



Original Article

Product type operators on (α, p) -Besov-Zygmund spaces

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ABSTRACT: In this paper, we consider the boundedness and compactness of operator $M_u C_\psi$ between (α, p) -Besov-Zygmund spaces in terms of Carleson-type measures. Also we obtain some equivalent statements for the boundedness and compactness of a generalized product type operator $T_{u_1, u_2, \psi}$ which is well-known as Stević-Sharma operator between (α, p) -Besov-Zygmund spaces.

Review History:

Received:17 October 2023

Revised:04 March 2025

Accepted:08 March 2025

Available Online:01 July 2026

Keywords:

Carleson measure

Boundedness

Compactness

MSC (2020):

30H10; 30H99; 47B38

1. Introduction

Let \mathbb{C} denotes the complex plane and \mathbb{D} be the unit disk of \mathbb{C} . Then by $\mathcal{S} = \mathcal{S}(\mathbb{D})$, we mean the family of all analytic self-maps of \mathbb{D} . Also $H(\mathbb{D})$ denotes the class of all complex valued functions analytic on \mathbb{D} . We denote by $H^\infty(\mathbb{D})$, the space of all bounded analytic functions with the norm

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|.$$

When dA be the normalized area measure on \mathbb{D} , $1 \leq p < \infty$ and $\alpha > -1$, the family of all analytic functions f on \mathbb{D} such that the norm

$$\|f\|_{A_{p,\alpha}} := \left(\int_{\mathbb{D}} (1 - |z|^2)^\alpha |f(z)|^p dA(z) \right)^{\frac{1}{p}}$$

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be finite, is known as α -weighted Bergman space $A_{p,\alpha}(\mathbb{D})$. In addition, the analytic (α, p) -Besov type space $\mathcal{B}_{p,\alpha}(\mathbb{D})$, is supposed to be the class of all analytic functions $f \in H(\mathbb{D})$, for which

$$\|f\|_{\mathcal{B}_{p,\alpha}(\mathbb{D})} = |f(0)| + \left(\int_{\mathbb{D}} (1 - |z|^2)^\alpha |f'(z)|^p dA(z) \right)^p < \infty.$$

We can see that $f \in \mathcal{B}_{p,\alpha}(\mathbb{D})$ if and only if $f' \in A_{p,\alpha}(\mathbb{D})$ and the defined norm $\|\cdot\|_{\mathcal{B}_{p,\alpha}(\mathbb{D})}$ makes (α, p) -Besov type space a Banach space. In special case if we consider $\alpha = p - 1$, we get the space $\mathcal{B}_{p,p-1}$ which is known as Dirichlet type space. For more information about α -weighted Bergman spaces, (α, p) -Besov types spaces and Dirichlet type spaces, we refer to [7, 8, 11]. In related studies, an analytic function $f \in H(\mathbb{D})$ has been considered to be in the Zygmund space \mathcal{Z} , if $\|f\|_{\mathcal{Z}} := \sup_{z \in \mathbb{D}} (1 - |z|^2) |f''(z)| < \infty$. Following the above definitions and notations, we introduce (α, p) -Besov-Zygmund space as the space of all analytic functions $f \in H(\mathbb{D})$ such that the following norm is finite,

$$\begin{aligned} \|f\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})}^p &= |f(0)| + \|f'\|_{\mathcal{B}_{p,\alpha}}^p \\ &= |f(0)| + |f'(0)| + \|f''\|_{A_{p,\alpha}}^p \\ &= |f(0)| + |f'(0)| + \left(\int_{\mathbb{D}} (1 - |z|^2)^\alpha |f''(z)|^p dA(z) \right)^p. \end{aligned}$$

It's clear that Besov-Zygmund space is a Banach space under the above norm. Furthermore, $f \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$ if and only if $f' \in \mathcal{B}_{p,\alpha}(\mathbb{D})$ or $f'' \in A_{p,\alpha}(\mathbb{D})$. To the best of our knowledge, [20], is the only work to study (α, p) -Besov-Zygmund spaces in the case that $\alpha = p - 1$, where the author studied some properties such as boundedness and compactness of weighted composition operator on the spaces $\mathcal{Z}_{p,p-1}(\mathbb{D})$. For more explanation of such spaces, we recall the definition of the minimal Möbius invariant space \mathcal{B}_1 , which is the space of all $f \in H(\mathbb{D})$ which can be represented in the form $f(z) = \sum_{j=1}^\infty b_j \sigma_{w_j}(z)$ for some $(b_w)_j \in l^1$, $w_j \in \mathbb{D}$ and $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ for $z, a \in \mathbb{D}$. In [2] the authors studied such spaces and showed that there exist positive constant $C > 0$, such that for any $f \in \mathcal{B}_1$

$$\frac{1}{C} \int_{\mathbb{D}} |f''| dA(z) \leq \|f - f(0) - f'(0)z\|_{\mathcal{B}_1} \leq C \int_{\mathbb{D}} |f''(z)| dA(z).$$

