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Original Article

A note on algebraic commutators in division rings with uncountable center

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ABSTRACT: Let D be a division ring with uncountable center C. Suppose that K is a sub-division ring of D containing C and that $a \in D \setminus C$. The purpose of this paper is to prove that if either $axa^{-1}x^{-1}$ or $xy - yx$ is right algebraic over K for all $x, y \in D \setminus \{0\}$, then D is also right algebraic over K. This result provides the affirmative answers to [8, Problems 1 and 5] for division rings with uncountable center.

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1. Introduction

In this paper, we are interested in the following two conjectures.

Conjecture 1.1 ([8, Problem 1]). *Let D be a division ring with center C. If every additive commutator $xy - yx$ is algebraic over C for every elements x and y in D, then D is algebraic, that is, every element in D is algebraic over C.*

Conjecture 1.2 ([8, Problem 3]). *Let D be a division ring with center C and $a \in D \setminus C$. If every multiplicative commutator $axa^{-1}x^{-1}$ is algebraic over C for all $x \in D \setminus \{0\}$, then D is algebraic.*

We will prove that if the center C is uncountable, then both of the above conjectures are true. In fact, we are particularly interested in one-sided algebraicity over sub-division rings. To provide clarity, we briefly recall some fundamental concepts.

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Let D be a division ring. Assume that C is the center of D and K is a sub-division ring of D. Let $f(t)$ be a polynomial in the variable t over D. For $a \in D$, the value $f(a)$ is understood as the result of substituting t with a. Consequently, the computed value may vary depending on how the coefficients are written. For example, suppose a and b are two elements in D such that $ab \neq ba$. Consider the "two" polynomials $f(t) = bt - ab$ and $g(t) = tb - ab$ over the division ring D. As polynomials, we have the identity $at - ab = ta - ab$, but evaluating at a gives

$$f(a) = ba - ab \neq 0 = ab - ab = g(a).$$

Polynomials over division rings are discussed in more detail in [11, Chapter 5]. This leads to the observation that algebraicity in noncommutative division rings lacks symmetry. Naturally, this gives rise to the concepts of left algebraicity and right algebraicity, defined as follows.

Let a be an element in D. We say that a is *left algebraic* (respectively, *right algebraic*) over K if there exist $\alpha_0, \alpha_1, \ldots, \alpha_n \in K$, not all zero, such that

$$\alpha_0 + \alpha_1 a + \cdots + \alpha_n a^n = 0,$$

(respectively, $\alpha_0 + a\alpha_1 + \cdots + a^n\alpha_n = 0$). In other words, a is left algebraic (respectively, right algebraic) over K if there exists a nonzero polynomial $f(t) \in K[t]$ whose coefficients are written on the left (respectively, on the right) such that $f(a) = 0$. Remark that if $a \neq 0$, then without loss of generality, we can choose $a_0 \neq 0$.

We call D *left algebraic* (respectively, *right algebraic*) over K if every element in D is left algebraic (respectively, right algebraic) over K. If $K \subseteq C$, then right algebraicity and left algebraicity over K are the same. In this case, we briefly call D *algebraic*. However, if K is not central, that is, $K \nsubseteq C$, one can find division rings $K \subseteq D$ such that D is left algebraic but D is not right algebraic over K. Notably, for certain division rings $K \subseteq D$ constructed by Cohn [9, Page 548], it can be shown that D is left algebraic and D is not right algebraic over K. The study of one-sided algebraicity in division rings has recently attracted significant attention (see, for example, [1, 4, 5, 6, 7, 10, 12]).

Our main goal is to address Conjectures 1.1 and 1.2 for division rings with uncountable center.

2. Results

We begin with the following important result. The idea for the proof is based on [3] and [10]. For a division ring D and a sub-division ring K, we can consider D as both a left and a right vector space over K. Hence, the element $a \in D$ is left algebraic (respectively, right algebraic) over K if and only if the set $\{a^n : n \in \mathbb{N}\}$ is left (respectively, right) linear dependent over K. Of course, in case K is contained in the center of D, then right vector space and the left vector space are the same.

Lemma 2.1. *Let D be a division ring with center C. Assume that K is a sub-division ring of D containing C and $a \in D \setminus C$. Then either a is left algebraic (respectively, right algebraic) over K, or the set*

$$\{(a - \alpha)^{-1} : \alpha \in C\},$$

is left (respectively, right) linearly independent over K.

Proof. We prove the result for left algebraicity; the argument for right algebraicity is entirely analogous.

Suppose that the element a is not left algebraic over K. Since $a \notin C$, the element $a - \alpha$ is non-zero for every $\alpha \in C$. We must show that the set

$$\{(a - \alpha)^{-1} : \alpha \in C\},$$

is left linearly independent over K.

