed in [22] to compute matrices which are row equivalent to Fischer-Clifford matrices of $B(m,n)$. The Fischer-Clifford matrices of $B_S(m,n)$ are then extracted from those of $B(m,n)$. In our current work, we adopt one of the combinatorial

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approaches by [22] to obtain the Fischer-Clifford matrices of $B_S(3,7) = 3^6{:}S_7$ from the Fischer-Clifford matrices of $B(3,7) = 3^7{:}S_7$.

In general, the special unitary group $SU_n(q)$ has a maximal imprimitive subgroup with the structure $(q+1)^{n-1}{:}S_n$ (see [3] or [9]) with $SU_3(5)$, $SU_4(3)$ and $SU_6(2)$ being the exceptions, where the imprimitive subgroup is not maximal. The simple group $SU_7(2) = U_7(2)$ [21] has an imprimitive maximal subgroup of structure $3^6{:}S_7$ which is identified with the group $B_S(3,7)$. In the rest of the paper, the group $3^6{:}S_7$ is denoted by $\overline{G}$ and is a split extension of $N = 3^6$ by $G = S_7$.

The Fischer-Clifford technique that is being applied to the group $\overline{G}$ draws its fundamentals from the process of obtaining the irreducible characters of $\overline{G}$ by way of induction of the irreducible characters of the inertia groups $\overline{H}_i = N{:}H_i$ of $\mathrm{Irr}(N)$, which have N in their kernels, to $\overline{G}$. In general, the character tables of $\overline{H}_i$ are much larger and complicated than the set $\mathrm{Irr}(\overline{G})$, so it suffices to just use the character tables of the inertia factors groups $H_i = \frac{\overline{H}_i}{N}$ and so-called Fischer-Clifford matrices of $\overline{G}$ to construct the ordinary character table of $\overline{G}$ (see Section 6). The coset-analysis technique (see Section 2) is used to obtain the classes of $\overline{G}$ coming from each class $[g]$ of G. Section 3 deals with the inertia factor groups H_i for $i \in \{1, 2, \ldots, 12\}$ and their fusion maps into $G = S_7$ while Section 5 reviews combinatorial objects necessary for determining the entries of Fischer-Clifford matrices. Section 6 is dedicated to the determination of a Fischer-Clifford matrix $M(g)$ corresponding to a class $[g]$ of G and all the Fischer-Clifford matrices of $\overline{G}$ while Section 7 deals with the computation of the full character table of $\overline{G}$. For all computations involving of the conjugacy classes and inertia factor groups of $\overline{G} = 3^6{:}S_7$, the computer algebra system GAP [7] is used.

2. The Conjugacy Classes of $\overline{G}$

A 6-dimensional matrix representation of $G = S_7$ over $GF(3)$ is obtained using generators g_1 and g_2 in Figure 1 taken from [21]. The generators are such that $o(g_1) = 2$, $o(g_2) = 6$ and $o(g_1g_2) = 7$. In this section, we use this representation of G to compute the classes of $\overline{G}$ by the coset-analysis method.

$$g_1 = \begin{pmatrix} 2&0&0&0&0&0\\ 0&0&2&0&0&0\\ 0&2&0&0&0&0\\ 0&0&0&2&0&0\\ 0&2&1&0&2&0\\ 0&1&2&0&0&2 \end{pmatrix}, \quad g_2 = \begin{pmatrix} 0&2&0&0&0&0\\ 1&2&0&0&0&0\\ 0&0&0&2&0&0\\ 0&0&0&0&2&0\\ 0&0&0&0&0&2\\ 2&0&2&2&0&1 \end{pmatrix}.$$

Figure 1: The generators of $G = S_7$

The group $G = \langle g_1, g_2\rangle$ has 12 orbits on the vector space $V_6(3) \cong 3^6$ having lengths 1, 105, 105, 21, 210, 21, 35, 140, 35, 7, 7 and 42 with corresponding orbit stabilizers $P_1 = S_7$, $P_2 = P_3 = C_2 \times S_4$, $P_4 = P_6 = C_2 \times S_5$, $P_5 = 2^2 \times S_3$, $P_7 = S_4 \times S_3$, $P_8 = S_3 \times S_3$, $P_9 = S_3 \times S_4$, $P_{10} = P_{11} = S_6$ and $P_{12} = S_5$. The orbit stabilizers were computed using GAP, and those identified as equal were verified directly within GAP. Since the stabilizers P_i are known, we proceed to compute the permutation character $\chi(G|N) = 1 + \sum_{i=2}^{12} I_{P_i}^G$ of S_7 acting on the classes of 3^6, where $I_{P_i}^G$ are the identity characters of the stabilizers P_i induced to G. The character $\chi(G|3^6)$ has degree 729 and is written as the sum of irreducible characters of G. For instance, to determine $I_{P_2}^G = I_{P_3}^G$ the inner product $\langle \chi_i, \psi_1\rangle$ of each $\chi_i \in \mathrm{Irr}(G)$ with the identity character ψ_1 of P_2 is computed, where $deg(\chi_i) < 105$. The values for $\langle \chi_i, \psi_1\rangle$ pertaining to each P_i are given in Table 1 and were computed in GAP.

Using the Frobenious Reciprocity theorem and the information in the last row of Table 1, it turns out that $\chi(G|N) = 12 \times 1a + 16 \times 6a + 8 \times 14a + 14 \times 14c + 5 \times 15b + 2 \times 21a + 21b + 5 \times 35b$. Evaluating $\chi(G|N)$ on a class representative $g \in G$ gives the number of elements of N which is fixed by $g \in G$. The method of coset-analysis [14] is used to compute the conjugacy classes of $\overline{G}$ coming from a coset Ng. The size of the centralizer $C_{\overline{G}}(x)$ of $x \in \overline{G}$ is given by $|C_{\overline{G}}(x)| = \frac{k|C_G(g)|}{f_j}$, where f_j is the number of orbits that fuse together under the action of $C_G(g)$ on the $\chi(G|N)(g) = k$ orbits of N on Ng. The constants f_j and orders of the classes $[x_j]$ of $\overline{G}$ corresponding to a class $[g]$ of G are computed with the Programme A in [4]. The information of the classes of $\overline{G}$ obtained by coset-analysis method is tabulated in Table 2.

Table 2: The Conjugacy Classes of $\overline{G} = 3^6{:}S_7$

$[g]_G$	k	f_j	m_j	$[x]_{\overline{G}}$	$\lvert C_{\overline{G}}(x)\rvert$	**2**	**3**	**5**	**7**	$\mapsto U_7(2)$	$[g]_G$	k	f_j	m_j	$[x]_{\overline{G}}$	$\lvert C_{\overline{G}}(x)\rvert$	**2**	**3**	**5**	**7**	$\mapsto U_7(2)$
1A	729	1	1	1A	3674160	1A	1A	1A	1A	1A	2A	243	1	3	2A	58320	1A	2A	2A	2A	2A

Continued on next page

Table 1: The inner product $\langle \chi_i, \psi_1 \rangle$

	χ_1	χ_2	χ_3	χ_4	χ_5	χ_6	χ_7	χ_8	χ_9	χ_{10}	χ_{11}	χ_{12}	χ_{13}	χ_{14}	χ_{15}
$\langle \chi_i, \psi_1 \rangle_{P_1}$	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-
$\langle \chi_i, \psi_1 \rangle_{P_.}$	1	0	2	0	1	0	0	2	0	1	0	0	0	0	1
$\langle \chi_i, \psi_1 \rangle_{P_3}$	1	0	2	0	1	0	0	2	0	1	0	0	0	0	1
$\langle \chi_i, \psi_1 \rangle_{P_.}$	1	0	1	0	0	0	0	1	0	0	0	0	0	-	-
$\langle \chi_i, \psi_1 \rangle_{P_5}$	1	0	2	0	2	0	0	3	0	1	0	1	1	0	2
$\langle \chi_i, \psi_1 \rangle_{P_.}$	1	0	1	0	0	0	0	1	0	0	0	0	0	-	-
$\langle \chi_i, \psi_1 \rangle_{P_7}$	1	0	1	0	1	0	0	1	0	0	0	0	0	0	0
$\langle \chi_i, \psi_1 \rangle_{P_.}$	1	0	2	0	2	0	0	2	0	1	0	1	0	0	1
$\langle \chi_i, \psi_1 \rangle_{P_9}$	1	0	1	0	1	0	0	1	0	0	0	0	0	0	0
$\langle \chi_i, \psi_1 \rangle_{P_.}$	1	0	1	0	-	-	-	-	-	-	-	-	-	-	-
$\langle \chi_i, \psi_1 \rangle_{P_{11}}$	1	0	1	0	-	-	-	-	-	-	-	-	-	-	-
$\langle \chi_i, \psi_1 \rangle_{P_.}$	1	0	2	0	0	0	0	1	0	1	0	0	0	0	0
	12	0	16	0	8	0	0	14	0	5	0	2	1	0	5

Table 2: The Conjugacy Classes of $\overline{G} = 3^6{:}S_7$ (Continued)

$[g]_G$	k	f_j	m_j	$[x]_{\overline{G}}$	$\lvert C_{\overline{G}}(x)\rvert$	**2**	**3**	**5**	**7**	$\mapsto U_7(2)$	$[g]_G$	k	f_j	m_j	$[x]_{\overline{G}}$	$\lvert C_{\overline{G}}(x)\rvert$	**2**	**3**	**5**	**7**	$\mapsto U_7(2)$
		7	7	3A	524880	3B	1A	3B	3A	3A			1	3	6A	58320	3D	2A	6B	6A	6C
		7	7	3B	524880	3A	1A	3A	3B	3B			1	3	6B	58320	3C	2A	6A	6B	6D
		21	21	3C	174960	3D	1A	3D	3C	3C			5	15	6C	11664	3B	2A	6D	6C	6B
		21	21	3D	174960	3C	1A	3C	3D	3D			5	15	6D	11664	3A	2A	6C	6D	6A
		35	35	3E	104976	3F	1A	3F	3E	3G			5	15	6E	11664	3H	2A	6G	6E	6N
		35	35	3F	104976	3E	1A	3E	3F	3F			5	15	6F	11664	3E	2A	6H	6F	6E
		42	42	3G	87480	3G	1A	3G	3G	3E			5	15	6G	11664	3I	2A	6E	6G	6O
		105	105	3H	34992	3I	1A	3I	3H	3H			5	15	6H	11664	3F	2A	6F	6H	6F
		105	105	3I	34992	3H	1A	3H	3I	3I			10	30	6I	5832	3D	2A	6J	6I	6G
		140	140	3J	26244	3J	1A	3J	3J	3J			10	30	6J	5832	3C	2A	6I	6J	6H
		210	210	3K	17496	3K	1A	3K	3K	3K			10	30	6K	5832	3K	2A	6N	6K	6AH
													10	30	6L	5832	3F	2A	6M	6L	6Q
													10	30	6M	5832	3E	2A	6L	6M	6P
													10	30	6N	5832	3K	2A	6K	6N	6AG
													30	90	6O	1944	3I	2A	6P	6O	6AD
													30	90	6P	1944	3H	2A	6O	6P	6AC
													30	90	6Q	1944	3K	2A	6Q	6Q	6AK
													20	60	6R	2916	3G	2A	6R	6R	6K
													20	60	6S	2916	3J	2A	6T	6S	6X
													20	60	6T	2916	3J	2A	6S	6T	6Y
2B	81	1	9	2B	3888	1A	2B	2B	2B	2B	2C	27	1	27	2C	1296	1A	2C	2C	2C	2C
		1	9	6U	3888	3F	2B	6V	6U	6W			1	27	6AN	1296	3A	2C	6AO	6AN	6R
		1	9	6V	3888	3E	2B	6U	6V	6V			1	27	6AO	1296	3B	2C	6AN	6AO	6S
		2	18	6W	1944	3C	2B	6X	6W	6L			3	81	6AP	432	3H	2C	6AS	6AP	6AU
		2	18	6X	1944	3D	2B	6W	6X	6M			3	81	6AQ	432	3C	2C	6AU	6AQ	6AF
		2	18	6Y	1944	3K	2B	6Y	6Y	6AL			3	81	6AR	432	3E	2C	6AT	6AR	6AN
		3	27	6Z	1296	3B	2B	6AA	6Z	6I			3	81	6AS	432	3I	2C	6AP	6AS	6AT
		3	27	6AA	1296	3A	2B	6Z	6AA	6J			3	81	6AT	432	3F	2C	6AR	6AT	6AM
		3	27	6AB	1296	3D	2B	6AC	6AB	6T			3	81	6AU	432	3D	2C	6AQ	6AU	6AE
		3	27	6AC	1296	3C	2B	6AB	6AC	6U			6	162	6AV	216	3K	2C	6AV	6AV	6AV
		3	27	6AD	1296	3I	2B	6AE	6AD	6AO											
		3	27	6AE	1296	3H	2B	6AD	6AE	6AQ											
		6	54	6AF	648	3H	2B	6AI	6AF	6AJ											
		6	54	6AG	648	3E	2B	6AH	6AG	6AA											
		6	54	6AH	648	3F	2B	6AG	6AH	6Z											
		6	54	6AI	648	3I	2B	6AF	6AI	6AI											
		6	54	6AJ	648	3G	2B	6AJ	6AJ	6AB											

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Table 2: The Conjugacy Classes of $\overline{G} = 3^6{:}S_7$ (Continued)

$[g]_G$	k	f_j	m_j	$[x]_{\overline{G}}$	$\lvert C_{\overline{G}}(x)\rvert$	**2**	**3**	**5**	**7**	$\mapsto U_7(2)$	$[g]_G$	k	f_j	m_j	$[x]_{\overline{G}}$	$\lvert C_{\overline{G}}(x)\rvert$	**2**	**3**	**5**	**7**	$\mapsto U_7(2)$
		6	54	6AK	648	3K	2B	6AL	6AK	6AR											
		6	54	6AL	648	3K	2B	6AK	6AL	6AS											
		12	108	6AM	324	3J	2B	6AM	6AM	6AP											
3A	81	1	9	3L	5832	3L	1A	3L	3L	3E	3B	9	1	81	3Q	162	3Q	1A	3Q	3Q	3K
		1	9	9A	5832	9B	3E	9B	9A	9B			1	81	9K	162	9L	3B	9L	9K	9H
		1	9	9B	5832	9A	3F	9A	9B	9A			1	81	9L	162	9K	3A	9K	9L	9G
		4	36	3M	1458	3N	1A	3N	3M	3I			2	162	9M	81	9O	3E	9O	9M	9L
		4	36	3N	1458	3M	1A	3M	3N	3H			2	162	9N	81	9N	3J	9N	9N	9M
		4	36	9C	1458	9F	3F	9F	9C	9C			2	162	9O	81	9M	3F	9M	9O	9K
		4	36	9D	1458	9E	3F	9E	9D	9D											
		4	36	9E	1458	9D	3E	9D	9E	9E											
		4	36	9F	1458	9C	3E	9C	9F	9F											
		6	54	3O	972	3O	1A	3O	3O	3J											
		6	54	9G	972	9H	3F	9H	9G	9I											
		6	54	9H	972	9G	3E	9G	9H	9J											
		12	108	3P	486	3P	1A	3P	3P	3K											
		12	108	9I	486	9J	3F	9J	9I	9K											
		12	108	9J	486	9I	3E	9I	9J	9L											
4A	9	1	81	4A	72	2B	4A	4A	4A	4F	4B	27	1	27	4B	648	2B	4B	4B	4B	4C
		1	81	12A	72	6AC	4A	12B	12A	12AX			1	27	12I	648	6U	4B	12J	12I	12W
		1	81	12B	72	6AB	4A	12A	12B	12AS			1	27	12J	648	6V	4B	12I	12J	12V
		1	81	12C	72	6AE	4A	12F	12C	12BE			3	81	12K	216	6AA	4B	12N	12K	12R
		1	81	12D	72	6U	4A	12H	12D	12AW			3	81	12L	216	6AB	4B	12P	12L	12AI
		1	81	12E	72	6Z	4A	12G	12E	12AF			3	81	12M	216	6AE	4B	12O	12M	12AZ
		1	81	12F	72	6AD	4A	12C	12F	12BF			3	81	12N	216	6Z	4B	12K	12N	12Q
		1	81	12G	72	6AA	4A	12E	12G	12AG			3	81	12O	216	6AD	4B	12M	12O	12BA
		1	81	12H	72	6V	4A	12D	12H	12AV			3	81	12P	216	6AC	4B	12L	12P	12AJ
													6	162	12Q	108	6AJ	4B	12Q	12Q	12AN
5A	9	1	81	5A	90	5A	5A	1A	5A	5A	6A	27	1	27	6AW	324	3L	2A	6AW	6AW	6K
		1	81	15A	90	15B	5A	3D	15A	15D			1	27	18A	324	9G	6L	18B	18A	18K
		1	81	15B	90	15A	5A	3C	15B	15E			1	27	18B	324	9H	6M	18A	18B	18J
		2	162	15C	45	15E	5A	3B	15C	15H			1	27	18C	324	9H	6M	18E	18C	18L
		2	162	15D	45	15D	5A	3G	15D	15I			1	27	6AX	324	3O	2A	6AY	6AX	6X
		2	162	15E	45	15C	5A	3A	15E	15C			1	27	18D	324	9B	6L	18F	18D	18A
													1	27	18E	324	9G	6L	18C	18E	18I
													1	27	18F	324	9A	6M	18D	18F	18B
													1	27	6AY	324	3O	2A	6AX	6AY	6Y
													2	54	18G	162	9C	6L	18H	18G	18E
													2	54	18H	162	9F	6M	18G	18H	18H
													2	54	18I	162	9J	6M	18J	18I	18N
													2	54	6AZ	162	3M	2A	6BA	6AZ	6AD
													2	54	6BA	162	3N	2A	6AZ	6BA	6AC
													2	54	18J	162	9I	6L	18I	18J	18M
													2	54	6BB	162	3P	2A	6BB	6BB	6AK
													2	54	18K	162	9D	6L	18L	18K	18F
													2	54	18L	162	9E	6M	18K	18L	18G
6B	9	1	81	6BC	216	3L	2B	6BC	6BC	6AB	6C	3	1	243	6BE	18	3Q	2C	6BE	6BE	6AV
		1	81	18M	216	9B	6U	18N	18M	18C			1	243	18Q	18	9L	6AN	18R	18Q	18Q
		1	81	18N	216	9A	6V	18M	18N	18D			1	243	18R	18	9K	6AO	18Q	18R	18R
		2	162	6BD	108	3O	2B	6BD	6BD	6AP											
		2	162	18O	108	9G	6U	18P	18O	18O											
		2	162	18P	108	9H	6V	18O	18P	18P											
7A	1	1	729	7A	7	7A	7A	7A	1A	7A	10A	3	1	243	10A	30	5A	10A	2A	10A	10A

