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Original Article

On compact pseudo-Riemannian manifolds admitting Killing vector fields

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ABSTRACT: We prove some theorems about the Killing vector fields on compact and connected pseudo-Riemannian manifolds. Among other results, we present a relationship between curvature of a compact pseudo-Riemannian manifold M and existence of Killing vector fields on M.

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1. Introduction

If M is a compact and connected pseudo-Riemannian manifold admitting Killing vector fields (smooth vector fields such that their flows are isometries), then its topology and geometry will strongly be restricted. There are many interesting results about the Killing vector fields on Riemannian manifolds in the literature [4, 8, 14, 15]. The causal character of the Killing vector fields on pseudo-Riemannian manifolds plays important role [1, 3, 5, 6, 7, 10, 11, 12, 13]. It is proved in [5] that a compact Lorentzian manifold that admits a timelike Killing vector field, must be geodesically complete. A. Romero and M. Sanchez in [12] prove that a connected pseudo-Riemannian manifold M^n_ν is complete if it admits a collection of $\{K_1, \dots, K_\nu\}$ of conformal vector fields satisfying some conditions, which implies the completeness of a compact pseudo-Riemannian manifold M^n_ν admitting a collection $\{K_1, \dots, K_\nu\}$ of linearly independent timelike Killing vector fields. We provide another direct and short proof of this result. In the second part of the article, we study the zero points of the Killing vector fields on pseudo-Riemannian manifolds. Study of the zero points for Killing vector fields is important because of their applications in relativity (Lorentzian case) and from the geometric point of view, it is important because existence of the zero points imposes global restrictions on K and M. There are many results about the zero points of Killing vector fields on Riemannian

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manifolds. Also, one can find interesting results about the zero points of Killing vector fields on Lorentzian manifolds (see [3]). But, in the more general case where the manifold is pseudo-Riemannian, there are many open problems on relationships between existence of Killing vector fields with zero points on a manifold and its geometric properties.

Following and inspired by the previous mentioned papers, we present some new results as follows:
Let M^n_ν be a connected pseudo-Riemannian manifold and K be a Killing vector field on M.

We show in Theorem 4.3 that n is even if K has an isolated zero point. If n is odd, each zero point of K belongs to a geodesic γ such that the restriction of K on γ is a zero vector field. If M is compact, we prove in Theorem 4.5 that the signature of $(n-2\nu)$ and the causal character of K are closely related to existence of the zero points for K. As an application of our theorems, we show in Theorem 5.2 that on a compact and connected Lorentzian manifold M^n_1, if there is a non-spacelike and non-trivial Killing vector field, then the timelike sectional curvature of M is not positive, and if there is a spacelike Killing vector field, then the spacelike curvature of M is not negative.

2. Preliminaries

We refer to [2] and [9] for preliminaries. Throughout the paper, (M^n_ν, g) is a (connected) pseudo-Riemannian manifold, ∇ and R are respectively its Levi-Civita connection and curvature tensor. Recall that a pseudo-Riemannian manifold (M^n_ν, g) is by definition a differentiable manifold M of dimension n equipped with a symmetric nondegenerate $(0,2)$ tensor field g of constant index ν, $0 \leq \nu < n$, i.e., g assigns to each point $p \in M$ a scalar product g_p on T_pM such that the index of g_p is equal to ν for all $p \in M$. We will use $\langle , \rangle$ as an alternative notation for g. If the index of M is one ($\nu = 1$), M is called a Lorentzian manifold and the corresponding metric is called Lorentzian. A tangent vector v of a pseudo-Riemannian manifold M is called spacelike (resp. timelike) if $v = 0$ or $g(v, v) > 0$ (resp. $g(v, v) < 0$). A vector v is called null if v is not zero and $g(v, v) = 0$.

Consider a two dimensional nondegenerate vector subspace P of $T_pM^n_\nu$. P is called a spacelike plane if it is generated by a basis $\{e_1, e_2\}$ of spacelike vectors. It is called a time plane if e_1 and e_2 are timelike, and P is called timelike plane if one of the vectors e_1, e_2 is timelike and the other one is spacelike. The sectional curvature $\kappa(e_1, e_2)$ is called spacelike curvature, time curvature and timelike curvature, respectively. Note that if M is a Lorentzian manifold, each nondegenerate plane tangent to M is either timelike or spacelike (there is no time plane).