The above inequality clarifies that in fact in the case of $p = 1$ and $\alpha = 0$, the two spaces \mathcal{B}_1 and $\mathcal{Z}_{1,0}$ equivalent. For $\psi \in \mathcal{S}$ and $u \in H(\mathbb{D})$, the weighted composition operator, induced by ψ and u is defined by

$$W_{u,\psi}(f) := u \cdot f \circ \psi, \quad f \in H(\mathbb{D}).$$

This operator can be considered as a generalization for multiplication operator M_u and composition operator C_ψ , where $M_u f = u \cdot f$ and $C_\psi f = f \circ \psi$. During the past decades, theory of composition operators and related topics, has been studied on various aspects. For more information see standard references [3, 12, 19] and also [20], where X. Zhu, studied the boundedness and compactness of operator $M_u C_\psi$ from (α, p) -Besov-Zygmund spaces into Zygmund type and Bloch type spaces, in this paper, one of our goals is to investigate the boundedness and compactness of operator $M_u C_\psi$ between Besov-Zygmund spaces.

The well-known differentiation operator D , which is unbounded on many analytic function spaces, is defined by $Df = f'$, for $f \in H(\mathbb{D})$. In the literature, any product of the three operators C_ψ , M_u and D , which can be obtained in following twelve ways, considered as product type operators

$$\begin{aligned} &M_u C_\psi, C_\psi M_u, M_u D, D M_u, C_\psi D, D C_\psi, \\ &M_u C_\psi D, C_\psi M_u D, M_u D C_\psi, C_\psi D M_u, D M_u C_\psi, D C_\psi M_u. \end{aligned}$$

For more studies about product type operators on different analytic function spaces, we refer to [4, 6, 9] and references therein.

The so-called Stević-Sharma operator $T_{u_1, u_2, \psi} = W_{u_1, \psi} + W_{u_2, \psi} D$, i.e.,

$$T_{u_1, u_2, \psi} f = u_1 \cdot f \circ \psi + u_2 \cdot f' \circ \psi,$$

which has been introduced by Stević and et al. is a generalized important operators for investigation. Since all the product type operators can be obtained form Stević-Sharma operator by applying proper u_1, u_2 . For example we have that

$$\begin{aligned} M_u D C_\psi &= T_{0, u\psi', \psi}, & M_u C_\psi D &= T_{0, u, \psi}, & D C_\psi M_u &= T_{(u' \circ \psi)\psi', (u \circ \psi)\psi', \psi}, \\ D M_u C_\psi &= T_{u', u\psi', \psi}, & C_\psi D M_u &= T_{u' \circ \psi, u \circ \psi, \psi}, & C_\psi M_u D &= T_{0, u \circ \psi, \psi}. \end{aligned} \tag{1}$$

More information about the Stević-Sharma operator on various spaces of analytic functions, can be found in [1, 10, 13, 14, 15, 16, 17].

In this paper, we aim to discuss the boundedness and compactness of the product type operator $M_u C_\psi$ between (α, p) -Besov-Zygmund spaces, in terms of (α, p) -Carleson measures. Then we find some equivalent statements for the boundedness and compactness of the Stević-Sharma operator $T_{u_1, u_2, \psi} = M_{u_1} C_\psi + M_{u_2} C_\psi$ between (α, p) -Besov-Zygmund spaces. Also as conclusions, according to relation (1), we have characterizations for the boundedness and compactness of other product type operators between (α, p) -Besov-Zygmund spaces.

We denoted by C , the constants which are positive and not necessary the same as each occurrence. Also by $A \preceq B$, we mean $A \leq CB$.

2. Boundedness

For a point ξ in $\partial\mathbb{D}$, the boundary of the unit disk, and $h > 0$, we define the Carleson set $S(\xi, h) := \{z \in \mathbb{D} : |z - \xi| < h\}$. Let $\alpha > -1$ and γ be a positive Borel measure on \mathbb{D} . For $0 < p < \infty$, it is well known (see [18, Section 2.4]) that

$$\text{the embedding } A_{p,\alpha}(\mathbb{D}) \subset L^p(d\gamma) \text{ is bounded} \Leftrightarrow \sup_{\xi \in \partial\mathbb{D}} \frac{\gamma(S(\xi, h))}{h^{\alpha+2}} < \infty, \tag{2}$$

for $h > 0$ and

$$\text{the embedding } A_{p,\alpha}(\mathbb{D}) \subset L^p(d\gamma) \text{ is compact} \Leftrightarrow \limsup_{h \rightarrow 0} \sup_{\partial\mathbb{D}} \frac{\gamma(S(\xi, h))}{h^{\alpha+2}} = 0. \tag{3}$$

We say that γ is an (α, p) -Carleson measure if either side of relation (2) holds. Also, we say that γ is a compact (α, p) -Carleson measure if either side of relation (3) holds.