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Table 2: The Conjugacy Classes of $\overline{G} = 3^6{:}S_7$ (Continued)

$[g]_G$	k	f_j	m_j	$[x]_{\overline{G}}$	$\lvert C_{\overline{G}}(x)\rvert$	**2**	**3**	**5**	**7**	$\mapsto U_7(2)$	$[g]_G$	k	f_j	m_j	$[x]_{\overline{G}}$	$\lvert C_{\overline{G}}(x)\rvert$	**2**	**3**	**5**	**7**	$\mapsto U_7(2)$
													1	243	30A	30	15A	10A	6A	30A	30D
													1	243	30B	30	15B	10A	6B	30B	30E
12A	3	1	243	12R	36	6BC	4B	12R	12R	12AN											
		1	243	36A	36	18N	12J	36B	36A	36H											
		1	243	36B	36	18M	12I	36A	36B	36G											

3. The Inertia Factor Groups of $\overline{G}$

By Brauer's theorem [8], the number of orbits of G on N is the same as the number of orbits of G on $\mathrm{Irr}(N)$. Hence it follows from Section 2 that G has also 12 orbits on $\mathrm{Irr}(N)$. The Brauer character table modulo 3 of G has two irreducible characters of degree six and hence two six-dimensional G-modules over $GF(3)$ exist which are isomorphic to each other. Moreover, N and its dual $N^* \cong \mathrm{Irr}(N)$ are equivalent as G- modules. Thus the action of G on N is identified with the action of G on $\mathrm{Irr}(N)$. Therefore, the orbit lengths and corresponding orbit stabilizers of the actions of G on N and $\mathrm{Irr}(N)$ coincide. The stabilizers of the action of G on $\mathrm{Irr}(N)$, called the inertia factors H_i, have the structures $H_1 = S_7$, $H_2 = H_3 = C_2 \times S_4$, $H_4 = H_6 = C_2 \times S_5$, $H_5 = 2^2 \times S_3$, $H_7 = S_4 \times S_3$, $H_8 = S_3 \times S_3$, $H_9 = S_3 \times S_4$, $H_{10} = H_{11} = S_6$ and $H_{12} = S_5$. The computation and identification of the aforementioned inertia factor groups were carried out with the assistance of GAP, and where they are shown to be equal, this was also confirmed within GAP. The fusion maps of the conjugacy classes of the inertia factors H_i into the classes of G are computed using the GAP function "FusionConjugacyClasses(H_i, G)" and listed in Tables 3 and 4.

Table 3: The fusion of $H_2 = H_3$, $H_4 = H_6$, H_5 and H_7 into $G = S_7$

$[h]_{H_2}$	$\longrightarrow [g]_{S_7}$	$[h]_{H_4}$	$\longrightarrow [g]_{S_7}$	$[h]_{H_5}$	$\longrightarrow [g]_{S_7}$	$[h]_{H_7}$	$\longrightarrow [g]_{S_7}$
$1A$	$1A$	$1A$	$1A$	$1A$	$1A$	$1A$	$1A$
$2A$	$2C$	$2A$	$2C$	$2A$	$2A$	$2A$	$2B$
$2B$	$2A$	$2B$	$2B$	$2B$	$2B$	$2B$	$2A$
$2C$	$2B$	$2C$	$2A$	$2C$	$2A$	$2C$	$2C$
$2D$	$2B$	$2D$	$2B$	$2D$	$2C$	$2D$	$2B$
$2E$	$2A$	$2E$	$2A$	$2E$	$2B$	$2E$	$2A$
$3A$	$3A$	$3A$	$3A$	$2F$	$2B$	$3A$	$3B$
$4A$	$4B$	$4A$	$4B$	$2G$	$2A$	$3B$	$3A$
$4B$	$4A$	$4B$	$4A$	$3A$	$3A$	$3C$	$3A$
$6A$	$6A$	$5A$	$5A$	$6A$	$6A$	$4A$	$4A$
		$6A$	$6A$	$6B$	$6B$	$4B$	$4B$
		$6B$	$6B$	$6C$	$6A$	$6A$	$6A$
		$6C$	$6A$			$6B$	$6B$
		$10A$	$10A$			$6C$	$6A$
						$12A$	$12A$

Table 4: The fusion of H_8, H_9, $H_{10} = H_{11}$ and H_{12} into $G = S_7$

$[h]_{H_8}$	$\longrightarrow [g]_{S_7}$	$[h]_{H_9}$	$\longrightarrow [g]_{S_7}$	$[h]_{H_{10}}$	$\longrightarrow [g]_{S_7}$	$[h]_{H_{12}}$	$\longrightarrow [g]_{S_7}$
$1A$	$1A$	$1A$	$1A$	$1A$	$1A$	$1A$	$1A$
$2A$	$2A$	$2A$	$2A$	$2A$	$2A$	$2A$	$2B$
$2B$	$2B$	$2B$	$2A$	$2B$	$2B$	$2B$	$2A$
$2C$	$2A$	$2C$	$2B$	$2C$	$2C$	$3A$	$3A$
$3A$	$3B$	$2D$	$2B$	$3A$	$3A$	$4A$	$4B$
$3B$	$3A$	$2E$	$2C$	$3B$	$3B$	$5A$	$5A$
$3C$	$3A$	$3A$	$3A$	$4A$	$4A$	$6A$	$6A$
$6A$	$6A$	$3B$	$3A$	$4B$	$4B$		
$6B$	$6A$	$3C$	$3B$	$5A$	$5A$		
		$4A$	$4A$	$6A$	$6A$		
		$6A$	$6A$	$6B$	$6C$		
		$6B$	$6A$				
		$6C$	$6B$				
		$12A$	$12A$				
		$4B$	$4B$				

4. Theory of Fischer-Clifford matrices for $\overline{G}$

In this section, a theoretical outline is given on the application of the Fischer-Clifford matrices technique to $\overline{G} = 3^6{:}S_7$. For a general and more detailed theory of Fischer-Clifford matrices, readers are referred to [2, 6, 12, 13, 14, 16, 17] and [18].

The action of $\overline{G}$ on $\mathrm{Irr}(N)$ gives rise to 12 orbits $\mathrm{O}_1, \mathrm{O}_2, \ldots, \mathrm{O}_{12}$ with lengths 1, 105, 105, 21, 210, 21, 35, 140, 35, 7, 7, 42 and corresponding inertia groups $\overline{H_1} = 3^6{:}S_7$, $\overline{H_2} = \overline{H_3} = 3^6{:}C_2 \times S_4$, $\overline{H_4} = \overline{H_6} = 3^6{:}C_2 \times S_5$, $\overline{H_5} = 3^6{:}2^2 \times S_3$, $\overline{H_7} = 3^6{:}S_4 \times S_3$, $\overline{H_8} = 3^6{:}S_3 \times S_3$, $\overline{H_9} = 3^6{:}S_3 \times S_4$, $\overline{H_{10}} = \overline{H_{11}} = 3^6{:}S_6$, $\overline{H_{12}} = 3^6{:}S_5$ respectively. The computation and identification of the above inertia groups were carried out using GAP, and where equalities are indicated, they were directly confirmed within GAP. The inertia groups are defined as $\overline{H_i} = N{:}H_i = \left\{x \in \overline{G} | \theta_i^x = \theta_i\right\}$, $i = 1, 2, \ldots, 12$, where $\theta_i \in \mathrm{O}_i$ are representatives of the orbits O_i. Since $N = 3^6$ is elementary abelian, each θ_i extends to a $\psi_i \in \mathrm{Irr}(\overline{H_i})$, i.e. $\psi_i \downarrow_N = \theta_i$, by Mackey's Theorem (see Theorem 5.1.15 in [14]). Also, using Theorem 5.1.7, Remark 5.1.8 and Theorem 5.1.19 in [14], an ordinary irreducible character $\chi = (\psi_i\overline{\beta})^{\overline{G}}$ of $\overline{G}$ is obtained by induction of $\psi_i\overline{\beta} \in \mathrm{Irr}(\overline{H_i})$ to $\overline{G}$, where N is contained in the kernel $\ker(\overline{\beta})$ of $\overline{\beta} \in \mathrm{Irr}(\overline{H_i})$. Note that $\overline{\beta} \in \mathrm{Irr}(\overline{H_i})$ is a lifting of $\beta \in \mathrm{Irr}(H_i)$ to $\overline{H_i}$. Therefore,

$$\mathrm{Irr}(\overline{G}) = \bigcup_{i=1}^{12} \{(\psi_i\overline{\beta})^{\overline{G}} | \overline{\beta} \in \mathrm{Irr}(\overline{H_i}), N \subseteq \ker(\beta)\} = \bigcup_{i=1}^{12} \{(\psi_i\overline{\beta})^{\overline{G}} | \beta \in \mathrm{Irr}(H_i)\}.$$

It turns out that the irreducible characters of $\overline{G}$ will be divided into 12 blocks corresponding to the 12 inertia groups given earlier. Note that $\overline{H_1}$ is taken to be equal to $\overline{G}$ and and the corresponding inertia factor group H_1 to be equal to G. The set

$$R(g) = \{(i, y_k) \,|\, 1 \leq i \leq 12, H_i \cap [g] \neq \emptyset, 1 \leq k \leq r\},$$

is defined where y_k, $k = 1, 2, \ldots, r$, are class representatives of conjugacy classes $[y_k]$ of H_i that fuse into a class $[g]$ of $H_1 = G$. Let y_{l_k} be representatives of the conjugacy classes of $\overline{H_i}$, where each y_{l_k} has y_k as an image under the homomorphism $\overline{H_i} \longrightarrow H_i$ whose kernel is 3^6 and the set $X(g) = \{x_1, x_2, \ldots, x_{c(g)}\}$ contain the class representatives of $\overline{G}$ coming from Ng. Then for $x_j \in X(g)$, we have

Lemma 4.1.

$$(\psi_i\overline{\beta})^{\overline{G}}(x_j) = \sum_{y_k:(i,y_k)\in R(g)} \left[\sum_{l}{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|} \psi_i(y_{l_k}) \right] \beta(y_k).$$

Proof. See [20]. □

The Fischer-Clifford matrix $M(g) = \left(a^j_{(i,y_k)}\right)$ is then defined as

$$\left(a^j_{(i,y_k)}\right) = \left(\sum_{l}{}' \frac{|C_{\overline{G}}(x_j)|}{|C_{\overline{H_i}}(y_{l_k})|} \psi_i(y_{l_k}) \right),$$

with columns indexed by $X(g)$ and rows indexed by $R(g)$ and where $\sum_l'$ is the summation over all l for which y_{l_k} is conjugate to x_j in $\overline{G}$. A Fischer-Clifford matrix $M(g)$ of $\overline{G}$ has the shape as depicted in Figure 2.

$$
\begin{array}{rc}
 & \begin{array}{cccc} |C_{\overline{G}}(x_1)| & |C_{\overline{G}}(x_2)| & \cdots & |C_{\overline{G}}(x_{c(g)})| \end{array} \\
\text{M(g)=} \begin{array}{c} |C_G(g)| \\ \\ |C_{H_2}(y_1)| \\ |C_{H_2}(y_2)| \\ \vdots \\ \\ |C_{H_i}(y_1)| \\ |C_{H_i}(y_2)| \\ \vdots \\ \\ |C_{H_{12}}(y_1)| \\ |C_{H_{12}}(y_2)| \\ \vdots \\ \\ \end{array} &
\left(\begin{array}{cccc}
1 & 1 & \cdots & 1 \\
\hline
\frac{|C_G(g)|}{|C_{H_2}(y_1)|} & a^2_{(2,y_1)} & \cdots & a^{c(g)}_{(2,y_1)} \\
\frac{|C_G(g)|}{|C_{H_2}(y_2)|} & a^2_{(2,y_2)} & \cdots & a^{c(g)}_{(2,y_2)} \\
\vdots & \vdots & \vdots & \vdots \\
\hline
\frac{|C_G(g)|}{|C_{H_i}(y_1)|} & a^2_{(i,y_1)} & \cdots & a^{c(g)}_{(i,y_1)} \\
\frac{|C_G(g)|}{|C_{H_i}(y_2)|} & a^2_{(i,y_2)} & \cdots & a^{c(g)}_{(i,y_2)} \\
\vdots & \vdots & \vdots & \vdots \\
\hline
\frac{|C_G(g)|}{|C_{H_{12}}(y_1)|} & a^2_{(12,y_1)} & \cdots & a^{c(g)}_{(12,y_1)} \\
\frac{|C_G(g)|}{|C_{H_{12}}(y_2)|} & a^2_{(12,y_2)} & \cdots & a^{c(g)}_{(12,y_2)} \\
\vdots & \vdots & \vdots & \vdots \\
\hline
\end{array}\right)
\end{array}
$$

Figure 2: The Fischer-Clifford Matrix $M(g)$

The Fischer-Clifford matrix $M(g)$ in Figure 2 has its rows partitioned into blocks $M_i(g)$, such that each block corresponds to an inertia group $\overline{H}_i$. If there is is no class fusion from an inertia factor H_i to $[g]$, then the block $M_i(g)$ is omitted from $M(g)$. The centralizer size $|C_{\overline{G}}(x_j)|$, for every $x_j \in X(g)$, is written at the top of the columns of $M(g)$ while on the left of each row we write $|C_{H_i}(y_k)|$, where the conjugacy classes $[y_k]$, $k = 1, 2, ..., r$, of an inertia factor H_i fuse into the conjugacy class $[g]$ of G. Since $|X(g)| = |R(g)|$ it follows that $M(g)$ is a square matrix of size $c(g)$. The properties of $M(g)$ which may be found in for instance [14], are usually used to determine the entries of $M(g)$. However, in this paper, we use a combinatorial approach described in sections 5 and 6 to determine the entries of the Fischer-Clifford matrices.

5. Combinatorics

The ordinary irreducible representations of the symmetric group S_n and extensions of symmetric groups (generalized symmetric groups) may be described combinatorially using partitions of a positive integer n (see [5]). In this section, a recap of the partition of a positive integer n called an m-set of partition $[\lambda]$ of n is given. These results will play a very critical role in determination of entries of Fischer-Clifford matrices using Theorem 6.1 in Section 6.

Proposition 5.1 ([19]). *An $m-$tuple $(a_1, a_2, \ldots, a_m)$ such that $\sum_{j=1}^{m} a_j = n$, where $a_j \in \mathbb{N} \cup \{0\}$ is called an $m-$composition of n and the number of $m-$compositions of n is given by*

$$\binom{n+m-1}{m-1}.$$

Proof. See Proposition 2.1.2 in [22]. □

The set of $m-$compositions of n is going to be denoted by $A(n, m)$.