The Hopf-Rinow theorem states that a connected Riemannian manifold is geodesically complete if and only if it is complete as a metric space. There is no analog of the Hopf-Rinow theorem in the pseudo-Riemannian case. So, completeness is defined as geodesic completeness. A pseudo-Riemannian manifold is geodesically complete if every geodesic extends for infinite time. A compact Riemannian manifold is (geodesically) complete. But, contrary to the Riemannian case, a compact pseudo-Riemannian manifold may be geodesically incomplete. This striking fact motivated the search of sufficient assumptions under which compactness implies (geodesically) completeness of such a manifolds. By Theorem 3.3, existence of sufficient number of timelike Killing vector fields implies completeness of a pseudo-Riemannian manifold. Recall that a vector field K is called a Killing vector field if its local flows consist in local isometries of M. It is known that K is Killing iff for all vector fields Y and Z,

$$g(\nabla_Y K, Z) + g(\nabla_Z K, Y) = 0.$$

If γ is a geodesic, the restriction of K to γ is a Jacobi field (see [9], page 251-252). Then,

$$K'' = R(K, \gamma')\gamma'.$$

3. Completeness

The following theorem gives an example of compact pseudo-Riemannian manifolds admitting timelike Killing vector fields.

We recall that if a group H acts on M, then for each $x \in M$, the orbit of H passing from x is $H(x) = \{hx : h \in H\}$. The stabilizer subgroup at x is $H_x = \{h \in H : hx = x\}$.

Theorem 3.1. *Let M be a connected differentiable manifold and H be a compact and connected subgroup of the diffeomorphisms of M. Consider the natural action of H on M and suppose that for all $x \in M$, $H_x = \{e\}$ and let $dimH = \nu$. Then, there is a pseudo-Riemannian metric g of index ν on M such that there are ν linearly independent timelike Killing vector fields on (M, g).*

Proof. If $x \in M$, then the H-orbit containing x, $H(x) = \{hx : h \in H\}$, is diffeomorphic to $\frac{H}{H_x}$. Since by assumption, $H_x = \{e\}$, then $H(x)$ is diffeomorphic to H. Consider a Riemannian metric g_1 on M and let μ be a Harr measure on H. For each $h \in H$, consider the map $h : M \to M$, $h(x) = h.x$, and for each $p \in M$, let $dh : T_pM \to T_{h.p}M$ be the differential of the map h at p. For $v, w \in T_pM$, put

$$g_2(v, w)(x) = \int_H (g_1(dh(v), dh(w))(hx))d\mu(h).$$

g_2 is an H-invariant Riemannian metric on M.
Because, if $h_1 \in H$, $x \in M$ and $y = h_1(x)$, then from the definition of Harr measure, $d\mu(hh_1) = d\mu(h)$. Since $h_1H = H$, then

$$\begin{aligned} g_2(dh_1(v), dh_1(w))(y) &= \int_H (g_1(dh(dh_1(v), dh(dh_1(w))(hy))d\mu(h) \\ &= \int_{h_1H} (g_1(d(hh_1)(v), d(hh_1)(w))(hh_1x))d\mu(hh_1) \\ &= \int_H (g_1(dh(v), dh(w))(hx))d\mu(h) = g_2(v, w)(x). \end{aligned}$$

Thus, H is a closed and connected subgroup of the isometries of (M, g_2). Let e be the identity member of H and $X_1(e), \dots, X_\nu(e)$ be a basis for T_eH. Consider the exponential map $\exp : T_eH \to H$(see [9], page 449), and put $\alpha_i(t) = \exp(tX_i(e))$. For each $x \in M$ put $\beta_i(t) = \alpha_i(t).x$, and let $Y_i(x) = \beta'(0)$. Since $\alpha_i(t) \in H$, then the flows of Y_i are isometries of (M, g_2). Then, Y_i is a Killing vector field on (M, g_2). It is easy to see that $\{Y_1, \dots, Y_\nu\}$ is a collection of linearly independent vector fields. In fact, for each $x \in M$, H is diffeomorphic to $H(x)$ by the diffeomorphism ϕ:

$$\phi : H \to H(x), \quad \phi(h) = h.x.$$

It is clear that $d\phi : T_eH \to T_xH(x)$, maps $X_i(e)$ to $Y_i(x)$. Since $X_1(e), \dots, X_\nu(e)$ are linearly independent, then $Y_1(x), \dots, Y_\nu(x)$ are linearly independent. For each point $x \in M$, let $p : T_xM \to T_x(H(x))$ be the g_2-orthogonal projection of T_xM on to $T_x(H(x))$. For $v, w \in T_xM$, put

$$g(v, w) = -g_2(pv, pw) + g_2(v - p(v), w - p(w)).$$

It is clear that g is nondegenerate on T_xM. If v is a non zero vector tangent to $H(x)$, then $g(v, v) = -g_2(v, v) < 0$ and if it is g_2-normal to $H(x)$, then $g(v, v) = g_2(v, v) > 0$. Thus the index of g is equal to $dim(H) = \nu$ and $Y_1, \dots, Y_\nu$ are timelike vector fields of (M, g). Since H is a subgroup of the isometries of (M, g_2), then for each $h \in H$, $dh(p(v)) = p(dh(v))$. Then,

$$\begin{aligned} g(dh(v), dh(w)) &= -g_2(p(dh(v)), p(dh(w))) + g_2(dh(v) - pdh(v), dh(w) - pdh(w)) \\ &= -g_2(dh(p(v)), dh(p(w))) + g_2(dh(v - p(v)), dh(w - p(w))) \\ &= -g_2(p(v), p(w)) + g_2(v - p(v), w - p(w)) = g(v, w). \end{aligned}$$

Thus, H is a subgroup of the isometries of (M, g). By similar reason as the above, Y_i is a Killing vector field of (M, g). □

The following lemma shows that the behaviour of a Killing vector field K of a pseudo-Riemannain manifold M^n_ν around its zero points, is closely related to the behaviour of a corresponding Killing vector field on R^n_ν.

Corollary 3.2. *Let M^n_ν be a pseudo-Riemannian manifold and K be a Killing vector field on M such that $K(p) = 0$, $p \in M$. Then, there is a Killing vector field $\tilde{K}$ on T_pM with the following property:*

$$\exp_p(q) = b \Rightarrow K(b) = d(\exp_p)_q(\tilde{K}(q)).$$

Proof. If v is a vector in T_pM, for each point $q \in T_pM$ we use the symbol v_q to denote the parallel transformation of v to $T_q(T_pM)$. Consider the local flow ϕ_t of K. $\phi_t : M \to M$ is isometry and $\phi_t(p) = p$. Thus, $d\phi_t : T_pM \to T_pM$ is a linear isometry. For each $q \in T_pM$ put $q' = d\phi_t(q)$. Since $d\phi_t$ is linear, then we have $d_q(d\phi_t) = d\phi_t$. That is:

$$d_q(d\phi_t) : T_q(T_pM) \longrightarrow T_{q'}(T_pM), \quad d_q(d\phi_t)(v_q) = (d\phi_t(v))_{q'}.$$

Consider the vector field $\tilde{K}$ on T_pM with the local flow $d\phi_t$. That is $\tilde{K}(q) = \frac{d}{dt}(d\phi_t(q))_{|t=0}$. Since $d\phi_t$ is isometry of T_pM, then $\tilde{K}$ is a Killing vector field on T_pM. Now, let $\exp_p(q) = b$. We have

$$\exp_p \circ d\phi_t = \phi_t \circ \exp_p .$$

Then,

$$\begin{gathered} \exp_p \circ d\phi_t(q) = \phi_t \circ \exp_p(q) \Rightarrow \\ \frac{d}{dt}(\exp_p \circ d\phi_t(q))_{|t=0} = \frac{d}{dt}(\phi_t \circ \exp_p(q))_{|t=0} \Rightarrow \\ d(\exp_p)((\frac{d}{dt}d\phi_t(q))_{|t=0}) = \frac{d}{dt}(\phi_t(b))_{|t=0} \Rightarrow \\ d(\exp_p)(\tilde{K}(q)) = K(b). \end{gathered}$$

□

It is known that a compact Lorentzian manifold admitting a timelike Killing vector field is geodesically complete (see [5]). The authors of [12] prove that a connected pseudo-Riemannian manifold M^n_ν is complete if it admits a collection of $\{K_1, \ldots, K_\nu\}$ of conformal vector fields satisfying some conditions, which implies the following theorem. We give a more simple proof for this theorem.