Suppose that

$$\beta_1(a - \alpha_1)^{-1} + \beta_2(a - \alpha_2)^{-1} + \cdots + \beta_n(a - \alpha_n)^{-1} = 0, \tag{1}$$

for some a positive integer n, $\beta_1, \beta_2, \cdots, \beta_n \in K$ and distinct $\alpha_1, \alpha_2, \ldots, \alpha_n \in C$. Consider the polynomial

$$f(t) = (t - \alpha_1)(t - \alpha_2)\ldots(t - \alpha_n),$$

in $C[t] \subseteq K[t]$ and for every $1 \leq i \leq n$, put $f_i(t) = \dfrac{f(t)}{t - \alpha_i}$. By multiplying the both sides of (1) on the right by $f(a)$, one has

$$\beta_1 f_1(a) + \beta_2 f_2(a) + \cdots + \beta_n f_n(a) = 0.$$

Hence, a is a solution of the polynomial

$$\beta_1 f_1(t) + \beta_2 f_2(t) + \cdots + \beta_n f_n(t).$$

This implies that if $g(t) = \beta_1 f_1(t) + \beta_2 f_2(t) + \cdots + \beta_n f_n(t)$, then $g(t)$ is the zero polynomial, as a is not left algebraic over K. Therefore, for each $1 \leq i \leq n$, we have

$$0 = g(\alpha_i) = \beta_i f_i(\alpha_i),$$

which implies that $\beta_i = 0$ for all $1 \leq i \leq n$. Thus, the set

$$\{(a - \alpha)^{-1} : \alpha \in C\},$$

is left linearly independent over K. □

We denote $D^* = D \setminus \{0\}$ as the group of invertible elements of the division ring D.

Based on the preceding lemma, we obtain the following result, which serves as the key lemma of this paper.

Lemma 2.2. *Let D be a division ring. Assume that the center C of D is uncountable and K is a sub-division ring of D containing C. If x and y are two elements in D^* such that $x(y + \alpha)$ is left algebraic (respectively, $(y + \alpha)x$ is right algebraic) over K for all $\alpha \in C$, then, y is left algebraic (respectively, right algebraic) over K.*

Proof. We will show the lemma for the left case; the right case follows similarly.

If $y \in C \subseteq K$, then it is evident that y is left algebraic over K. Therefore, we now assume that $y \notin C$. For $\alpha \in C$, since $x(y + \alpha)$ is left algebraic over K, there exist a positive integer n and $\alpha_0, \alpha_1, \ldots, \alpha_n \in K$, not all zero, such that

$$\alpha_0 + \alpha_1 x(y + \alpha) + \cdots + \alpha_n (x(y + \alpha))^n = 0.$$

Since $y \notin C$, one has $x(y + \alpha) \neq 0$. According to a previous remark, without loss of generality, suppose $\alpha_0 \neq 0$. We then have

$$-\alpha_0^{-1}(\alpha_1 x(y + \alpha) + \cdots + \alpha_n (x(y + \alpha))^n) = 1.$$

This implies that

$$h(x, y)(y + \alpha) = 1,$$

where $h(t_1, t_2)$ is an element of the free algebra $K[t_1, t_2]$ generated by two noncommutating indeterminates t_1 and t_2. Consequently,

$$(y + \alpha)^{-1} = h(x, y),$$

lies in the left vector space over K spanned by all the images $w(x, y)$ of words $w(t_1, t_2)$ from the monoid $[t_1, t_2]$ generated by t_1, t_2. Since $[t_1, t_2]$ is countable, the algebra $K[t_1, t_2]$ is a left vector space over K with countable dimension, which implies that the left vector subspace of D spanned by $\{(y+\alpha)^{-1} : \alpha \in C\}$ is of countable dimension over K. On the other hand, since C is uncountably, so is $\{(y+\alpha)^{-1} : \alpha \in C\}$. Therefore, $\{(y+\alpha)^{-1} : \alpha \in C\}$ must be left linearly dependent over K. By Lemma 2.1, y is left algebraic over K. □

Now, we will prove the first main theorem. The proof idea is taken from [2], but we will need to modify several parts.

Theorem 2.3. *Let D be a division ring with uncountable center C. Assume that K is a sub-division ring of D containing C. If every additive commutator $xy - yx$ is left algebraic (respectively, right algebraic) over K for every two elements x and y in D, then is so D.*

Proof. Let $a \in D$. Since $C \subseteq K$, it is obvious that if $a \in C$, then a is left and right algebraic over K, so we assume that $a \in D \setminus C$. We must show that if every additive commutator $xy - yx$ is left algebraic (respectively, right algebraic) over K for every two elements x and y in D, then is so a.
Because $a \notin C$, there exists an element b in D such that $ab \neq ba$. Put $u = ab - ba$. We consider two cases.

Case 1. *$xy - yx$ is left algebraic over K for every elements x and y in D.* For every $c \in D$ such that $ac = ca$, one has $uc = a(bc) - (bc)a$ which is left algebraic over K. Hence,

$$u(c + \alpha) = a(b(c + \alpha)) - (b(c + \alpha))a,$$

is also left algebraic over K for all $\alpha \in C$. By applying Lemma 2.2, we conclude that c is left algebraic over K. Therefore, every element that commutes with a is left algebraic over K. In particular, a itself is left algebraic over K.