Definition 5.2. *For $n \in \mathbb{N}$, $[\lambda] = [1_1^{\lambda} 2_2^{\lambda} \cdots n_n^{\lambda}]$ is referred to as a partition of n if $\sum_{s=1}^{n} s\lambda_n = n$, where $\lambda_s \in \mathbb{N} \cup \{0\}$.*

Definition 5.3. *Let $[\lambda]$ be defined as above and $\overline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the n-tuple of powers of $[\lambda]$.*

i) $\alpha = ((a_{11}, a_{12}, \ldots, a_{1n}), (a_{21}, a_{22}, \ldots, a_{2n}), \ldots, (a_{m1}, a_{m2}, \ldots, a_{mn}))$, *such that* $\sum_{i=1}^{m} a_{is} = \lambda_s$ *is called a partition* $[\lambda]$. *for computation purposes,* α *may be expressed in the following form*

$$\alpha = a_{is} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ & \vdots & & \\ a_{m1} & a_{m2} & \cdots & a_{mn}. \end{pmatrix}$$

ii) α *is said to be a type-zero* $m-$*set of partition* $[\lambda]$ *if* $\sum_{i=1}^{m}\sum_{s=1}^{n}(i-1)a_{is} \equiv 0 (mod\ m)$.

iii) $\beta = ((b_{11}, b_{12}, \cdots, b_{1n}), (b_{21}, b_{22}, \cdots, b_{2n}), \cdots, (b_{m1}, b_{m2}, \cdots, b_{mn}))$ *be an* $m-$*set of partition* $[\lambda]$. *Then* α *and* β *are said to be equivalent if* β *can be obtained by cyclically re-arranging the rows of* α. *The number of* $m-$*sets of partition* $[\lambda]$ *equivalent to* α *is* m.

iv) *If* $\lambda_s = 0$ *and the lengths* s *are clear, we will write the* m*-sets of partition* $[\lambda]$ *omitting the zero columns to which such* s *refers.*

The GAP routine, Programme 2.2.3 in [22], is used to compute the m-sets for all partitions of n.

6. The Fischer-Clifford matrices of $\overline{G}$

It is important to note that most of the studies on application of Fischer-Clifford theory to group extensions make use of the properties Fischer matrices to determine their entries. In this section, a combinatorial approach is adopted to compute the entries of the Fischer-Clifford matrices of $B(3,7)$ from which the Fischer-Clifford matrices of $B_S(3,7)$ are obtained. The results used in this chapter are found in [1] and [11].

Theorem 6.1 ([1]). *The entries of the Fischer-Clifford matrix* $F(m, 1^n)$, *which are in the column labeled* $(1^{a_1}2^{a_2}\cdots m^{a_m})$ *and rows labeled* $(1^{k_1}2^{k_2}\cdots m^{k_m})$ *where* $(k_1, k_2, \cdots, k_m) \in A(n, m)$ *are given by the coefficients of* $x_1^{k_1}x_2^{k_2}\cdots x_m^{k_m}$ *in the expansion*

$$\prod_{i=1}^{m}(\sum_{j=1}^{m}\xi^{(i-1)(j-1)}x_j)^{a_i}.$$

Proof. See Theorem 4.3.2 in [22] □

Theorem 6.2. *For a conjugacy class of the group* S_n *of the form* $(1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n})$. *The matrix*

$$(F'(m, 1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n}) = \otimes_{s=n}^{1}F(m, 1^{\lambda_s}),$$

where $\otimes$ *is the tensor product over* s *is equivalent to the Fischer-Clifford matrix* $(F(m, 1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n})$ *of* $B(m,n)$ *at* $[g]$.

Proof. See Theorem 4.3.5 in [22] □

The above two theorems are used in determining the entries of a Fischer-Clifford matrix of the generalized symmetric group $B(m,n)$. Furthermore, using Theorem 7.1.6 and Proposition 7.1.12 in [22], it is evident that a type-zero p-set of partition $[\lambda]$ indexes a column of the Fischer-Clifford matrix $F_S(p, 1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n})$ of $B_S(m,n)$ which is obtained from $F(p, 1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n})$ of $B(m,n)$. A method of extracting a Fischer-Clifford matrix of $B_S(m,n)$ from corresponding matrices of $B(m,n)$ is given by the following proposition.

Proposition 6.3. *Suppose* g *is a conjugacy class representative of* S_n *of type* $(1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n})$, *then a Fischer-Clifford matrix* $F_S(p, 1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n})$ *of* $B_S(m,n)$ *is obtained from that of* $B(m,n)$ *by;*

i) *Selecting the columns of* $F(p, 1^{\lambda_1}2^{\lambda_2}\cdots n^{\lambda_n})$ *indexed by type-zero columns.*

ii) *Selecting the rows (of the resulting matrix in (i)) corresponding to the representatives of sets of equivalent* m*-sets of partition* $[\lambda]$ *(also referred to as permissible rows).*

iii) *Having the rows of the resulting matrix in (ii) above that correspond to self-equivalent* $p-$*sets of partition* $[\lambda]$ *divided by* p.

For finer details on combinatorial treatment of Fischer-Clifford matrices, interested readers are referred to [1, 10, 11] and [22]. We now illustrate how to determine the Fischer-Clifford matrices of $B_S(3,7)$ from those of $B(3,7)$ using Theorems 6.1, 6.2 and Proposition 6.3. For $M(1A)$ of $B(3,7)$ which is hereby denoted by $F(3,1^7)$, $\lambda_1 = 7$. The thirty six 3-sets of partition $[1^7]$ denoting the columns of $F(3,1^7)$ are as in Figure 3. This implies that $F(3,1^7)$ has order 36.

$$\left\{ \begin{array}{c} \begin{pmatrix}7\\0\\0\end{pmatrix}, \begin{pmatrix}6\\1\\0\end{pmatrix}, \begin{pmatrix}6\\0\\1\end{pmatrix}, \begin{pmatrix}5\\2\\0\end{pmatrix}, \begin{pmatrix}5\\1\\1\end{pmatrix}, \begin{pmatrix}5\\0\\2\end{pmatrix}, \begin{pmatrix}4\\3\\0\end{pmatrix}, \begin{pmatrix}4\\2\\1\end{pmatrix}, \begin{pmatrix}4\\1\\2\end{pmatrix}, \begin{pmatrix}4\\0\\3\end{pmatrix}, \begin{pmatrix}3\\4\\0\end{pmatrix}, \begin{pmatrix}3\\3\\1\end{pmatrix} \\ \begin{pmatrix}3\\2\\2\end{pmatrix}, \begin{pmatrix}3\\1\\3\end{pmatrix}, \begin{pmatrix}3\\0\\4\end{pmatrix}, \begin{pmatrix}2\\5\\0\end{pmatrix}, \begin{pmatrix}2\\4\\1\end{pmatrix}, \begin{pmatrix}2\\3\\2\end{pmatrix}, \begin{pmatrix}2\\2\\3\end{pmatrix}, \begin{pmatrix}2\\1\\4\end{pmatrix}, \begin{pmatrix}2\\0\\5\end{pmatrix}, \begin{pmatrix}1\\6\\0\end{pmatrix}, \begin{pmatrix}1\\5\\1\end{pmatrix}, \begin{pmatrix}1\\4\\2\end{pmatrix} \\ \begin{pmatrix}1\\3\\3\end{pmatrix}, \begin{pmatrix}1\\2\\4\end{pmatrix}, \begin{pmatrix}1\\1\\5\end{pmatrix}, \begin{pmatrix}1\\0\\6\end{pmatrix}, \begin{pmatrix}0\\7\\0\end{pmatrix}, \begin{pmatrix}0\\6\\1\end{pmatrix}, \begin{pmatrix}0\\5\\2\end{pmatrix}, \begin{pmatrix}0\\4\\3\end{pmatrix}, \begin{pmatrix}0\\3\\4\end{pmatrix}, \begin{pmatrix}0\\2\\5\end{pmatrix}, \begin{pmatrix}0\\1\\6\end{pmatrix}, \begin{pmatrix}0\\0\\7\end{pmatrix} \end{array} \right\}$$

Figure 3: The 3-sets of partition $[1^7]$

Now by Theorem 6.1 and letting $p_1 = x+y+z$, $p_2 = x+\xi y+\xi^2 z$ and $p_3 = x+\xi^2 y+\xi^4 z$, where $x = x_1$, $y = x_2$, $z = x_3$ and ξ is the primitive third root of unity, we expand $p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3}$, where $\alpha_1 = 7$, $\alpha_2 = 0$ and $\alpha_3 = 0$ to get,

$$\begin{aligned} p_1^7 p_2^0 p_3^0 = {} & x^7 + 7x^6y + 7x^6z + 21x^5y^2 + 42x^5yz + 21x^5z^2 + 35x^4y^3 + 105x^4y^2z \\ & + 105x^4yz^2 + 35x^4z^3 + 35x^3y^4 + 140x^3y^3z + 210x^3y^2z^2 + 140x^3yz^3 \\ & + 35x^3z^4 + 21x^2y^5 + 105x^2y^4z + 210x^2y^3z^2 + 210x^2y^2z^3 + 105x^2yz^4 \\ & + 21x^2z^5 + 7xy^6 + 42xy^5z + 105xy^4z^2 + 140xy^3z^3 + 105xy^2z^4 + 42xyz^5 \\ & + 7xz^6 + y^7 + 7y^6z + 21y^5z^2 + 35y^4z^3 + 35y^3z^4 + 21y^2z^5 + 7yz^6 + z^7. \end{aligned}$$

The coefficients for the expansion $p_1^7 p_2^0 p_3^0$ are

$$\begin{gathered} 1, 7, 7, 21, 42, 21, 35, 105, 105, 35, 35, 140, 210, 140, 35, 21, 105, 210, 210, 105, \\ 21, 7, 42, 105, 140, 105, 42, 7, 1, 7, 21, 35, 35, 21, 7, 1, \end{gathered}$$

which form the first column of $F(3,1^7)$. Similarly for $\alpha_1 = 6$, $\alpha_2 = 1$ and $\alpha_3 = 0$ we have,

$$\begin{aligned} p_1^6 p_2^1 p_3^0 = {} & x^7 + (-5\xi - 6\xi^2)x^6y + (-6\xi - 5\xi^2)x^6z + (-9\xi - 15\xi^2)x^5y^2 + 24x^5yz \\ & + (-15\xi - 9\xi^2)x^5z^2 + (-5\xi - 20\xi^2)x^4y^3 + (-30\xi - 45\xi^2)x^4y^2z + (-45\xi - 30\xi^2)x^4yz^2 \\ & + (-20\xi - 5\xi^2)x^4z^3 + (5\xi - 15\xi^2)x^3y^4 - 40\xi^2x^3y^3z + 30x^3y^2z^2 - 40\xi x^3yz^3 \\ & + (-15\xi + 5\xi^2)x^3z^4 + (9\xi - 6\xi^2)x^2y^5 + (30\xi - 15\xi^2)x^2y^4z + 30\xi x^2y^3z^2 \\ & + 30\xi^2x^2y^2z^3 + (-15\xi + 30\xi^2)x^2yz^4 + (-6\xi + 9\xi^2)x^2z^5 + (5\xi - \xi^2)xy^6 \\ & + 24\xi xy^5z + (45\xi + 15\xi^2)xy^4z^2 - 40xy^3z^3 + (15\xi + 45\xi^2)xy^2z^4 + 24\xi^2xyz^5 \\ & + (-\xi + 5\xi^2)xz^6 + \xi y^7 + (6\xi + \xi^2)y^6z + (15\xi + 6\xi^2)y^5z^2 + (20\xi + 15\xi^2)y^4z^3 \\ & + (15\xi + 20\xi^2)y^3z^4 + (6\xi + 15\xi^2)y^2z^5 + (\xi + 6\xi^2)yz^6 + \xi^2z^7, \end{aligned}$$

giving column 2 entries of $F(3,1^7)$ as

$$\begin{gathered} 1, -5\xi - 6\xi^2, -6\xi - 5\xi^2, -9\xi - 15\xi^2, 24, -15\xi - 9\xi^2, -5\xi - 20\xi^2, -30\xi - 45\xi^2, -45\xi - 30\xi^2, \\ -20\xi - 5\xi^2, 5\xi - 15\xi^2, -40\xi^2, 30, -40\xi, -15\xi + 5\xi^2, 9\xi - 6\xi^2, 30\xi - 15\xi^2, 30\xi, 30\xi^2, -15\xi + 30\xi^2, \\ -6\xi + 9\xi^2, 5\xi - \xi^2, 24\xi, 45\xi + 15\xi^2, -40, 15\xi + 45\xi^2, 24\xi^2, -\xi + 5\xi^2, \xi, 6\xi + \xi^2, 15\xi + 6\xi^2, \\ 20\xi + 15\xi^2, 15\xi + 20\xi^2, 6\xi + 15\xi^2, \xi + 6\xi^2, \xi^2. \end{gathered}$$

For purposes of implementing such expansions, specifically for this group, the following GAP sub-routine is used. The coefficients are then extracted from the resulting expansion taking into consideration the cases where the coefficient of the variables is zero.

```
gap> x:=X(Rationals,"x");;
gap> y:=X(Rationals,"y");;
gap> z:=X(Rationals,"z");;
gap> p1:=(x+y+z);;
gap> p2:=x+E(3)*y+E(3)^2*z;;$
gap> p3:=x+E(3)^2*y+E(3)^4*z;;$
gap> p:=((p1)^6)*((p2)^1)*((p3)^0);;$
gap> coeffs:=function(p,ord)  local cs,c,m;
> cs:=[];
>     while not(IsZero(p)) do
>       m:=LeadingMonomialOfPolynomial(p,ord);
>         c:=LeadingCoefficientOfPolynomial(p,ord);
>          Add(cs,c);
>         p:=p-c*m;
>      od;
>    return cs;
>  end;
function( p, ord ) ... end
gap> coeffs(p,MonomialLexOrdering());
```

Similarly, for the remaining 3-sets of partition $[1^7]$, the other sets of coefficients are obtained giving rise to the Fischer-Clifford matrix $F(3,1^7)$ shown below. Note that this matrix has been split severally with a $|$ to fit in a4 paper. The primitive third root of unity is denoted by α in our matrices.