Theorem 3.3 ([12]). *If M^n_ν is a compact and connected pseudo-Riemannian manifold with ν linearly independent timelike killing vector fields, then M is geodesically complete.*

Proof. Step 1. Let $K_1, \ldots, K_\nu$ be linearly independent timelike vectors in $\mathbb{R}^n_\nu$ and $c_1, \ldots, c_\nu$ be constant numbers. Let

$$\Omega = \{v \in R^n_\nu : \langle v, v\rangle = -1, \quad \langle v, K_i\rangle = c_i, 1 \le i \le \nu\}.$$

Consider the linear subspace S of $\mathbb{R}^n_\nu$ generated by $K_1(\circ), \ldots, K_\nu(\circ)$ (the vector at the origin) and let $e_1(\circ), \ldots, e_\nu(\circ)$ be an orthonormal basis for S and $e_1, \ldots, e_\nu$ be the vector fields on R^n_ν obtained by translations of the vectors $e_1(\circ), \ldots, e_\nu(\circ)$. Then, for some constant numbers a_{ij}, $e_i(\circ) = \sum_j a_{ij} K_j(\circ)$. Consequently, if $v \in \Omega$, then $\langle v, e_i\rangle = \sum_j a_{ij} c_j$. Thus, $\langle v, e_i\rangle$ is constant, which we denote it by d_i. Consider an orthonormal basis $\{e_{\nu+1}(\circ), \ldots, e_n(\circ)\}$ for $S^\perp$. Then, $\{e_1(\circ), \ldots, , e_\nu(\circ), e_{\nu+1}(\circ), \ldots, e_n(\circ)\}$ is an orthonormal basis for $\mathbb{R}^n_\nu$ ($e_1, \ldots, e_\nu$ timelike, $e_{\nu+1}, \ldots, e_n$ spacelike). If $v = \sum_i v_i e_i \in \Omega$, then we have:

$$\langle v, v\rangle = -1 \Rightarrow -\sum_{i=1}^{\nu} v_i^2 + \sum_{i=\nu+1}^{n} v_i^2 = -1.$$

$$\langle v, e_i\rangle = d_i \Rightarrow v_i = d_i, \quad 1 \le i \le \nu.$$

Thus,

$$\Omega = \{v = (d_1, \ldots, d_\nu, v_{\nu+1}, \ldots, v_n) : \sum_{i=\nu+1}^{n} v_i^2 = -1 + \sum_{i=1}^{\nu} d_i^2\}.$$

$-1 + \sum_{i=1}^{\nu} d_i^2$ is a non-negative constant number which we denote it by c. This means that Ω is homeomorphic to the set $\{(v_{\nu+1}, \ldots, v_n) : \sum_{i=\nu+1}^{n} v_i^2 = c\}$ of $\mathbb{R}^{n-\nu}$. Therefore, Ω is compact.
Step 2. Let $\gamma : I \to M$ be a unite speed timelike geodesic in M. Since K_i is a Killing vector field, $\langle K_i, \gamma'(t)\rangle$ is constant along γ which we denote it by c_i. Put

$$\Omega_p = \{v \in T_pM : \quad \langle v, v\rangle = -1, \quad \langle v, K_i\rangle = c_i\}, \quad \bar{\Omega} = \bigcup_{p\in M} \Omega_p.$$

By Step 1, Ω_p is a compact subset of T_pM. Since M is compact, then $\bar{\Omega}$ is also compact in TM. We have $\{(\gamma(t), \gamma'(t)) : t \in I\} \subset \bar{\Omega}$. Therefore, γ is complete. A similar proof works if γ is spacelike or null. □

4. Zero points

The study of the zero points of Killing vector fields is an important subject in differential geometry. Because, existence of the zero points for a Killing vector field K usually imposes strong restrictions on the global properties of K. Consequently, it has useful applications in the subjects where the Killing vector fields play role, including pseudo Riemannian G-manifolds and their applications in physics. In what follows, we mention some useful results which are generalizations of the similar results previously proved for Riemannian and Lorentzian manifolds. It is proved in [3] that if K is a Killing vector field on R^n_1 with an isolated zero at a point p, then n is even and in each neighborhood $U(p)$ of p there are points at which K is timelike, spacelike and null. We use the ideas of the proof of Proposition 3.3 in [3] to prove the following Theorem.