Case 2. *$xy - yx$ is right algebraic over K for every elements x and y in D.* For every $c \in D$ such that $ac = ca$, one has $a(cb) - (cb)a = cu$ is right algebraic over K. Using the same arguments as in Case 1, one has c is right algebaic over K. In particular, a is right algebraic over K.

The proof is complete. □

The following corollary provides the answer of Conjecture 1.1 for the case of division rings with uncountable center.

Corollary 2.4. *Let D be a division ring. If the center C of D is uncountable and every additive commutator $xy - yx$ is algebraic over C for all $x, y \in D$, then D is algebraic.*

Proof. The corollary follows directly from Theorem 2.3. □

Now, we turn to the multiplicative commutator. It is important to note that if a non-zero element a in D is left algebraic (respectively, right algebraic) over K, then a^{-1} is also left algebraic (respectively, right algebraic) over K. To see this, we assume that a is left algebraic over K. Then, there exist a positive integer n and $\alpha_0, \alpha_1, \dots, \alpha_n \in K$, not all zero, such that

$$\alpha_0 + \alpha_1 a + \cdots + \alpha_n a^n = 0.$$

Then, multiplying both sides on the right with a^{-n}, one has

$$\alpha_0 (a^{-1})^n + \alpha_1 (a^{-1})^{n-1} + \cdots + \alpha_n = 0,$$

which implies that a^{-1} is left algebraic over K.

This observation will be used in the reasoning below.

Theorem 2.5. *Let D be a division ring. Assume that the center C of D is uncountable, K is a sub-division ring of D containing C and $a \in D \setminus C$. If every multiplicative commutator $axa^{-1}x^{-1}$ is right algebraic (respectively, $a^{-1}x^{-1}ax$ is left algebraic) over K for each $x \in D^*$, then D is also right algebraic (respectively, left algebraic) over K.*

Proof. Assume that $axa^{-1}x^{-1}$ is right algebraic over K for every $x \in D^*$. Let $b \in D$. We must show b is right algebraic over K. Put $d = ab - ba$. Then, for every $\alpha \in C$,

$$\begin{aligned} d = ab - ba &= a(b+\alpha) - (b+\alpha)a \\ &= (a(b+\alpha)a^{-1}(b+\alpha)^{-1} - 1)(b+\alpha)a \\ &= (c-1)(b+\alpha)a, \end{aligned}$$

where $c = a(b+\alpha)a^{-1}(b+\alpha)^{-1}$.

Case 1. $ab \neq ba$. Then, d and $c - 1$ are different from 0. Since c is right algebraic over K, obviously so is the element $c - 1$, which implies that

$$(c-1)^{-1} = (b+\alpha)ad^{-1},$$

is right algebraic over K for every $\alpha \in C$. According to Lemma 2.2, b is right algebraic over K.

Case 2. $ab = ba$. Since a is noncentral, one has there exists $c \in D$ such that $ca \neq ac$. Moreover, for every $\alpha \in C$, one has $a(b+\alpha) = (b+\alpha)a$, which implies that $a((b+\alpha)c) \neq ((b+\alpha)c)a$. According to Case 1, $(b+\alpha)c$ is right algebraic over K. Using Lemma 2.2 again, b is right algebraic over K.

Two cases lead b is right algebraic over K. Thus, D is right algebraic over K.
The case when $a^{-1}x^{-1}ax$ is left algebraic over K for each $x \in D^*$ is similar. To see this, firstly, for every $x \in D^*$, since $a^{-1}x^{-1}ax$ is left algebraic over K and $x^{-1}a^{-1}xa = (a^{-1}x^{-1}ax)^{-1}$, one has the element $x^{-1}a^{-1}xa$ is also left algebraic over K. Secondly, for every $b \in D$ and $\alpha \in C$, we have

$$\begin{aligned} d = ab - ba &= a(b+\alpha) - (b+\alpha)a \\ &= a(b+\alpha)(1 - (b+\alpha)^{-1}a^{-1}(b+\alpha)a) \\ &= a(b+\alpha)(1-c), \end{aligned}$$

where $c = (b+\alpha)^{-1}a^{-1}(b+\alpha)a$. Now using the same arguments as the right case, we conclude that b is left algebraic over K for every $b \in D$. □

The following result provides the answer of Conjecture 1.2 in the case of division rings whose centers are uncountable.

Corollary 2.6. *Let D be a division ring. Suppose that a is noncentral element in D. If the center C of D is uncountable and every multiplicative commutator $axa^{-1}x^{-1}$ is algebraic over C for every $x \in D^*$, then D is algebraic.*

Proof. The corollary follows directly from Theorem 2.5. □

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