$$\left(\begin{array}{rrrrrrrrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
7 & -5\alpha-6\overline{\alpha} & -6\alpha-5\overline{\alpha} & -3\alpha-5\overline{\alpha} & 4 & -5\alpha-3\overline{\alpha} & -\alpha-4\overline{\alpha} & -2\alpha-3\overline{\alpha} & -3\alpha-2\overline{\alpha} \\
7 & -6\alpha-5\overline{\alpha} & -5\alpha-6\overline{\alpha} & -5\alpha-3\overline{\alpha} & 4 & -3\alpha-5\overline{\alpha} & -4\alpha-\overline{\alpha} & -3\alpha-2\overline{\alpha} & -2\alpha-3\overline{\alpha} \\
21 & -9\alpha-15\overline{\alpha} & -15\alpha-9\overline{\alpha} & -9\overline{\alpha} & 6 & -9\alpha & 6\alpha-3\overline{\alpha} & -3\overline{\alpha} & -3\alpha \\
42 & 24 & 24 & 12 & 9 & 12 & 6 & 0 & 0 \\
21 & -15\alpha-9\overline{\alpha} & -9\alpha-15\overline{\alpha} & -9\alpha & 6 & -9\overline{\alpha} & -3\alpha+6\overline{\alpha} & -3\alpha & -3\overline{\alpha} \\
35 & -5\alpha-20\overline{\alpha} & -20\alpha-5\overline{\alpha} & 10\alpha-5\overline{\alpha} & 5 & -5\alpha+10\overline{\alpha} & 13\alpha+7\overline{\alpha} & \alpha-2\overline{\alpha} & -2\alpha+\overline{\alpha} \\
105 & -30\alpha-45\overline{\alpha} & -45\alpha-30\overline{\alpha} & -15\overline{\alpha} & 0 & -15\alpha & 3\alpha-6\overline{\alpha} & -9 & -9 \\
105 & -45\alpha-30\overline{\alpha} & -30\alpha-45\overline{\alpha} & -15\alpha & 0 & -15\overline{\alpha} & -6\alpha+3\overline{\alpha} & -9 & -9 \\
35 & -20\alpha-5\overline{\alpha} & -5\alpha-20\overline{\alpha} & -5\alpha+10\overline{\alpha} & 5 & 10\alpha-5\overline{\alpha} & 7\alpha+13\overline{\alpha} & -2\alpha+\overline{\alpha} & \alpha-2\overline{\alpha} \\
35 & 5\alpha-15\overline{\alpha} & -15\alpha+5\overline{\alpha} & 15\alpha+5\overline{\alpha} & 5 & 5\alpha+15\overline{\alpha} & 7\alpha+13\overline{\alpha} & -\alpha-3\overline{\alpha} & -3\alpha-\overline{\alpha} \\
140 & -40\overline{\alpha} & -40\alpha & 20\alpha & -10 & 20\overline{\alpha} & -4 & 8\overline{\alpha} & 8\alpha \\
210 & 30 & 30 & 0 & -30 & 0 & 12 & -12 & -12 \\
140 & -40\alpha & -40\overline{\alpha} & 20\overline{\alpha} & -10 & 20\alpha & -4 & 8\alpha & 8\overline{\alpha} \\
35 & -15\alpha+5\overline{\alpha} & 5\alpha-15\overline{\alpha} & 5\alpha+15\overline{\alpha} & 5 & 15\alpha+5\overline{\alpha} & 13\alpha+7\overline{\alpha} & -3\alpha-\overline{\alpha} & -\alpha-3\overline{\alpha} \\
21 & 9\alpha-6\overline{\alpha} & -6\alpha+9\overline{\alpha} & -9 & 6 & -9 & -3\alpha+6\overline{\alpha} & -3\overline{\alpha} & -3\alpha \\
105 & 30\alpha-15\overline{\alpha} & -15\alpha+30\overline{\alpha} & -15 & 0 & -15 & -6\alpha+3\overline{\alpha} & -9\alpha & -9\overline{\alpha} \\
210 & 30\alpha & 30\overline{\alpha} & 0 & -30 & 0 & 12 & -12\alpha & -12\overline{\alpha} \\
210 & 30\overline{\alpha} & 30\alpha & 0 & -30 & 0 & 12 & -12\overline{\alpha} & -12\alpha \\
105 & -15\alpha+30\overline{\alpha} & 30\alpha-15\overline{\alpha} & -15 & 0 & -15 & 3\alpha-6\overline{\alpha} & -9\overline{\alpha} & -9\alpha \\
21 & -6\alpha+9\overline{\alpha} & 9\alpha-6\overline{\alpha} & -9 & 6 & -9 & 6\alpha-3\overline{\alpha} & -3\alpha & -3\overline{\alpha} \\
7 & 5\alpha-\overline{\alpha} & -\alpha+5\overline{\alpha} & 2\alpha+5\overline{\alpha} & 4 & 5\alpha+2\overline{\alpha} & -4\alpha-\overline{\alpha} & 2\alpha-\overline{\alpha} & -\alpha+2\overline{\alpha} \\
42 & 24\alpha & 24\overline{\alpha} & 12\overline{\alpha} & 9 & 12\alpha & 6 & 0 & 0 \\
105 & 45\alpha+15\overline{\alpha} & 15\alpha+45\overline{\alpha} & -15\alpha & 0 & -15\overline{\alpha} & 3\alpha-6\overline{\alpha} & -9\alpha & -9\overline{\alpha} \\
140 & -40 & -40 & 20 & -10 & 20 & -4 & 8 & 8 \\
105 & 15\alpha+45\overline{\alpha} & 45\alpha+15\overline{\alpha} & -15\overline{\alpha} & 0 & -15\alpha & -6\alpha+3\overline{\alpha} & -9\overline{\alpha} & -9\alpha \\
42 & 24\overline{\alpha} & 24\alpha & 12\alpha & 9 & 12\overline{\alpha} & 6 & 0 & 0 \\
7 & -\alpha+5\overline{\alpha} & 5\alpha-\overline{\alpha} & 5\alpha+2\overline{\alpha} & 4 & 2\alpha+5\overline{\alpha} & -\alpha-4\overline{\alpha} & -\alpha+2\overline{\alpha} & 2\alpha-\overline{\alpha} \\
1 & \alpha & \overline{\alpha} & \overline{\alpha} & 1 & \alpha & 1 & \alpha & \overline{\alpha} \\
7 & 6\alpha+\overline{\alpha} & \alpha+6\overline{\alpha} & -2\alpha+3\overline{\alpha} & 4 & 3\alpha-2\overline{\alpha} & -\alpha-4\overline{\alpha} & 3\alpha+\overline{\alpha} & \alpha+3\overline{\alpha} \\
21 & 15\alpha+6\overline{\alpha} & 6\alpha+15\overline{\alpha} & -9\alpha & 6 & -9\overline{\alpha} & 6\alpha-3\overline{\alpha} & -3 & -3 \\
35 & 20\alpha+15\overline{\alpha} & 15\alpha+20\overline{\alpha} & -15\alpha-10\overline{\alpha} & 5 & -10\alpha-15\overline{\alpha} & 13\alpha+7\overline{\alpha} & 2\alpha+3\overline{\alpha} & 3\alpha+2\overline{\alpha} \\
35 & 15\alpha+20\overline{\alpha} & 20\alpha+15\overline{\alpha} & -10\alpha-15\overline{\alpha} & 5 & -15\alpha-10\overline{\alpha} & 7\alpha+13\overline{\alpha} & 3\alpha+2\overline{\alpha} & 2\alpha+3\overline{\alpha} \\
21 & 6\alpha+15\overline{\alpha} & 15\alpha+6\overline{\alpha} & -9\overline{\alpha} & 6 & -9\alpha & -3\alpha+6\overline{\alpha} & -3 & -3 \\
7 & \alpha+6\overline{\alpha} & 6\alpha+\overline{\alpha} & 3\alpha-2\overline{\alpha} & 4 & -2\alpha+3\overline{\alpha} & -4\alpha-\overline{\alpha} & \alpha+3\overline{\alpha} & 3\alpha+\overline{\alpha} \\
1 & \overline{\alpha} & \alpha & \alpha & 1 & \overline{\alpha} & 1 & \overline{\alpha} & \alpha
\end{array}\right|$$

$$\left|\begin{array}{rrrrrrrrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-4\alpha-\overline{\alpha} & \alpha-3\overline{\alpha} & -2\overline{\alpha} & 1 & -2\alpha & -3\alpha+\overline{\alpha} & 3\alpha-2\overline{\alpha} & 2\alpha-\overline{\alpha} & \alpha \\
-\alpha-4\overline{\alpha} & -3\alpha+\overline{\alpha} & -2\alpha & 1 & -2\overline{\alpha} & \alpha-3\overline{\alpha} & -2\alpha+3\overline{\alpha} & -\alpha+2\overline{\alpha} & \overline{\alpha} \\
-3\alpha+6\overline{\alpha} & 9\alpha+3\overline{\alpha} & 3\alpha & 0 & 3\overline{\alpha} & 3\alpha+9\overline{\alpha} & -9 & -3 & 0 \\
6 & 6 & -3 & -6 & -3 & 6 & 12 & 0 & -6 \\
6\alpha-3\overline{\alpha} & 3\alpha+9\overline{\alpha} & 3\overline{\alpha} & 0 & 3\alpha & 9\alpha+3\overline{\alpha} & -9 & -3 & 0 \\
7\alpha+13\overline{\alpha} & 7\alpha+13\overline{\alpha} & -1 & 2 & -1 & 13\alpha+7\overline{\alpha} & -5\alpha+10\overline{\alpha} & -2\alpha+\overline{\alpha} & 2 \\
-6\alpha+3\overline{\alpha} & -3\alpha-9\overline{\alpha} & 6\overline{\alpha} & -6 & 6\alpha & -9\alpha-3\overline{\alpha} & -15\overline{\alpha} & -9\alpha & -6\alpha \\
3\alpha-6\overline{\alpha} & -9\alpha-3\overline{\alpha} & 6\alpha & -6 & 6\overline{\alpha} & -3\alpha-9\overline{\alpha} & -15\alpha & -9\overline{\alpha} & -6\overline{\alpha} \\
13\alpha+7\overline{\alpha} & 13\alpha+7\overline{\alpha} & -1 & 2 & -1 & 7\alpha+13\overline{\alpha} & 10\alpha-5\overline{\alpha} & \alpha-2\overline{\alpha} & 2 \\
13\alpha+7\overline{\alpha} & -7\alpha+6\overline{\alpha} & -\overline{\alpha} & 2 & -\alpha & 6\alpha-7\overline{\alpha} & -15\alpha-10\overline{\alpha} & \alpha-2\overline{\alpha} & 2\alpha \\
-4 & -4\overline{\alpha} & -13\alpha & 2 & -13\overline{\alpha} & -4\alpha & 20\alpha & 8 & 2\overline{\alpha} \\
12 & 12 & 3 & 9 & 3 & 12 & 0 & -12 & 9 \\
-4 & -4\alpha & -13\overline{\alpha} & 2 & -13\alpha & -4\overline{\alpha} & 20\overline{\alpha} & 8 & 2\alpha \\
7\alpha+13\overline{\alpha} & 6\alpha-7\overline{\alpha} & -\alpha & 2 & -\overline{\alpha} & -7\alpha+6\overline{\alpha} & -10\alpha-15\overline{\alpha} & -2\alpha+\overline{\alpha} & 2\overline{\alpha} \\
6\alpha-3\overline{\alpha} & -9\alpha-6\overline{\alpha} & 3\alpha & 0 & 3\overline{\alpha} & -6\alpha-9\overline{\alpha} & -9\overline{\alpha} & -3 & 0 \\
3\alpha-6\overline{\alpha} & 3\alpha-6\overline{\alpha} & 6 & -6 & 6 & -6\alpha+3\overline{\alpha} & -15 & -9\overline{\alpha} & -6 \\
12 & 12\alpha & 3\overline{\alpha} & 9 & 3\alpha & 12\overline{\alpha} & 0 & -12 & 9\alpha \\
12 & 12\overline{\alpha} & 3\alpha & 9 & 3\overline{\alpha} & 12\alpha & 0 & -12 & 9\overline{\alpha} \\
-6\alpha+3\overline{\alpha} & -6\alpha+3\overline{\alpha} & 6 & -6 & 6 & 3\alpha-6\overline{\alpha} & -15 & -9\alpha & -6 \\
-3\alpha+6\overline{\alpha} & -6\alpha-9\overline{\alpha} & 3\overline{\alpha} & 0 & 3\alpha & -9\alpha-6\overline{\alpha} & -9\alpha & -3 & 0 \\
-\alpha-4\overline{\alpha} & -\alpha-4\overline{\alpha} & -2 & 1 & -2 & -4\alpha-\overline{\alpha} & 5\alpha+2\overline{\alpha} & -\alpha+2\overline{\alpha} & 1 \\
6 & 6\alpha & -3\overline{\alpha} & -6 & -3\alpha & 6\overline{\alpha} & 12\overline{\alpha} & 0 & -6\alpha \\
-6\alpha+3\overline{\alpha} & 9\alpha+6\overline{\alpha} & 6\alpha & -6 & 6\overline{\alpha} & 6\alpha+9\overline{\alpha} & -15\alpha & -9\alpha & -6\overline{\alpha} \\
-4 & -4 & -13 & 2 & -13 & -4 & 20 & 8 & 2 \\
3\alpha-6\overline{\alpha} & 6\alpha+9\overline{\alpha} & 6\overline{\alpha} & -6 & 6\alpha & 9\alpha+6\overline{\alpha} & -15\overline{\alpha} & -9\overline{\alpha} & -6\alpha \\
6 & 6\overline{\alpha} & -3\alpha & -6 & -3\overline{\alpha} & 6\alpha & 12\alpha & 0 & -6\overline{\alpha} \\
-4\alpha-\overline{\alpha} & -4\alpha-\overline{\alpha} & -2 & 1 & -2 & -\alpha-4\overline{\alpha} & 2\alpha+5\overline{\alpha} & 2\alpha-\overline{\alpha} & 1 \\
1 & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} & \overline{\alpha} & 1 & \alpha \\
-4\alpha-\overline{\alpha} & 3\alpha+4\overline{\alpha} & -2\alpha & 1 & -2\overline{\alpha} & 4\alpha+3\overline{\alpha} & -5\alpha-3\overline{\alpha} & 2\alpha-\overline{\alpha} & \overline{\alpha} \\
-3\alpha+6\overline{\alpha} & -3\alpha+6\overline{\alpha} & 3 & 0 & 3 & 6\alpha-3\overline{\alpha} & -9\overline{\alpha} & -3 & 0 \\
7\alpha+13\overline{\alpha} & -13\alpha-6\overline{\alpha} & -\overline{\alpha} & 2 & -\alpha & -6\alpha-13\overline{\alpha} & 15\alpha+5\overline{\alpha} & -2\alpha+\overline{\alpha} & 2\alpha \\
13\alpha+7\overline{\alpha} & -6\alpha-13\overline{\alpha} & -\alpha & 2 & -\overline{\alpha} & -13\alpha-6\overline{\alpha} & 5\alpha+15\overline{\alpha} & \alpha-2\overline{\alpha} & 2\overline{\alpha} \\
6\alpha-3\overline{\alpha} & 6\alpha-3\overline{\alpha} & 3 & 0 & 3 & -3\alpha+6\overline{\alpha} & -9\alpha & -3 & 0 \\
-\alpha-4\overline{\alpha} & 4\alpha+3\overline{\alpha} & -2\overline{\alpha} & 1 & -2\alpha & 3\alpha+4\overline{\alpha} & -3\alpha-5\overline{\alpha} & -\alpha+2\overline{\alpha} & \alpha \\
1 & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha & \alpha & 1 & \overline{\alpha}
\end{array}\right|$$