Theorem 4.1. *Let K be a killing vector field on R^n_υ and $K(\circ) = \circ$.*

(1) If n is odd, then K is zero along a line L passing from the origin.

(2) If n is even, then for all non-degenerate linear hyperplanes P, K is normal to P along a one dimensional linear subspace L of P. In this case, there is a vector field U of constant length and normal to P such that for a parametrization $\gamma(s)$, $s \in R$, for L, we have $K(\gamma(s)) = sU$.

Proof. If a is a vector in R^n_υ and x is a point in R^n_υ, we denote by a_x the translation of the vector a from the origin to the point x. Each Killing vector field K on R^n_υ is in the form $K(x) = a_x + (S(x))_x$, where a is a constant vector and S is a skew adjoint operator on R^n_υ (see [9], p. 253). If $K(\circ) = \circ$, then $a = 0$ and $K(x) = (S(x))_x$ for all $x \in R^n_\upsilon$. S is skew adjoint and its matrix in the standard basis of $\mathbb{R}^n_\nu$ obeys the equality ${}^tS = -\epsilon S \epsilon$ (see

[9], p. 235, Lemma 3). Where, ϵ, is the diagonal matrix whose diagonal entries are $\epsilon_{11} = \cdots = \epsilon_{\nu\nu} = -1$ and $\epsilon_{(\nu+1)(\nu+1)} = \cdots = \epsilon_{nn} = 1$. Thus, we have:

$$\det S = (-1)^n(-1)^{2\nu} detS = (-1)^n detS.$$

Then, for odd numbers n, S is singular and it has at least one zero eigenvalue. Let v be the corresponding (non-zero)eigenvector. For each number $r \in R$, we have $K(rv) = rS(v) = 0$. Consequently, K is zero along the line $L = \{rv : r \in R\}$.
If n is even, we have the following argument:
Let P be a non-degenerate linear subspace of $\mathbb{R}^n_\nu$ and $dimP = n-1$ (hyperplane). Put $Y = tan_P K$, where tan_P denotes the normal projection map on P. Y is a killing vector field along P. Since $n-1$ is odd, then by the above argument, there is a linear subspace L of P such that Y is zero along L. Then, K is normal to P along L. Let W be the unite normal vector field on P. Parameterize L as a geodesic $\gamma(s)$, $s \in R$ such that $\gamma(0) = \circ$. We have along L, $K = f(s)W$, where f is a differentiable function on R. Since K is a Jaccobi field along γ, then $K'' = R_{K\gamma'}\gamma'$. Thus,

$$f''(s)W = f(s)R_{w\gamma'}\gamma'.$$

Since the curvature is zero, then $g(R_{W\gamma'}\gamma', W) = 0$. Thus, $f''(s) = 0$ and $f(s) = cs + d$, $c, d \in R$. Since $K(\circ) = \circ$, then $f(0) = 0$ and $f(s) = cs$. Therefore, $K = csW$ for a constant number c. Put $U = cW$ and get the result (2). □

Theorem 4.2. *Let M^n_ν be a flat pseudo-Riemannian manifold and K be a Killing vector field on M with a zero point $p \in M$.*

(1) If n is odd then there is a geodesic γ passing from p such that K is zero along γ.

(2) If p is an isolated zero point for K, then n is even and K becomes timelike and spacelike in each neighborhood of p.