$$
\left|\begin{array}{rrrrrrrrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\overline{\alpha} & -\alpha+2\overline{\alpha} & -2\alpha+3\overline{\alpha} & 5\alpha-\overline{\alpha} & 4\alpha & 3\alpha+\overline{\alpha} & -2 & \alpha+3\overline{\alpha} & 4\overline{\alpha} \\
\alpha & 2\alpha-\overline{\alpha} & 3\alpha-2\overline{\alpha} & -\alpha+5\overline{\alpha} & 4\overline{\alpha} & \alpha+3\overline{\alpha} & -2 & 3\alpha+\overline{\alpha} & 4\alpha \\
0 & -3 & -9 & 6\alpha+15\overline{\alpha} & 6\overline{\alpha} & -3\alpha & 3 & -3\overline{\alpha} & 6\alpha \\
-6 & 0 & 12 & 24 & 9 & 0 & -3 & 0 & 9 \\
0 & -3 & -9 & 15\alpha+6\overline{\alpha} & 6\alpha & -3\overline{\alpha} & 3 & -3\alpha & 6\overline{\alpha} \\
2 & \alpha-2\overline{\alpha} & 10\alpha-5\overline{\alpha} & -20\alpha-5\overline{\alpha} & 5 & \alpha-2\overline{\alpha} & -1 & -2\alpha+\overline{\alpha} & 5 \\
-6\overline{\alpha} & -9\overline{\alpha} & -15\alpha & 30\alpha-15\overline{\alpha} & 0 & -9\alpha & 6 & -9\overline{\alpha} & 0 \\
-6\alpha & -9\alpha & -15\overline{\alpha} & -15\alpha+30\overline{\alpha} & 0 & -9\overline{\alpha} & 6 & -9\alpha & 0 \\
2 & -2\alpha+\overline{\alpha} & -5\alpha+10\overline{\alpha} & -5\alpha-20\overline{\alpha} & 5 & -2\alpha+\overline{\alpha} & -1 & \alpha-2\overline{\alpha} & 5 \\
2\overline{\alpha} & -2\alpha+\overline{\alpha} & -10\alpha-15\overline{\alpha} & -5\alpha-20\overline{\alpha} & 5\alpha & 3\alpha+2\overline{\alpha} & -1 & 2\alpha+3\overline{\alpha} & 5\overline{\alpha} \\
2\alpha & 8 & 20\overline{\alpha} & -40 & -10\overline{\alpha} & 8\alpha & -13 & 8\overline{\alpha} & -10\alpha \\
9 & -12 & 0 & 30 & -30 & -12 & 3 & -12 & -30 \\
2\overline{\alpha} & 8 & 20\alpha & -40 & -10\alpha & 8\overline{\alpha} & -13 & 8\alpha & -10\overline{\alpha} \\
2\alpha & \alpha-2\overline{\alpha} & -15\alpha-10\overline{\alpha} & -20\alpha-5\overline{\alpha} & 5\overline{\alpha} & 2\alpha+3\overline{\alpha} & -1 & 3\alpha+2\overline{\alpha} & 5\alpha \\
0 & -3 & -9\alpha & 15\alpha+6\overline{\alpha} & 6\overline{\alpha} & -3\alpha & 3 & -3\overline{\alpha} & 6\alpha \\
-6 & -9\alpha & -15 & -15\alpha+30\overline{\alpha} & 0 & -9\alpha & 6 & -9\overline{\alpha} & 0 \\
9\overline{\alpha} & -12 & 0 & 30 & -30\alpha & -12\overline{\alpha} & 3 & -12\alpha & -30\overline{\alpha} \\
9\alpha & -12 & 0 & 30 & -30\overline{\alpha} & -12\alpha & 3 & -12\overline{\alpha} & -30\alpha \\
-6 & -9\overline{\alpha} & -15 & 30\alpha-15\overline{\alpha} & 0 & -9\overline{\alpha} & 6 & -9\alpha & 0 \\
0 & -3 & -9\overline{\alpha} & 6\alpha+15\overline{\alpha} & 6\alpha & -3\overline{\alpha} & 3 & -3\alpha & 6\overline{\alpha} \\
1 & 2\alpha-\overline{\alpha} & 2\alpha+5\overline{\alpha} & -\alpha+5\overline{\alpha} & 4 & 2\alpha-\overline{\alpha} & -2 & -\alpha+2\overline{\alpha} & 4 \\
-6\overline{\alpha} & 0 & 12\alpha & 24 & 9\alpha & 0 & -3 & 0 & 9\overline{\alpha} \\
-6\alpha & -9\overline{\alpha} & -15\overline{\alpha} & 30\alpha-15\overline{\alpha} & 0 & -9 & 6 & -9 & 0 \\
2 & 8 & 20 & -40 & -10 & 8 & -13 & 8 & -10 \\
-6\overline{\alpha} & -9\alpha & -15\alpha & -15\alpha+30\overline{\alpha} & 0 & -9 & 6 & -9 & 0 \\
-6\alpha & 0 & 12\overline{\alpha} & 24 & 9\overline{\alpha} & 0 & -3 & 0 & 9\alpha \\
1 & -\alpha+2\overline{\alpha} & 5\alpha+2\overline{\alpha} & 5\alpha-\overline{\alpha} & 4 & -\alpha+2\overline{\alpha} & -2 & 2\alpha-\overline{\alpha} & 4 \\
\overline{\alpha} & 1 & \alpha & 1 & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} \\
\alpha & -\alpha+2\overline{\alpha} & -3\alpha-5\overline{\alpha} & 5\alpha-\overline{\alpha} & 4\overline{\alpha} & -2\alpha-3\overline{\alpha} & -2 & -3\alpha-2\overline{\alpha} & 4\alpha \\
0 & -3 & -9\alpha & 6\alpha+15\overline{\alpha} & 6 & -3 & 3 & -3 & 6 \\
2\overline{\alpha} & \alpha-2\overline{\alpha} & 5\alpha+15\overline{\alpha} & -20\alpha-5\overline{\alpha} & 5\alpha & -3\alpha-\overline{\alpha} & -1 & -\alpha-3\overline{\alpha} & 5\overline{\alpha} \\
2\alpha & -2\alpha+\overline{\alpha} & 15\alpha+5\overline{\alpha} & -5\alpha-20\overline{\alpha} & 5\overline{\alpha} & -\alpha-3\overline{\alpha} & -1 & -3\alpha-\overline{\alpha} & 5\alpha \\
0 & -3 & -9\overline{\alpha} & 15\alpha+6\overline{\alpha} & 6 & -3 & 3 & -3 & 6 \\
\overline{\alpha} & 2\alpha-\overline{\alpha} & -5\alpha-3\overline{\alpha} & -\alpha+5\overline{\alpha} & 4\alpha & -3\alpha-2\overline{\alpha} & -2 & -2\alpha-3\overline{\alpha} & 4\overline{\alpha} \\
\alpha & 1 & \overline{\alpha} & 1 & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha
\end{array}\right|
$$

$$
\left|\begin{array}{rrrrrrrrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-\alpha+5\overline{\alpha} & 7\alpha & 6\alpha+\overline{\alpha} & 5\alpha+2\overline{\alpha} & 4\alpha+3\overline{\alpha} & 3\alpha+4\overline{\alpha} & 2\alpha+5\overline{\alpha} & \alpha+6\overline{\alpha} & 7\overline{\alpha} \\
5\alpha-\overline{\alpha} & 7\overline{\alpha} & \alpha+6\overline{\alpha} & 2\alpha+5\overline{\alpha} & 3\alpha+4\overline{\alpha} & 4\alpha+3\overline{\alpha} & 5\alpha+2\overline{\alpha} & 6\alpha+\overline{\alpha} & 7\alpha \\
15\alpha+6\overline{\alpha} & 21\overline{\alpha} & -6\alpha+9\overline{\alpha} & -9\alpha & -9\alpha-6\overline{\alpha} & -6\alpha-9\overline{\alpha} & -9\overline{\alpha} & 9\alpha-6\overline{\alpha} & 21\alpha \\
24 & 42 & 24 & 12 & 6 & 6 & 12 & 24 & 42 \\
6\alpha+15\overline{\alpha} & 21\alpha & 9\alpha-6\overline{\alpha} & -9\overline{\alpha} & -6\alpha-9\overline{\alpha} & -9\alpha-6\overline{\alpha} & -9\alpha & -6\alpha+9\overline{\alpha} & 21\overline{\alpha} \\
-5\alpha-20\overline{\alpha} & 35 & -5\alpha-20\overline{\alpha} & 10\alpha-5\overline{\alpha} & 13\alpha+7\overline{\alpha} & 7\alpha+13\overline{\alpha} & -5\alpha+10\overline{\alpha} & -20\alpha-5\overline{\alpha} & 35 \\
-15\alpha+30\overline{\alpha} & 105\alpha & 45\alpha+15\overline{\alpha} & -15 & 6\alpha+9\overline{\alpha} & 9\alpha+6\overline{\alpha} & -15 & 15\alpha+45\overline{\alpha} & 105\overline{\alpha} \\
30\alpha-15\overline{\alpha} & 105\overline{\alpha} & 15\alpha+45\overline{\alpha} & -15 & 9\alpha+6\overline{\alpha} & 6\alpha+9\overline{\alpha} & -15 & 45\alpha+15\overline{\alpha} & 105\alpha \\
-20\alpha-5\overline{\alpha} & 35 & -20\alpha-5\overline{\alpha} & -5\alpha+10\overline{\alpha} & 7\alpha+13\overline{\alpha} & 13\alpha+7\overline{\alpha} & 10\alpha-5\overline{\alpha} & -5\alpha-20\overline{\alpha} & 35 \\
-20\alpha-5\overline{\alpha} & 35\alpha & 15\alpha+20\overline{\alpha} & -5\alpha+10\overline{\alpha} & -13\alpha-6\overline{\alpha} & -6\alpha-13\overline{\alpha} & 10\alpha-5\overline{\alpha} & 20\alpha+15\overline{\alpha} & 35\overline{\alpha} \\
-40 & 140\overline{\alpha} & -40\alpha & 20 & -4\overline{\alpha} & -4\alpha & 20 & -40\overline{\alpha} & 140\alpha \\
30 & 210 & 30 & 0 & 12 & 12 & 0 & 30 & 210 \\
-40 & 140\alpha & -40\overline{\alpha} & 20 & -4\alpha & -4\overline{\alpha} & 20 & -40\alpha & 140\overline{\alpha} \\
-5\alpha-20\overline{\alpha} & 35\overline{\alpha} & 20\alpha+15\overline{\alpha} & 10\alpha-5\overline{\alpha} & -6\alpha-13\overline{\alpha} & -13\alpha-6\overline{\alpha} & -5\alpha+10\overline{\alpha} & 15\alpha+20\overline{\alpha} & 35\alpha \\
6\alpha+15\overline{\alpha} & 21\overline{\alpha} & -15\alpha-9\overline{\alpha} & -9\overline{\alpha} & 9\alpha+3\overline{\alpha} & 3\alpha+9\overline{\alpha} & -9\alpha & -9\alpha-15\overline{\alpha} & 21\alpha \\
30\alpha-15\overline{\alpha} & 105 & 30\alpha-15\overline{\alpha} & -15 & -6\alpha+3\overline{\alpha} & 3\alpha-6\overline{\alpha} & -15 & -15\alpha+30\overline{\alpha} & 105 \\
30 & 210\alpha & 30\overline{\alpha} & 0 & 12\alpha & 12\overline{\alpha} & 0 & 30\alpha & 210\overline{\alpha} \\
30 & 210\overline{\alpha} & 30\alpha & 0 & 12\overline{\alpha} & 12\alpha & 0 & 30\overline{\alpha} & 210\alpha \\
-15\alpha+30\overline{\alpha} & 105 & -15\alpha+30\overline{\alpha} & -15 & 3\alpha-6\overline{\alpha} & -6\alpha+3\overline{\alpha} & -15 & 30\alpha-15\overline{\alpha} & 105 \\
15\alpha+6\overline{\alpha} & 21\alpha & -9\alpha-15\overline{\alpha} & -9\alpha & 3\alpha+9\overline{\alpha} & 9\alpha+3\overline{\alpha} & -9\overline{\alpha} & -15\alpha-9\overline{\alpha} & 21\overline{\alpha} \\
5\alpha-\overline{\alpha} & 7 & 5\alpha-\overline{\alpha} & 2\alpha+5\overline{\alpha} & -4\alpha-\overline{\alpha} & -\alpha-4\overline{\alpha} & 5\alpha+2\overline{\alpha} & -\alpha+5\overline{\alpha} & 7 \\
24 & 42\alpha & 24\overline{\alpha} & 12 & 6\alpha & 6\overline{\alpha} & 12 & 24\alpha & 42\overline{\alpha} \\
-15\alpha+30\overline{\alpha} & 105\overline{\alpha} & -30\alpha-45\overline{\alpha} & -15 & -9\alpha-3\overline{\alpha} & -3\alpha-9\overline{\alpha} & -15 & -45\alpha-30\overline{\alpha} & 105\alpha \\
-40 & 140 & -40 & 20 & -4 & -4 & 20 & -40 & 140 \\
30\alpha-15\overline{\alpha} & 105\alpha & -45\alpha-30\overline{\alpha} & -15 & -3\alpha-9\overline{\alpha} & -9\alpha-3\overline{\alpha} & -15 & -30\alpha-45\overline{\alpha} & 105\overline{\alpha} \\
24 & 42\overline{\alpha} & 24\alpha & 12 & 6\overline{\alpha} & 6\alpha & 12 & 24\overline{\alpha} & 42\alpha \\
-\alpha+5\overline{\alpha} & 7 & -\alpha+5\overline{\alpha} & 5\alpha+2\overline{\alpha} & -\alpha-4\overline{\alpha} & -4\alpha-\overline{\alpha} & 2\alpha+5\overline{\alpha} & 5\alpha-\overline{\alpha} & 7 \\
1 & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} \\
-\alpha+5\overline{\alpha} & 7\overline{\alpha} & -5\alpha-6\overline{\alpha} & 5\alpha+2\overline{\alpha} & -3\alpha+\overline{\alpha} & \alpha-3\overline{\alpha} & 2\alpha+5\overline{\alpha} & -6\alpha-5\overline{\alpha} & 7\alpha \\
15\alpha+6\overline{\alpha} & 21 & 15\alpha+6\overline{\alpha} & -9\alpha & 6\alpha-3\overline{\alpha} & -3\alpha+6\overline{\alpha} & -9\overline{\alpha} & 6\alpha+15\overline{\alpha} & 21 \\
-5\alpha-20\overline{\alpha} & 35\alpha & -15\alpha+5\overline{\alpha} & 10\alpha-5\overline{\alpha} & -7\alpha+6\overline{\alpha} & 6\alpha-7\overline{\alpha} & -5\alpha+10\overline{\alpha} & 5\alpha-15\overline{\alpha} & 35\overline{\alpha} \\
-20\alpha-5\overline{\alpha} & 35\overline{\alpha} & 5\alpha-15\overline{\alpha} & -5\alpha+10\overline{\alpha} & 6\alpha-7\overline{\alpha} & -7\alpha+6\overline{\alpha} & 10\alpha-5\overline{\alpha} & -15\alpha+5\overline{\alpha} & 35\alpha \\
6\alpha+15\overline{\alpha} & 21 & 6\alpha+15\overline{\alpha} & -9\overline{\alpha} & -3\alpha+6\overline{\alpha} & 6\alpha-3\overline{\alpha} & -9\alpha & 15\alpha+6\overline{\alpha} & 21 \\
5\alpha-\overline{\alpha} & 7\alpha & -6\alpha-5\overline{\alpha} & 2\alpha+5\overline{\alpha} & \alpha-3\overline{\alpha} & -3\alpha+\overline{\alpha} & 5\alpha+2\overline{\alpha} & -5\alpha-6\overline{\alpha} & 7\overline{\alpha} \\
1 & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha
\end{array}\right)
$$

We proceed using Proposition 6.3 to get the Fischer Clifford matrix $M(1A)$ of $\overline{G} = 3^6{:}S_7$ from the matrix $F(3,1^7)$. For $\lambda = 7$ and $p = 3$, we have 36, 3−sets partitions of $[1^7]$ given in Figure 3 from which we select the following twelve type-zero 3-sets of partition $[1^7]$ that give rise to the columns of $F_S(3,1^7) = M(1A)$.

$$\left\{ \begin{pmatrix}7\\0\\0\end{pmatrix}, \begin{pmatrix}5\\1\\1\end{pmatrix}, \begin{pmatrix}4\\3\\0\end{pmatrix}, \begin{pmatrix}4\\0\\3\end{pmatrix}, \begin{pmatrix}3\\2\\2\end{pmatrix}, \begin{pmatrix}2\\4\\1\end{pmatrix}, \begin{pmatrix}2\\1\\4\end{pmatrix}, \begin{pmatrix}1\\6\\0\end{pmatrix}, \begin{pmatrix}1\\3\\3\end{pmatrix}, \begin{pmatrix}1\\0\\6\end{pmatrix}, \begin{pmatrix}0\\5\\2\end{pmatrix}, \begin{pmatrix}0\\2\\5\end{pmatrix} \right\}$$

Figure 4: Type-zero 3-sets of partition $[1^7]$

The Fischer-Clifford matrix $F_S(3,1^7) = M(1A)$ is thus of order 12. The following twelve representatives of sets of equivalent 3-sets of partition $[1^7]$ which index the rows of $F_S(3,1^7)$ are also selected.

$$\left\{ \begin{pmatrix}7\\0\\0\end{pmatrix}, \begin{pmatrix}6\\1\\0\end{pmatrix}, \begin{pmatrix}6\\0\\1\end{pmatrix}, \begin{pmatrix}5\\2\\0\end{pmatrix}, \begin{pmatrix}5\\1\\1\end{pmatrix}, \begin{pmatrix}5\\0\\2\end{pmatrix}, \begin{pmatrix}4\\3\\0\end{pmatrix}, \begin{pmatrix}4\\2\\1\end{pmatrix}, \begin{pmatrix}4\\1\\2\end{pmatrix}, \begin{pmatrix}4\\0\\3\end{pmatrix}, \begin{pmatrix}3\\3\\1\end{pmatrix}, \begin{pmatrix}3\\2\\2\end{pmatrix} \right\}$$