Proof. (1) Since M is flat, R^n_ν is its universal covering manifold. Consider the covering map $\pi : R^n_\nu \to M$ and let $\tilde{p} \in \pi^{-1}(p)$. Without lose of the generality we can suppose that $\tilde{p} = \circ$. Let $\tilde{U}$, U be connected neighborhoods of $\circ$ and p such that $\pi : \tilde{U} \to U$ is isometry. Consider the vector field $\tilde{K}$ on $\tilde{U}$ defined by $d\pi(\tilde{K}) = K$. $\tilde{K}$ is a Killing vector field on $\tilde{U}$ with the zero point at $\circ$. Without lose of the generality, we can assume that $\tilde{U} = R^n_\nu$. By the above theorem, if n is odd, there is a line L containing $\tilde{p}(= \circ)$, such that $\tilde{K}$ is zero along L. Put $\gamma = \pi(L)$. γ is the image of a geodesic in M such that K is zero along γ.
(2) If p is an isolated zero point of K, then $\circ$ is an isolated zero point of $\tilde{K}$ and by the above theorem (part 1), n must be even. In proof the above theorem part 2, choose P an affine non-degenerate hyperplane with the index $v-1$. Then, the vector field U must be timelike and $\tilde{K}$ will be timelike along a line in P containing $\circ$. Then, K is timelike along a geodesic γ of M. If we choose P with the index ν, then U will be spacelike and similarly, K will be spacelike along γ. □

Note that by using of Corollary 3.2, and Theorem 3.1, we can prove a theorem similar to Theorem 4.2, in general for all pseudo Riemannian manifolds.

Theorem 4.3. *Let M^n_ν be a pseudo-Riemannian manifold and K be a Killing vector field on M with a zero point $p \in M$.*
If n is odd then there is a geodesic γ passing from p such that K is zero along γ, and if p is an isolated zero point for K, then n is even.

Lemma 4.4. *Let $T : R^n_v \to R^{n'}_{v'}$ be a linear map.*
(a) If $n > v + v'$, then there is a spacelike vector $v \in R^n_v$ such that $T(v)$ is not timelike.
(b) If $n' < v + v'$, then there is a timelike vector $v \in R^n_v$ such that $T(v)$ is not spacelike.

Proof. (a) Consider the following linear maps:

$$T_1 : R^{n-\nu} \longrightarrow R^n_\nu, \quad T_1(v_1, \ldots, v_{n-\nu}) = (0, \ldots, 0, v_1, \ldots, v_{n-v}) \in R^v_v \times R^{n-v} = R^n_\nu,$$

and

$$T_2 : R^{n'}_{\nu'} \longrightarrow R^{\nu'}, \quad T_2(a_1, \ldots, a_{\nu'}, \ldots, a_{n'}) = (a_1, \ldots, a_{\nu'}).$$

It is clear that if for a vector $w \in R^{n'}_{\nu'}$, $T_2(w) = 0$ then w is not timelike. Put $T_3 : R^{n-\nu} \to R^{\nu'}$, $T_3 = T_2 \circ T \circ T_1$. Since $n - v > v'$, then $\ker(T_3) \neq \emptyset$. Thus, there is a non-zero vector $(v_1, \ldots, v_{n-\nu})$ in $R^{n-\nu}$ such that $T_3(v_1, \ldots, v_{n-\nu}) = 0$. Then,

$$T_2 \circ T \circ T_1(v_1, \ldots, v_{n-\nu}) = 0 \Longrightarrow T_2 \circ T(0, 0, \ldots, 0, v_1, \ldots, v_{n-\nu}) = 0.$$

Thus, $T(0,0,\ldots,0,v_1,\ldots,v_{n-\nu})$ is not timlike, while $(0,0,\ldots,0,v_1,\ldots,v_{n-\nu})$ is spacelike in R^n_ν.
(b) The proof is similar to (a). Consider a suitable map $R^v \to R^{n'-v'}$. □

Theorem 4.5. *Let K be a killing vector field on a compact and connected pseudo-Riemannian manifold M^n_ν. If one of the following assertions is valid, then at least in one point $q \in M$ we have $K(q) = 0$.*
(a) M has positive timelike curvature, $\nu < \frac{n}{2}$, K is not spacelike.
(b) M has negative spacelike curvature, $\nu < \frac{n}{2}$, K is not timelike.
(c) M has positive time curvature, $\nu > \frac{n}{2}$, K is not spacelike.
(d) M has negative timelike curvature, $\nu > \frac{n}{2}$, K is not timelike.