Figure 5: Representatives of equivalent 3-sets of partition $[1^7]$

Selecting columns $1, 5, 7, 10, 13, 17, 20, 22, 25, 28, 31$ and 34 and rows $1, \ldots, 10, 12$ and 13 corresponding to the type zero 3−sets and the twelve representatives of sets of equivalent 3-sets of partition $[1^7]$ respectively, we get $F_S(3,1^7)$ shown in Figure 6. Since there is no self-equivalent 3-set of partition $[1^7]$, no row of the resulting matrix is divided by $p = 3$.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
7 & 4 & -\alpha-4\overline{\alpha} & -4\alpha-\overline{\alpha} & 1 & 2\alpha-\overline{\alpha} & -\alpha+2\overline{\alpha} & 5\alpha-\overline{\alpha} & -2 & -\alpha+5\overline{\alpha} & 5\alpha+2\overline{\alpha} & 2\alpha+5\overline{\alpha} \\
7 & 4 & -4\alpha-\overline{\alpha} & -\alpha-4\overline{\alpha} & 1 & -\alpha+2\overline{\alpha} & 2\alpha-\overline{\alpha} & -\alpha+5\overline{\alpha} & -2 & 5\alpha-\overline{\alpha} & 2\alpha+5\overline{\alpha} & 5\alpha+2\overline{\alpha} \\
21 & 6 & 6\alpha-3\overline{\alpha} & -3\alpha+6\overline{\alpha} & 0 & -3 & -3 & 6\alpha+15\overline{\alpha} & 3 & 15\alpha+6\overline{\alpha} & -9\alpha & -9\overline{\alpha} \\
42 & 9 & 6 & 6 & -6 & 0 & 0 & 24 & -3 & 24 & 12 & 12 \\
21 & 6 & -3\alpha+6\overline{\alpha} & 6\alpha-3\overline{\alpha} & 0 & -3 & -3 & 15\alpha+6\overline{\alpha} & 3 & 6\alpha+15\overline{\alpha} & -9\overline{\alpha} & -9\alpha \\
35 & 5 & 13\alpha+7\overline{\alpha} & 7\alpha+13\overline{\alpha} & 2 & -2\alpha+\overline{\alpha} & \alpha-2\overline{\alpha} & -20\alpha-5\overline{\alpha} & -1 & -5\alpha-20\overline{\alpha} & 10\alpha-5\overline{\alpha} & -5\alpha+10\overline{\alpha} \\
105 & 0 & 3\alpha-6\overline{\alpha} & -6\alpha+3\overline{\alpha} & -6 & -9\alpha & -9\overline{\alpha} & 30\alpha-15\overline{\alpha} & 6 & -15\alpha+30\overline{\alpha} & -15 & -15 \\
105 & 0 & -6\alpha+3\overline{\alpha} & 3\alpha-6\overline{\alpha} & -6 & -9\overline{\alpha} & -9\alpha & -15\alpha+30\overline{\alpha} & 6 & 30\alpha-15\overline{\alpha} & -15 & -15 \\
35 & 5 & 7\alpha+13\overline{\alpha} & 13\alpha+7\overline{\alpha} & 2 & \alpha-2\overline{\alpha} & -2\alpha+\overline{\alpha} & -5\alpha-20\overline{\alpha} & -1 & -20\alpha-5\overline{\alpha} & -5\alpha+10\overline{\alpha} & 10\alpha-5\overline{\alpha} \\
140 & -10 & -4 & -4 & 2 & 8 & 8 & -40 & -13 & -40 & 20 & 20 \\
210 & -30 & 12 & 12 & 9 & -12 & -12 & 30 & 3 & 30 & 0 & 0
\end{pmatrix}$$

Figure 6: Fischer-Clifford matrix $F_S(3,1^7)$

Lastly, we note that the rows and columns of the matrix $M(1A)$ require to be re-arranged to give the actual Fischer-Clifford matrix. This is achieved by use of the partial fusion of $\overline{G}$ into the parent group $U_7(2)$, the power maps of $\overline{G}$ as well as restrictions of irreducible characters of $U_7(2)$ to $\overline{G}$. The actual $M(1A)$, after re-arrangement of rows and columns is displayed as Figure 7. The other Fischer-Clifford matrices are computed in a similar manner and listed immediately after $M(1A)$. The orders of the Fischer-Clifford matrices so obtained ranges from 1 to 21. Again, note that some matrices have been split with a | to fit in a4 paper.

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
105 & -15\alpha+30\overline{\alpha} & 30\alpha-15\overline{\alpha} & -15 & -15 & 3\alpha-6\overline{\alpha} & -6\alpha+3\overline{\alpha} & 0 & -9\overline{\alpha} & -9\alpha & 6 & -6 \\
105 & 30\alpha-15\overline{\alpha} & -15\alpha+30\overline{\alpha} & -15 & -15 & -6\alpha+3\overline{\alpha} & 3\alpha-6\overline{\alpha} & 0 & -9\alpha & -9\overline{\alpha} & 6 & -6 \\
21 & 6\alpha+15\overline{\alpha} & 15\alpha+6\overline{\alpha} & -9\overline{\alpha} & -9\alpha & -3\alpha+6\overline{\alpha} & 6\alpha-3\overline{\alpha} & 6 & -3 & -3 & 3 & 0 \\
21 & 15\alpha+6\overline{\alpha} & 6\alpha+15\overline{\alpha} & -9\alpha & -9\overline{\alpha} & 6\alpha-3\overline{\alpha} & -3\alpha+6\overline{\alpha} & 6 & -3 & -3 & 3 & 0 \\
210 & 30 & 30 & 0 & 0 & 12 & 12 & -30 & -12 & -12 & 3 & 9 \\
35 & -20\alpha-5\overline{\alpha} & -5\alpha-20\overline{\alpha} & -5\alpha+10\overline{\alpha} & 10\alpha-5\overline{\alpha} & 7\alpha+13\overline{\alpha} & 13\alpha+7\overline{\alpha} & 5 & -2\alpha+\overline{\alpha} & \alpha-2\overline{\alpha} & -1 & 2 \\
140 & -40 & -40 & 20 & 20 & -4 & -4 & -10 & 8 & 8 & -13 & 2 \\
35 & -5\alpha-20\overline{\alpha} & -20\alpha-5\overline{\alpha} & 10\alpha-5\overline{\alpha} & -5\alpha+10\overline{\alpha} & 13\alpha+7\overline{\alpha} & 7\alpha+13\overline{\alpha} & 5 & \alpha-2\overline{\alpha} & -2\alpha+\overline{\alpha} & -1 & 2 \\
7 & -\alpha+5\overline{\alpha} & 5\alpha-\overline{\alpha} & 5\alpha+2\overline{\alpha} & 2\alpha+5\overline{\alpha} & -\alpha-4\overline{\alpha} & -4\alpha-\overline{\alpha} & 4 & -\alpha+2\overline{\alpha} & 2\alpha-\overline{\alpha} & -2 & 1 \\
7 & 5\alpha-\overline{\alpha} & -\alpha+5\overline{\alpha} & 2\alpha+5\overline{\alpha} & 5\alpha+2\overline{\alpha} & -4\alpha-\overline{\alpha} & -\alpha-4\overline{\alpha} & 4 & 2\alpha-\overline{\alpha} & -\alpha+2\overline{\alpha} & -2 & 1 \\
42 & 24 & 24 & 12 & 12 & 6 & 6 & 9 & 0 & 0 & -3 & -6
\end{pmatrix}$$

Figure 7: Final Fischer-Clifford matrix $M(1A)$

$$M(2A)=\left(\begin{array}{rrrrrrrrrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
30 & 30\alpha & 30\bar{\alpha} & -6\alpha & -6\bar{\alpha} & -6 & -6\alpha & -6 & -6\bar{\alpha} & 3\bar{\alpha}\\
5 & 5 & 5 & \alpha+4\bar{\alpha} & 4\alpha+\bar{\alpha} & \alpha+4\bar{\alpha} & 4\alpha+\bar{\alpha} & 4\alpha+\bar{\alpha} & \alpha+4\bar{\alpha} & \alpha-2\bar{\alpha}\\
30 & 30\bar{\alpha} & 30\alpha & -6\bar{\alpha} & -6\alpha & -6 & -6\bar{\alpha} & -6 & -6\alpha & 3\alpha\\
5 & 5 & 5 & 4\alpha+\bar{\alpha} & \alpha+4\bar{\alpha} & 4\alpha+\bar{\alpha} & \alpha+4\bar{\alpha} & \alpha+4\bar{\alpha} & 4\alpha+\bar{\alpha} & -2\alpha+\bar{\alpha}\\
10 & 10\alpha & 10\bar{\alpha} & 4\alpha+6\bar{\alpha} & 6\alpha+4\bar{\alpha} & 2\alpha-4\bar{\alpha} & -2\alpha-6\bar{\alpha} & -4\alpha+2\bar{\alpha} & -6\alpha-2\bar{\alpha} & -3\alpha-5\bar{\alpha}\\
1 & \bar{\alpha} & \alpha & \bar{\alpha} & \alpha & 1 & \bar{\alpha} & 1 & \alpha & \alpha\\
10 & 10\bar{\alpha} & 10\alpha & 6\alpha+4\bar{\alpha} & 4\alpha+6\bar{\alpha} & -4\alpha+2\bar{\alpha} & -6\alpha-2\bar{\alpha} & 2\alpha-4\bar{\alpha} & -2\alpha-6\bar{\alpha} & -5\alpha-3\bar{\alpha}\\
1 & \alpha & \bar{\alpha} & \alpha & \bar{\alpha} & 1 & \alpha & 1 & \bar{\alpha} & \bar{\alpha}\\
10 & 10\alpha & 10\bar{\alpha} & -2\alpha-6\bar{\alpha} & -6\alpha-2\bar{\alpha} & -4\alpha+2\bar{\alpha} & 4\alpha+6\bar{\alpha} & 2\alpha-4\bar{\alpha} & 6\alpha+4\bar{\alpha} & 3\alpha-2\bar{\alpha}\\
30 & 30 & 30 & -6 & -6 & -6 & -6 & -6 & -6 & 3\\
10 & 10\bar{\alpha} & 10\alpha & -6\alpha-2\bar{\alpha} & -2\alpha-6\bar{\alpha} & 2\alpha-4\bar{\alpha} & 6\alpha+4\bar{\alpha} & -4\alpha+2\bar{\alpha} & 4\alpha+6\bar{\alpha} & -2\alpha+3\bar{\alpha}\\
5 & 5\alpha & 5\bar{\alpha} & -4\alpha-3\bar{\alpha} & -3\alpha-4\bar{\alpha} & \alpha+4\bar{\alpha} & -\alpha+3\bar{\alpha} & 4\alpha+\bar{\alpha} & 3\alpha-\bar{\alpha} & -3\alpha-\bar{\alpha}\\
10 & 10 & 10 & -4\alpha+2\bar{\alpha} & 2\alpha-4\bar{\alpha} & -4\alpha+2\bar{\alpha} & 2\alpha-4\bar{\alpha} & 2\alpha-4\bar{\alpha} & -4\alpha+2\bar{\alpha} & 2\alpha+5\bar{\alpha}\\
20 & 20\alpha & 20\bar{\alpha} & 8\alpha & 8\bar{\alpha} & 8 & 8\alpha & 8 & 8\bar{\alpha} & 2\bar{\alpha}\\
20 & 20\bar{\alpha} & 20\alpha & 8\bar{\alpha} & 8\alpha & 8 & 8\bar{\alpha} & 8 & 8\alpha & 2\alpha\\
10 & 10 & 10 & 2\alpha-4\bar{\alpha} & -4\alpha+2\bar{\alpha} & 2\alpha-4\bar{\alpha} & -4\alpha+2\bar{\alpha} & -4\alpha+2\bar{\alpha} & 2\alpha-4\bar{\alpha} & 5\alpha+2\bar{\alpha}\\
5 & 5\bar{\alpha} & 5\alpha & -3\alpha-4\bar{\alpha} & -4\alpha-3\bar{\alpha} & 4\alpha+\bar{\alpha} & 3\alpha-\bar{\alpha} & \alpha+4\bar{\alpha} & -\alpha+3\bar{\alpha} & -\alpha-3\bar{\alpha}\\
5 & 5\alpha & 5\bar{\alpha} & -\alpha+3\bar{\alpha} & 3\alpha-\bar{\alpha} & 4\alpha+\bar{\alpha} & -4\alpha-3\bar{\alpha} & \alpha+4\bar{\alpha} & -3\alpha-4\bar{\alpha} & 3\alpha+2\bar{\alpha}\\
5 & 5\bar{\alpha} & 5\alpha & 3\alpha-\bar{\alpha} & -\alpha+3\bar{\alpha} & \alpha+4\bar{\alpha} & -3\alpha-4\bar{\alpha} & 4\alpha+\bar{\alpha} & -4\alpha-3\bar{\alpha} & 2\alpha+3\bar{\alpha}\\
20 & 20 & 20 & 8 & 8 & 8 & 8 & 8 & 8 & 2
\end{array}\right|$$

$$\left|\begin{array}{rrrrrrrrrrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
3\alpha & 3\alpha & 3 & 3 & 3\bar{\alpha} & 3\alpha & 3\bar{\alpha} & 3 & -6 & -6\alpha & -6\bar{\alpha}\\
-2\alpha+\bar{\alpha} & \alpha-2\bar{\alpha} & -2\alpha+\bar{\alpha} & \alpha-2\bar{\alpha} & -2\alpha+\bar{\alpha} & -1 & -1 & -1 & 2 & 2 & 2\\
3\bar{\alpha} & 3\bar{\alpha} & 3 & 3 & 3\alpha & 3\bar{\alpha} & 3\alpha & 3 & -6 & -6\bar{\alpha} & -6\alpha\\
\alpha-2\bar{\alpha} & -2\alpha+\bar{\alpha} & \alpha-2\bar{\alpha} & -2\alpha+\bar{\alpha} & \alpha-2\bar{\alpha} & -1 & -1 & -1 & 2 & 2 & 2\\
-5\alpha-3\bar{\alpha} & -2\alpha+3\bar{\alpha} & 2\alpha+5\bar{\alpha} & 5\alpha+2\bar{\alpha} & 3\alpha-2\bar{\alpha} & \alpha & \bar{\alpha} & 1 & 1 & \alpha & \bar{\alpha}\\
\bar{\alpha} & \bar{\alpha} & 1 & 1 & \alpha & \bar{\alpha} & \alpha & 1 & 1 & \bar{\alpha} & \alpha\\
-3\alpha-5\bar{\alpha} & 3\alpha-2\bar{\alpha} & 5\alpha+2\bar{\alpha} & 2\alpha+5\bar{\alpha} & -2\alpha+3\bar{\alpha} & \bar{\alpha} & \alpha & 1 & 1 & \bar{\alpha} & \alpha\\
\alpha & \alpha & 1 & 1 & \bar{\alpha} & \alpha & \bar{\alpha} & 1 & 1 & \alpha & \bar{\alpha}\\
-2\alpha+3\bar{\alpha} & -5\alpha-3\bar{\alpha} & 5\alpha+2\bar{\alpha} & 2\alpha+5\bar{\alpha} & -3\alpha-5\bar{\alpha} & \alpha & \bar{\alpha} & 1 & 1 & \alpha & \bar{\alpha}\\
3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & -6 & -6 & -6\\
3\alpha-2\bar{\alpha} & -3\alpha-5\bar{\alpha} & 2\alpha+5\bar{\alpha} & 5\alpha+2\bar{\alpha} & -5\alpha-3\bar{\alpha} & \bar{\alpha} & \alpha & 1 & 1 & \bar{\alpha} & \alpha\\
-\alpha-3\bar{\alpha} & 2\alpha+3\bar{\alpha} & -2\alpha+\bar{\alpha} & \alpha-2\bar{\alpha} & 3\alpha+2\bar{\alpha} & -\alpha & -\bar{\alpha} & -1 & 2 & 2\alpha & 2\bar{\alpha}\\
5\alpha+2\bar{\alpha} & 2\alpha+5\bar{\alpha} & 5\alpha+2\bar{\alpha} & 2\alpha+5\bar{\alpha} & 5\alpha+2\bar{\alpha} & 1 & 1 & 1 & 1 & 1 & 1\\
2\alpha & 2\alpha & 2 & 2 & 2\bar{\alpha} & -4\alpha & -4\bar{\alpha} & -4 & -1 & -\alpha & -\bar{\alpha}\\
2\bar{\alpha} & 2\bar{\alpha} & 2 & 2 & 2\alpha & -4\bar{\alpha} & -4\alpha & -4 & -1 & -\bar{\alpha} & -\alpha\\
2\alpha+5\bar{\alpha} & 5\alpha+2\bar{\alpha} & 2\alpha+5\bar{\alpha} & 5\alpha+2\bar{\alpha} & 2\alpha+5\bar{\alpha} & 1 & 1 & 1 & 1 & 1 & 1\\
-3\alpha-\bar{\alpha} & 3\alpha+2\bar{\alpha} & \alpha-2\bar{\alpha} & -2\alpha+\bar{\alpha} & 2\alpha+3\bar{\alpha} & -\bar{\alpha} & -\alpha & -1 & 2 & 2\bar{\alpha} & 2\alpha\\
2\alpha+3\bar{\alpha} & -\alpha-3\bar{\alpha} & \alpha-2\bar{\alpha} & -2\alpha+\bar{\alpha} & -3\alpha-\bar{\alpha} & -\alpha & -\bar{\alpha} & -1 & 2 & 2\alpha & 2\bar{\alpha}\\
3\alpha+2\bar{\alpha} & -3\alpha-\bar{\alpha} & -2\alpha+\bar{\alpha} & \alpha-2\bar{\alpha} & -\alpha-3\bar{\alpha} & -\bar{\alpha} & -\alpha & -1 & 2 & 2\bar{\alpha} & 2\alpha\\
2 & 2 & 2 & 2 & 2 & -4 & -4 & -4 & -1 & -1 & -1
\end{array}\right),$$