Proof. (a) Let K be nonzero at all points of M. We will get a contradiction as follows: Put $f : M \to R$, $f(x) = \langle K(x), K(x)\rangle$. Since M is compact, f has a maximum point p. Put

$$S : T_pM(= R^n_\nu) \to T_pM(= R^n_\nu), \quad S(v) = \nabla_v K.$$

Since by assumption $n > 2\nu$, by Lemma 3.4 (a), there is a spacelike vector $v \in T_pM$ such that $\nabla_v K$ is not timelike. We can suppose that v is normal to K (if not, decompose v as $v = \lambda K + w$, where w is normal to K and use w instead of v). Thus, the sectional curvature along the plane generated by K and v is

$$\kappa(v,k) = \frac{\langle R(K,v)K, v\rangle}{\langle K,K\rangle\langle v,v\rangle}.$$

Now, consider the following computations:

$$\nabla_v\nabla_v f = 2\nabla_v\langle\nabla_v K, K\rangle = 2\langle\nabla_v\nabla_v K, K\rangle + 2\langle\nabla_v K, \nabla_v K\rangle. \tag{1}$$

Since K is Killing, its restriction along a geodesic with the initial speed v is a Jacobi field. Thus,

$$\nabla_v\nabla_v K = R(K,v)v.$$

Then,

$$\langle\nabla_v\nabla_v K, K\rangle = \langle R(K,v)v, K\rangle = -\langle R(K,v)K, v\rangle. \tag{2}$$

We get from (1) and (2) that

$$\nabla_v\nabla_v f = -2\langle R(K,v)K,v\rangle + 2\langle\nabla_v K,\nabla_v K\rangle = -2\kappa(v,K)(\langle K,K\rangle\langle v,v\rangle) + 2\langle\nabla_v K,\nabla_v K\rangle. \tag{3}$$

Since $\nabla_v K$ is not timelike, then $\langle\nabla_v K,\nabla_v K\rangle$ is nongegative. Thus, by our assumption of (a), (3) yields to $\nabla_v\nabla_v f > 0$. This is in contrast with the fact that p is a maximum point of f.
(b). Proof is similar to (a).
(c), (d). The proof is similar to (a) but consider a minimum point p for the function f and show that for some vector $v \in T_pM$, $\nabla_v\nabla_v f < 0$ and get a contradiction. □

5. An application to Lorentzian manifolds

We mention an application of our theorem on Lorentzian manifolds. Although such results can be directly proved for Lorentzian manifolds, but it seems to be useful to see an application of our theorem.

Let K be a Killing vector field on a connected Lorentzian manifold M^n_1 such that $K(p) = \circ$ at a point $p \in M$. Let q be another point on M joined to p by a timelike geodesic $\gamma : [0,1] \to M$ $(\gamma(0) = p, \gamma(1) = q)$. We get from the constancy of $\langle\gamma', K\rangle$ along γ, that $\langle\gamma'(1), K(q)\rangle = 0$. Since γ' is timelike, $K(q)$ must be zero or spacelike. If M is connected then each point q of M can be joined to p by a timelike broken geodesic. Thus, we get the following Remark [3].

Remark 5.1. *If M^n_1 is a connected Lorentzian manifold and K is a non-spacelike Killing vector field on M with a zero point, then K is zero at all points.*

If in Theorem 4.5, M is lorentzian $(\nu = 1)$ and $n > 2$, then $n > 2\nu$. Thus, by Theorem 4.5 and Remark 5.1, we get the following Theorem.

Theorem 5.2. *Let $M^n_1, n > 2$, be a connected and compact Lorentzian manifold.*
(a) If there is a non-trivial and non-spacelike Killing vector field on M, then the timelike curvature is not positive.
(b) If there is a spacelike Killing vector field on M, then the space curvature is not negative.

Proof. a) Let K be a non-trivial and non-spacelike Killing vector field on M and let the timelike curvature be positive at all points. By Theorem 4.5 (a), K has a zero point. Thus, by Remark 5.1, K is zero everywhere and this is in contrast with assumption that K is non-trivial.
b) Let K be a spacelike Killing vector field on M and let the space curvature be negative at all points. By Theorem 4.5 (b), K has zero points which is a contradiction. $\square$

Remark 5.3. *By the above theorem part (b), In compact and connected Lorentzian manifolds with the negative space curvature, there is no spacelike Killing vector fields. This is similar to the well known theorem about compact Riemannian manifolds which states: There is no non-trivial Killing vector fields on compact Riemannian manifolds of negative curvature.*

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