$$M(2B)=\left(\begin{array}{rrrrrrrrrr}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
3 & 3\alpha & 3\bar{\alpha} & 3\bar{\alpha} & 3\alpha & 3 & -\alpha-2\bar{\alpha} & -2\alpha-\bar{\alpha} & \alpha-\bar{\alpha} & -\alpha+\bar{\alpha}\\
6 & 6\bar{\alpha} & 6\alpha & -3\alpha & -3\bar{\alpha} & -3 & 2\alpha+4\bar{\alpha} & 4\alpha+2\bar{\alpha} & -2\alpha-4\bar{\alpha} & -4\alpha-2\bar{\alpha}\\
3 & 3\bar{\alpha} & 3\alpha & 3\alpha & 3\bar{\alpha} & 3 & -2\alpha-\bar{\alpha} & -\alpha-2\bar{\alpha} & -\alpha+\bar{\alpha} & \alpha-\bar{\alpha}\\
6 & 6\alpha & 6\bar{\alpha} & -3\bar{\alpha} & -3\alpha & -3 & 4\alpha+2\bar{\alpha} & 2\alpha+4\bar{\alpha} & -4\alpha-2\bar{\alpha} & -2\alpha-4\bar{\alpha}\\
3 & 3\alpha & 3\bar{\alpha} & 3\bar{\alpha} & 3\alpha & 3 & 2\alpha+\bar{\alpha} & \alpha+2\bar{\alpha} & -2\alpha-\bar{\alpha} & -\alpha-2\bar{\alpha}\\
2 & 2 & 2 & -1 & -1 & -1 & 2\bar{\alpha} & 2\alpha & 2\alpha & 2\bar{\alpha}\\
3 & 3\bar{\alpha} & 3\alpha & 3\alpha & 3\bar{\alpha} & 3 & \alpha+2\bar{\alpha} & 2\alpha+\bar{\alpha} & -\alpha-2\bar{\alpha} & -2\alpha-\bar{\alpha}\\
2 & 2 & 2 & -1 & -1 & -1 & 2\alpha & 2\bar{\alpha} & 2\bar{\alpha} & 2\alpha\\
6 & 6\alpha & 6\bar{\alpha} & -3\bar{\alpha} & -3\alpha & -3 & -2\alpha+2\bar{\alpha} & 2\alpha-2\bar{\alpha} & 2\alpha+4\bar{\alpha} & 4\alpha+2\bar{\alpha}\\
2 & 2 & 2 & -1 & -1 & -1 & 2 & 2 & 2 & 2\\
6 & 6\bar{\alpha} & 6\alpha & -3\alpha & -3\bar{\alpha} & -3 & 2\alpha-2\bar{\alpha} & -2\alpha+2\bar{\alpha} & 4\alpha+2\bar{\alpha} & 2\alpha+4\bar{\alpha}\\
6 & 6\bar{\alpha} & 6\alpha & -3\alpha & -3\bar{\alpha} & -3 & -4\alpha-2\bar{\alpha} & -2\alpha-4\bar{\alpha} & -2\alpha+2\bar{\alpha} & 2\alpha-2\bar{\alpha}\\
1 & 1 & 1 & 1 & 1 & 1 & \bar{\alpha} & \alpha & \alpha & \bar{\alpha}\\
12 & 12 & 12 & -6 & -6 & -6 & 0 & 0 & 0 & 0\\
6 & 6\alpha & 6\bar{\alpha} & -3\bar{\alpha} & -3\alpha & -3 & -2\alpha-4\bar{\alpha} & -4\alpha-2\bar{\alpha} & 2\alpha-2\bar{\alpha} & -2\alpha+2\bar{\alpha}\\
1 & 1 & 1 & 1 & 1 & 1 & \alpha & \bar{\alpha} & \bar{\alpha} & \alpha\\
3 & 3\alpha & 3\bar{\alpha} & 3\bar{\alpha} & 3\alpha & 3 & -\alpha+\bar{\alpha} & \alpha-\bar{\alpha} & \alpha+2\bar{\alpha} & 2\alpha+\bar{\alpha}\\
3 & 3\bar{\alpha} & 3\alpha & 3\alpha & 3\bar{\alpha} & 3 & \alpha-\bar{\alpha} & -\alpha+\bar{\alpha} & 2\alpha+\bar{\alpha} & \alpha+2\bar{\alpha}\\
6 & 6 & 6 & 6 & 6 & 6 & 0 & 0 & 0 & 0
\end{array}\right|$$

$$
\left|\begin{matrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2\alpha+\overline{\alpha} & \alpha+2\overline{\alpha} & -\alpha+\overline{\alpha} & \alpha+2\overline{\alpha} & 2\alpha+\overline{\alpha} & \alpha-\overline{\alpha} & 0 & -2\alpha-\overline{\alpha} & -\alpha-2\overline{\alpha} & 0 \\
2\alpha-2\overline{\alpha} & -2\alpha+2\overline{\alpha} & 2\alpha+\overline{\alpha} & \alpha-\overline{\alpha} & -\alpha+\overline{\alpha} & \alpha+2\overline{\alpha} & 0 & -2\alpha-\overline{\alpha} & -\alpha-2\overline{\alpha} & 0 \\
\alpha+2\overline{\alpha} & 2\alpha+\overline{\alpha} & \alpha-\overline{\alpha} & 2\alpha+\overline{\alpha} & \alpha+2\overline{\alpha} & -\alpha+\overline{\alpha} & 0 & -\alpha-2\overline{\alpha} & -2\alpha-\overline{\alpha} & 0 \\
-2\alpha+2\overline{\alpha} & 2\alpha-2\overline{\alpha} & \alpha+2\overline{\alpha} & -\alpha+\overline{\alpha} & \alpha-\overline{\alpha} & 2\alpha+\overline{\alpha} & 0 & -\alpha-2\overline{\alpha} & -2\alpha-\overline{\alpha} & 0 \\
-\alpha+\overline{\alpha} & \alpha-\overline{\alpha} & -\alpha-2\overline{\alpha} & \alpha-\overline{\alpha} & -\alpha+\overline{\alpha} & -2\alpha-\overline{\alpha} & 0 & \alpha+2\overline{\alpha} & 2\alpha+\overline{\alpha} & 0 \\
2\overline{\alpha} & 2\alpha & -\overline{\alpha} & -\alpha & -\overline{\alpha} & -\alpha & 2 & -\alpha & -\overline{\alpha} & -1 \\
\alpha-\overline{\alpha} & -\alpha+\overline{\alpha} & -2\alpha-\overline{\alpha} & -\alpha+\overline{\alpha} & \alpha-\overline{\alpha} & -\alpha-2\overline{\alpha} & 0 & 2\alpha+\overline{\alpha} & \alpha+2\overline{\alpha} & 0 \\
2\alpha & 2\overline{\alpha} & -\alpha & -\overline{\alpha} & -\alpha & -\overline{\alpha} & 2 & -\overline{\alpha} & -\alpha & -1 \\
-2\alpha-4\overline{\alpha} & -4\alpha-2\overline{\alpha} & -2\alpha-\overline{\alpha} & 2\alpha+\overline{\alpha} & \alpha+2\overline{\alpha} & -\alpha-2\overline{\alpha} & 0 & -\alpha+\overline{\alpha} & \alpha-\overline{\alpha} & 0 \\
2 & 2 & -1 & -1 & -1 & -1 & 2 & -1 & -1 & -1 \\
-4\alpha-2\overline{\alpha} & -2\alpha-4\overline{\alpha} & -\alpha-2\overline{\alpha} & \alpha+2\overline{\alpha} & 2\alpha+\overline{\alpha} & -2\alpha-\overline{\alpha} & 0 & \alpha-\overline{\alpha} & -\alpha+\overline{\alpha} & 0 \\
2\alpha+4\overline{\alpha} & 4\alpha+2\overline{\alpha} & -\alpha+\overline{\alpha} & -2\alpha-\overline{\alpha} & -\alpha-2\overline{\alpha} & \alpha-\overline{\alpha} & 0 & \alpha+2\overline{\alpha} & 2\alpha+\overline{\alpha} & 0 \\
\overline{\alpha} & \alpha & \overline{\alpha} & \alpha & \overline{\alpha} & \alpha & 1 & \alpha & \overline{\alpha} & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & -6 & 0 & 0 & 3 \\
4\alpha+2\overline{\alpha} & 2\alpha+4\overline{\alpha} & \alpha-\overline{\alpha} & -\alpha-2\overline{\alpha} & -2\alpha-\overline{\alpha} & -\alpha+\overline{\alpha} & 0 & 2\alpha+\overline{\alpha} & \alpha+2\overline{\alpha} & 0 \\
\alpha & \overline{\alpha} & \alpha & \overline{\alpha} & \alpha & \overline{\alpha} & 1 & \overline{\alpha} & \alpha & 1 \\
-\alpha-2\overline{\alpha} & -2\alpha-\overline{\alpha} & 2\alpha+\overline{\alpha} & -2\alpha-\overline{\alpha} & -\alpha-2\overline{\alpha} & \alpha+2\overline{\alpha} & 0 & \alpha-\overline{\alpha} & -\alpha+\overline{\alpha} & 0 \\
-2\alpha-\overline{\alpha} & -\alpha-2\overline{\alpha} & \alpha+2\overline{\alpha} & -\alpha-2\overline{\alpha} & -2\alpha-\overline{\alpha} & 2\alpha+\overline{\alpha} & 0 & -\alpha+\overline{\alpha} & \alpha-\overline{\alpha} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 & 0 & 0 & -3
\end{matrix}\right),
$$

$$
M(2C)=\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
3 & 3\overline{\alpha} & 3\alpha & -\alpha-2\overline{\alpha} & -\alpha+\overline{\alpha} & 2\alpha+\overline{\alpha} & -2\alpha-\overline{\alpha} & \alpha+2\overline{\alpha} & \alpha-\overline{\alpha} & 0 \\
3 & 3\alpha & 3\overline{\alpha} & -2\alpha-\overline{\alpha} & \alpha-\overline{\alpha} & \alpha+2\overline{\alpha} & -\alpha-2\overline{\alpha} & 2\alpha+\overline{\alpha} & -\alpha+\overline{\alpha} & 0 \\
3 & 3\alpha & 3\overline{\alpha} & \alpha-\overline{\alpha} & \alpha+2\overline{\alpha} & -2\alpha-\overline{\alpha} & -\alpha+\overline{\alpha} & -\alpha-2\overline{\alpha} & 2\alpha+\overline{\alpha} & 0 \\
3 & 3\overline{\alpha} & 3\alpha & -\alpha+\overline{\alpha} & 2\alpha+\overline{\alpha} & -\alpha-2\overline{\alpha} & \alpha-\overline{\alpha} & -2\alpha-\overline{\alpha} & \alpha+2\overline{\alpha} & 0 \\
6 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \\
3 & 3\alpha & 3\overline{\alpha} & \alpha+2\overline{\alpha} & -2\alpha-\overline{\alpha} & \alpha-\overline{\alpha} & 2\alpha+\overline{\alpha} & -\alpha+\overline{\alpha} & -\alpha-2\overline{\alpha} & 0 \\
3 & 3\overline{\alpha} & 3\alpha & 2\alpha+\overline{\alpha} & -\alpha-2\overline{\alpha} & -\alpha+\overline{\alpha} & \alpha+2\overline{\alpha} & \alpha-\overline{\alpha} & -2\alpha-\overline{\alpha} & 0 \\
1 & 1 & 1 & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} & \alpha & \alpha & \alpha & 1 \\
1 & 1 & 1 & \alpha & \alpha & \alpha & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} & 1
\end{pmatrix},
$$

$$
M(3A)=\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
12 & 12\overline{\alpha} & 12\alpha & 3 & 3 & 3\alpha & 3\alpha & 3\overline{\alpha} & 3\overline{\alpha} & 0 & 0 & 0 & -3 & -3\alpha & -3\overline{\alpha} \\
12 & 12\alpha & 12\overline{\alpha} & 3 & 3 & 3\overline{\alpha} & 3\overline{\alpha} & 3\alpha & 3\alpha & 0 & 0 & 0 & -3 & -3\overline{\alpha} & -3\alpha \\
6 & 6\overline{\alpha} & 6\alpha & -3 & -3 & -3\alpha & -3\alpha & -3\overline{\alpha} & -3\overline{\alpha} & 3 & 3\alpha & 3\overline{\alpha} & 0 & 0 & 0 \\
6 & 6\alpha & 6\overline{\alpha} & -3 & -3 & -3\overline{\alpha} & -3\overline{\alpha} & -3\alpha & -3\alpha & 3 & 3\overline{\alpha} & 3\alpha & 0 & 0 & 0 \\
6 & 6 & 6 & -3 & -3 & -3 & -3 & -3 & -3 & 3 & 3 & 3 & 0 & 0 & 0 \\
1 & \overline{\alpha} & \alpha & 1 & 1 & \alpha & \alpha & \overline{\alpha} & \overline{\alpha} & 1 & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} \\
4 & 4 & 4 & 2\alpha-\overline{\alpha} & -\alpha+2\overline{\alpha} & 2\alpha-\overline{\alpha} & -\alpha+2\overline{\alpha} & 2\alpha-\overline{\alpha} & -\alpha+2\overline{\alpha} & -2 & -2 & -2 & 1 & 1 & 1 \\
4 & 4\overline{\alpha} & 4\alpha & -\alpha+2\overline{\alpha} & 2\alpha-\overline{\alpha} & -2\alpha-3\overline{\alpha} & \alpha+3\overline{\alpha} & 3\alpha+\overline{\alpha} & -3\alpha-2\overline{\alpha} & -2 & -2\alpha & -2\overline{\alpha} & 1 & \alpha & \overline{\alpha} \\
4 & 4\alpha & 4\overline{\alpha} & 2\alpha-\overline{\alpha} & -\alpha+2\overline{\alpha} & -3\alpha-2\overline{\alpha} & 3\alpha+\overline{\alpha} & \alpha+3\overline{\alpha} & -2\alpha-3\overline{\alpha} & -2 & -2\overline{\alpha} & -2\alpha & 1 & \overline{\alpha} & \alpha \\
1 & \alpha & \overline{\alpha} & 1 & 1 & \overline{\alpha} & \overline{\alpha} & \alpha & \alpha & 1 & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha \\
4 & 4 & 4 & -\alpha+2\overline{\alpha} & 2\alpha-\overline{\alpha} & -\alpha+2\overline{\alpha} & 2\alpha-\overline{\alpha} & -\alpha+2\overline{\alpha} & 2\alpha-\overline{\alpha} & -2 & -2 & -2 & 1 & 1 & 1 \\
4 & 4\overline{\alpha} & 4\alpha & 2\alpha-\overline{\alpha} & -\alpha+2\overline{\alpha} & \alpha+3\overline{\alpha} & -2\alpha-3\overline{\alpha} & -3\alpha-2\overline{\alpha} & 3\alpha+\overline{\alpha} & -2 & -2\alpha & -2\overline{\alpha} & 1 & \alpha & \overline{\alpha} \\
4 & 4\alpha & 4\overline{\alpha} & -\alpha+2\overline{\alpha} & 2\alpha-\overline{\alpha} & 3\alpha+\overline{\alpha} & -3\alpha-2\overline{\alpha} & -2\alpha-3\overline{\alpha} & \alpha+3\overline{\alpha} & -2 & -2\overline{\alpha} & -2\alpha & 1 & \overline{\alpha} & \alpha \\
12 & 12 & 12 & 3 & 3 & 3 & 3 & 3 & 3 & 0 & 0 & 0 & -3 & -3 & -3
\end{pmatrix},
$$

$$
M(3B)=\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2\overline{\alpha} & 2\alpha & -\alpha & -1 & -\overline{\alpha} \\
2 & 2 & 2 & -1 & -1 & -1 \\
2 & 2\alpha & 2\overline{\alpha} & -\overline{\alpha} & -1 & -\alpha \\
1 & \alpha & \overline{\alpha} & \overline{\alpha} & 1 & \alpha \\
1 & \overline{\alpha} & \alpha & \alpha & 1 & \overline{\alpha}
\end{pmatrix}, \qquad
M(4A)=\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} & \alpha & \alpha & \alpha \\
1 & 1 & 1 & \alpha & \alpha & \alpha & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} \\
1 & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha \\
1 & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} \\
1 & \alpha & \overline{\alpha} & \overline{\alpha} & 1 & \alpha & \alpha & \overline{\alpha} & 1 \\
1 & \overline{\alpha} & \alpha & \alpha & 1 & \overline{\alpha} & \overline{\alpha} & \alpha & 1 \\
1 & \alpha & \overline{\alpha} & \alpha & \overline{\alpha} & 1 & \overline{\alpha} & 1 & \alpha \\
1 & \overline{\alpha} & \alpha & \overline{\alpha} & \alpha & 1 & \alpha & 1 & \overline{\alpha}
\end{pmatrix},
$$

$$M(4B)=\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
3 & 3\overline{\alpha} & 3\alpha & -\alpha-2\overline{\alpha} & -\alpha+\overline{\alpha} & 2\alpha+\overline{\alpha} & -2\alpha-\overline{\alpha} & \alpha+2\overline{\alpha} & \alpha-\overline{\alpha} & 0\\
3 & 3\alpha & 3\overline{\alpha} & -2\alpha-\overline{\alpha} & \alpha-\overline{\alpha} & \alpha+2\overline{\alpha} & -\alpha-2\overline{\alpha} & 2\alpha+\overline{\alpha} & -\alpha+\overline{\alpha} & 0\\
3 & 3\overline{\alpha} & 3\alpha & 2\alpha+\overline{\alpha} & -\alpha-2\overline{\alpha} & -\alpha+\overline{\alpha} & \alpha+2\overline{\alpha} & \alpha-\overline{\alpha} & -2\alpha-\overline{\alpha} & 0\\
3 & 3\alpha & 3\overline{\alpha} & \alpha+2\overline{\alpha} & -2\alpha-\overline{\alpha} & \alpha-\overline{\alpha} & 2\alpha+\overline{\alpha} & -\alpha+\overline{\alpha} & -\alpha-2\overline{\alpha} & 0\\
1 & 1 & 1 & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} & \alpha & \alpha & \alpha & 1\\
1 & 1 & 1 & \alpha & \alpha & \alpha & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} & 1\\
3 & 3\overline{\alpha} & 3\alpha & -\alpha+\overline{\alpha} & 2\alpha+\overline{\alpha} & -\alpha-2\overline{\alpha} & \alpha-\overline{\alpha} & -2\alpha-\overline{\alpha} & \alpha+2\overline{\alpha} & 0\\
3 & 3\alpha & 3\overline{\alpha} & \alpha-\overline{\alpha} & \alpha+2\overline{\alpha} & -2\alpha-\overline{\alpha} & -\alpha+\overline{\alpha} & -\alpha-2\overline{\alpha} & 2\alpha+\overline{\alpha} & 0\\
6 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & -3
\end{pmatrix},$$

$$M(6A)=\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\\
2 & 2\overline{\alpha} & 2\alpha & 2\alpha & 2 & 2\overline{\alpha} & 2\overline{\alpha} & 2\alpha & 2 & -\overline{\alpha} & -\alpha & -\alpha & -1 & -1 & -\overline{\alpha} & -1 & -\overline{\alpha} & -\alpha\\
2 & 2\alpha & 2\overline{\alpha} & 2\overline{\alpha} & 2 & 2\alpha & 2\alpha & 2\overline{\alpha} & 2 & -\alpha & -\overline{\alpha} & -\overline{\alpha} & -1 & -1 & -\alpha & -1 & -\alpha & -\overline{\alpha}\\
1 & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} & 1 & 1 & \alpha & \alpha & \overline{\alpha} & \overline{\alpha} & 1 & \alpha & \overline{\alpha}\\
1 & 1 & 1 & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} & \alpha & \alpha & \alpha & \alpha & \overline{\alpha} & \alpha & \overline{\alpha} & \alpha & \overline{\alpha} & 1 & 1 & 1\\
1 & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha & 1 & 1 & \overline{\alpha} & \overline{\alpha} & \alpha & \alpha & 1 & \overline{\alpha} & \alpha\\
1 & 1 & 1 & \alpha & \alpha & \alpha & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} & \overline{\alpha} & \alpha & \overline{\alpha} & \alpha & \overline{\alpha} & \alpha & 1 & 1 & 1\\
1 & \alpha & \overline{\alpha} & \alpha & \overline{\alpha} & 1 & \overline{\alpha} & 1 & \alpha & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha & 1 & 1 & \alpha & \overline{\alpha}\\
1 & \overline{\alpha} & \alpha & \overline{\alpha} & \alpha & 1 & \alpha & 1 & \overline{\alpha} & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} & 1 & 1 & \overline{\alpha} & \alpha\\
1 & \overline{\alpha} & \alpha & \alpha & 1 & \overline{\alpha} & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha & \alpha & 1 & 1 & \overline{\alpha} & 1 & \overline{\alpha} & \alpha\\
2 & 2\overline{\alpha} & 2\alpha & 2\overline{\alpha} & 2\alpha & 2 & 2\alpha & 2 & 2\overline{\alpha} & -\alpha & -\overline{\alpha} & -1 & -\alpha & -\overline{\alpha} & -1 & -1 & -\overline{\alpha} & -\alpha\\
2 & 2 & 2 & 2\alpha & 2\alpha & 2\alpha & 2\overline{\alpha} & 2\overline{\alpha} & 2\overline{\alpha} & -\overline{\alpha} & -\alpha & -\overline{\alpha} & -\alpha & -\overline{\alpha} & -\alpha & -1 & -1 & -1\\
2 & 2 & 2 & 2\overline{\alpha} & 2\overline{\alpha} & 2\overline{\alpha} & 2\alpha & 2\alpha & 2\alpha & -\alpha & -\overline{\alpha} & -\alpha & -\overline{\alpha} & -\alpha & -\overline{\alpha} & -1 & -1 & -1\\
1 & \alpha & \overline{\alpha} & \overline{\alpha} & 1 & \alpha & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha} & \overline{\alpha} & 1 & 1 & \alpha & 1 & \alpha & \overline{\alpha}\\
2 & 2\alpha & 2\overline{\alpha} & 2\alpha & 2\overline{\alpha} & 2 & 2\overline{\alpha} & 2 & 2\alpha & -\overline{\alpha} & -\alpha & -1 & -\overline{\alpha} & -\alpha & -1 & -1 & -\alpha & -\overline{\alpha}\\
2 & 2\alpha & 2\overline{\alpha} & 2 & 2\alpha & 2\overline{\alpha} & 2 & 2\alpha & 2\overline{\alpha} & -1 & -1 & -\alpha & -\alpha & -\overline{\alpha} & -\overline{\alpha} & -1 & -\alpha & -\overline{\alpha}\\
2 & 2\overline{\alpha} & 2\alpha & 2 & 2\overline{\alpha} & 2\alpha & 2 & 2\overline{\alpha} & 2\alpha & -1 & -1 & -\overline{\alpha} & -\overline{\alpha} & -\alpha & -\alpha & -1 & -\overline{\alpha} & -\alpha\\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1
\end{pmatrix},$$

$$M(5A)=\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1\\
1 & \overline{\alpha} & \alpha & \alpha & 1 & \overline{\alpha}\\
1 & \alpha & \overline{\alpha} & \overline{\alpha} & 1 & \alpha\\
2 & 2\overline{\alpha} & 2\alpha & -\alpha & -1 & -\overline{\alpha}\\
2 & 2\alpha & 2\overline{\alpha} & -\overline{\alpha} & -1 & -\alpha\\
2 & 2 & 2 & -1 & -1 & -1
\end{pmatrix},\qquad
M(6B)=\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1\\
2 & 2\overline{\alpha} & 2\alpha & -1 & -\overline{\alpha} & -\alpha\\
2 & 2\alpha & 2\overline{\alpha} & -1 & -\alpha & -\overline{\alpha}\\
2 & 2 & 2 & -1 & -1 & -1\\
1 & \overline{\alpha} & \alpha & 1 & \overline{\alpha} & \alpha\\
1 & \alpha & \overline{\alpha} & 1 & \alpha & \overline{\alpha}
\end{pmatrix},$$

$$M(6C)=\begin{pmatrix}
1 & 1 & 1\\
1 & \alpha & \overline{\alpha}\\
1 & \overline{\alpha} & \alpha
\end{pmatrix},\qquad M(7A)=\begin{pmatrix}1\end{pmatrix},\qquad
M(10A)=\begin{pmatrix}
1 & 1 & 1\\
1 & \alpha & \overline{\alpha}\\
1 & \overline{\alpha} & \alpha
\end{pmatrix},\qquad
M(12A)=\begin{pmatrix}
1 & 1 & 1\\
1 & \overline{\alpha} & \alpha\\
1 & \alpha & \overline{\alpha}
\end{pmatrix},$$

where $\alpha=\dfrac{-1+\sqrt{-3}}{2}$.

7. The Character Table of $\overline{G}$

Having obtained the fusion maps of classes of the inertia factors H_i into G, the character tables of the inertia factors H_i which are stored in the GAP library and the Fischer-Clifford matrices of $\overline{G}$, we proceed to construct the ordinary character table of $\overline{G}=3^6{:}S_7$. A partial character table (see Figure 8) of $\overline{G}$ is constructed by multiplying the columns $C_i(g)$ for $i\in\{1,2,3,\dots,12\}$ of the ordinary character tables of the inertia factors H_i associated with the classes of H_i fusing into $[g]$ by the rows of the Fischer-Clifford matrices in a block $M_i(g)$.

$$\begin{pmatrix}
C_1(g)\,M_1(g)\\
C_2(g)\,M_2(g)\\
C_3(g)\,M_3(g)\\
\vdots\\
C_{12}(g)\,M_{12}(g)
\end{pmatrix}$$

Figure 8: Partial character table of $\overline{G}$

To construct the set $\mathrm{Irr}(G)$, we append the partial character tables coming from each class $[g]$ of G as depicted in Figure 9.

$$\left(\begin{array}{c|c|c|c} C_1(g_1)\,M_1(g_1) & C_1(g_2)\,M_1(g_2) & \cdots & C_1(g_{15})\,M_1(g_{15}) \\ C_2(g_1)\,M_2(g_1) & C_2(g_2)\,M_2(g_2) & \cdots & C_2(g_{15})\,M_2(g_{15}) \\ C_3(g_1)\,M_3(g_1) & C_3(g_2)\,M_3(g_2) & \cdots & C_3(g_{15})\,M_3(g_{15}) \\ \vdots & \vdots & \vdots & \vdots \\ C_{12}(g_1)\,M_{12}(g_1) & C_{12}(g_2)\,M_{12}(g_2) & \cdots & C_{12}(g_{15})\,M_{12}(g_{15}) \end{array}\right)$$

Figure 9: Structure of $\mathrm{Irr}(\overline{G})$

The character table of $\overline{G}$ obtained in this manner is a 143×143-complex valued square matrix with the irreducible characters partitioned into blocks Δ_i, for $i \in \{1, 2, 3, \ldots, 12\}$, such that $\Delta_1 = \{\chi_i | 1 \leq i \leq 15\}$, $\Delta_2 = \{\chi_i | 16 \leq i \leq 25\}$, $\Delta_3 = \{\chi_i | 26 \leq i \leq 35\}$, $\Delta_4 = \{\chi_i | 36 \leq i \leq 49\}$, $\Delta_5 = \{\chi_i | 50 \leq i \leq 63\}$, $\Delta_6 = \{\chi_i | 64 \leq i \leq 75\}$, $\Delta_7 = \{\chi_i | 76 \leq i \leq 90\}$, $\Delta_8 = \{\chi_i | 90 \leq i \leq 99\}$, $\Delta_9 = \{\chi_i | 100 \leq i \leq 114\}$, $\Delta_{10} = \{\chi_i | 115 \leq i \leq 125\}$, $\Delta_{11} = \{\chi_i | 126 \leq i \leq 136\}$ and $\Delta_{12} = \{\chi_i | 137 \leq i \leq 143\}$ where $\chi_i \in \mathrm{Irr}(\overline{G})$. The fusion of the classes of the character table obtained into the parent group is obtained with the aid of GAP and captured in Table 2.

A GAP routine in [15] was implemented for checking the consistency and accuracy of the character table of $\overline{G}$. Since the character table of this group is dominated by complex entries, the short GAP routine below was written to diagnose areas of concern whenever the error message "irreducible characters inconsistent with the centralizer orders" arose during its testing in GAP. The routine computes the sizes of the centralizers of the character table being tested, which are then compared with the centralizer sizes in Table 2.

```
gap> mat:=Irr(ct);;
gap> tmat:=TransposedMat(mat);;
gap> CO:=[];;mult:=[];;
gap> for i in [1..Size(tmat)] do
> t:=tmat[i];
> for g in [1..Size(t)] do
> s:=t[g]*ComplexConjugate(t[g]);
> Add(mult,s);
> od; od;
gap> L:=List([0..Size(tmat)-1],n->mult{[Size(tmat)*n+1..Size(tmat)*(n+1)]});;
gap> for l in L do
> s2:=Sum(l);
> Add(CO,s2);
> od;
gap> CO;
```

Readers are referred to the links below for a PDF document containing the full character table of $\overline{G}$ and also a latex file containing the class orders, centralizer orders and the irreducible characters of $\overline{G}$ for self testing in GAP.

https://drive.google.com/file/d/1KVRWyImmXVcLTqIcnvcQszUVsC5Z0BOA/view?usp=share_link

https://drive.google.com/file/d/1dq0vcm09hZKIPCLibIaCgQKFHW4X2G1m/view?usp=share_link

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References

[1] M. S. O. Almestady, *Fischer Matrices for Generalised Symmetric Groups–A Combinatorial Approach*, PhD thesis, University of Wales, Aberystwyth, 1998.

[2] A. B. M. Basheer and J. Moori, *A survey on Clifford-Fischer theory*, in Groups St Andrews 2013, vol. 422 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2015, pp. 160–172.

[3] J. N. Bray, D. F. Holt, and C. M. Roney-Dougal, *The maximal subgroups of the low-dimensional finite classical groups*, vol. 407 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 2013. With a foreword by Martin Liebeck.

[4] C. Chileshe, *Irreducible characters of Sylow p-Subgroups associated with some classical linear groups*, PhD thesis, North-West University, South Africa, 2016.

[5] C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Pure and Applied Mathematics, Vol. XI, Interscience Publishers (a division of John Wiley & Sons, Inc.), New York-London, 1962.

[6] B. Fischer, *Clifford-matrices*, in Representation theory of finite groups and finite-dimensional algebras (Bielefeld, 1991), vol. 95 of Progr. Math., Birkhäuser, Basel, 1991, pp. 1–16.

[7] GAP Group, *GAP – Groups, Algorithms, and Programming, Version 4.10.2*, 2022. Version 4.12.2.

[8] D. Gorenstein, *Finite groups*, Harper & Row, Publishers, New York-London, 1968.

[9] P. Kleidman and M. Liebeck, *The subgroup structure of the finite classical groups*, vol. 129 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1990.

[10] R. J. List, *On the characters of* $2^{n-\epsilon} \cdot S_n$, Arch. Math. (Basel), 51 (1988), pp. 118–124.

[11] R. J. List and I. M. I. Mahmoud, *Fischer matrices for wreath products* $G\,w\,S_n$, Arch. Math. (Basel), 50 (1988), pp. 394–401.

[12] J. Moori and T. Seretlo, *On the Fischer-Clifford matrices of a maximal subgroup of the Lyons group Ly*, Bull. Iranian Math. Soc., 39 (2013), pp. 1037–1052.

[13] J. Moori and T. Seretlo, *On 2 nonsplit extension groups associated with* HS *and* $HS{:}2$, Turkish J. Math., 38 (2014), pp. 60–78.

[14] Z. E. Mpono, *Fischer-Clifford Theory and Character Tables of Group Extensions*, PhD thesis, University of Natal, Pietermaritzburg, 1998.

[15] A. L. Prins, *Fischer-Clifford Matrices and Character Tables of Inertia Groups of Maximal Subgroups of Finite Simple Groups of Extension Type*, PhD thesis, University of the Western Cape, 2011.

[16] ———, *The character table of an involution centralizer in the Dempwolff group* $2^{5\cdot}GL_5(2)$, Quaest. Math., 39 (2016), pp. 561–576.

[17] ———, *A maximal subgroup* $2^{4+6}{:}(A_5 \times 3)$ *of* $G_2(4)$ *treated as a non-split extension* $\overline{G} = 2^{6\cdot}(2^4{:}(A_5 \times 3))$, Adv. Group Theory Appl., 10 (2020), pp. 43–66.

[18] A. L. Prins, R. L. Monaledi, and R. L. Fray, *On a maximal subgroup* $2^9{:}(L_3(4)){:}3$ *of the automorphism group* $U_6(2){:}3$ *of* $U_6(2)$, Afr. Mat., 31 (2020), pp. 1311–1336.

[19] R. P. Stanley, *Enumerative combinatorics. Volume 1*, vol. 49 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, second ed., 2012.

[20] N. S. Whitley, *Fischer matrices and character tables of group extensions*, Master's thesis, 1994.

[21] R. Wilson, P. Walsh, J. Tripp, I. Suleiman, R. Parker, S. Norton, S. Nickerson, S. Linton, J. Bray, and R. Abbott, Atlas *of finite group representation.* http://brauer.maths.qmul.ac.uk/Atlas/v3/. http://brauer.maths.qmul.ac.uk/Atlas/v3/.

[22] K. Zimba, *Fischer-Clifford Matrices of the Generalized Symmetric Group and some Associated Groups*, PhD thesis, University of KwaZulu-Natal, Pietermaritzburg, 2005.

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