The connection between composition operators and Carleson measures comes from the standard identity (see [5, P. 163])

$$\int_{\mathbb{D}} (h \circ \psi)(z)(1 - |z|^2)^\alpha dA(z) = \int_{\mathbb{D}} h(z) d\gamma(z), \tag{4}$$

valid for $\psi \in \mathcal{S}$ and Borel functions $h \geq 0$ on \mathbb{D} . Here, $d\gamma$ denotes the pullback measure defined by

$$\gamma(E) = \int_{\psi^{-1}(E)} (1 - |z|^2)^\alpha dA(z),$$

for Borel sets $E \subset \mathbb{D}$. In particular, one can easily observe from (4) that $C_\psi : A_{p,\alpha} \rightarrow A_{p,\alpha}$ is (compact) bounded if and only if γ is (compact) (α, p) -Carleson measure for $A_{p,\alpha}$.

Assume $1 \leq p < \infty$ and $-1 < \alpha$. Let ψ be a self map on \mathbb{D} , and $u \in H(\mathbb{D})$. Denote by $E_{\psi,w}$ the number of zeros of $\psi(z) - w$ on \mathbb{D} . Then $E_{p,\alpha,u}(\psi, w)$ on \mathbb{D} is defined as follows:

$$E_{p,\alpha,u}(\psi, w) = \sum \frac{(1 - |z|^2)^\alpha |u(z)|^p}{|\psi'(z)|^{1-p}},$$

when repeated by multiplicity, the sum extends over the zeros of $\psi - w$. In particular, for $w \notin \psi(\mathbb{D})$, we have that $E_{p,\alpha,u}(\psi, w) = 0$. Also for $u = 1$, $\alpha = 0$ and $p = 1$, we get $E_{\psi,w}$. Now define the measure $\gamma_{p,\alpha,u}$ on \mathbb{D} , by

$$d\gamma_{p,\alpha,u}(w) = E_{p,\alpha,u}(\psi, w)dA(w).$$

Lemma 2.1 ([7]). *Let $-1 < \alpha$, $0 < p$ and $f \in H(\mathbb{D})$. Then $f \in A_{p,\alpha}(\mathbb{D})$ if and only if $f' \in A_{p,\alpha+p}(\mathbb{D})$ and $A_{p,\alpha}(\mathbb{D}) \cong \mathcal{B}_{p,\alpha+p}(\mathbb{D})$.*

Lemma 2.2 ([11]). *Suppose that $-1 < \alpha < \infty$ and $p > \alpha + 2$. Then for any $f \in \mathcal{B}_{p,\alpha}(\mathbb{D})$, there exists a constant C such that*

$$|f(z)| \leq C \|f\|_{\mathcal{B}_{p,\alpha}(\mathbb{D})}.$$

Using the two above Lemmas, we obtain the following key Lemma which is a useful characterization for $f \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$ and will help us to prove our main results.

Lemma 2.3. *Suppose that $-1 < \alpha < \infty$. Then we have the two following statements:*

(a) *For $p > \alpha + 2$ and any $f \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$ and $i \in \{0, 1, 2\}$*

$$\|f^{(i)}\|_\infty \leq C \|f\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})}.$$

(b) For $1 \leq p < \infty$, we have that $\mathcal{Z}_{p,\alpha}(\mathbb{D}) \subset \mathcal{B}_{p,\alpha}(\mathbb{D})$.

Proof. (a) Let $f \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$, then $f' \in \mathcal{B}_{p,\alpha}(\mathbb{D})$ and according to Lemma 2.2, for $p > \alpha + 2$,

$$|f'(z)| \leq C \|f'\|_{\mathcal{B}_{p,\alpha}(\mathbb{D})}. \tag{5}$$

Hence

$$\|f'\|_{\infty} = \sup_{z \in \mathbb{D}} |f'(z)| \leq C \|f\|_{\mathcal{Z}_{p,\alpha}}.$$

Also (5) yields that

$$|f(z) - f(0)| \leq \int_0^1 |zf'(tz)| dt \leq \int_0^1 |z| |f'(tz)| dt \leq C \|f'\|_{\mathcal{B}_{p,\alpha}(\mathbb{D})}.$$

Therefore, we have

$$|f(z)| \leq C(\|f'\|_{\mathcal{B}_{p,\alpha}(\mathbb{D})} + |f(0)|),$$

and thus

$$\|f\|_{\infty} \leq C \|f\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})}.$$

Also by applying lemma 2.1, we get that $f'' \in A_{p,\alpha}(\mathbb{D}) \cong \mathcal{B}_{p,\alpha+p}(\mathbb{D})$. Then Lemma 2.2 implies

$$\|f''\|_{\infty} \preceq \|f''\|_{\mathcal{B}_{p,\alpha+p}(\mathbb{D})} \preceq \|f''\|_{A_{p,\alpha}(\mathbb{D})} \preceq \|f\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})}.$$

(b) Suppose that $f \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$, then

$$\begin{aligned} \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^\alpha dA(z) &= \int_{\mathbb{D}} \left| \int_0^z f''(w) dw + f'(0) \right|^p (1 - |z|^2)^\alpha dA(z) \\ &\leq C |f'(0)|^p + \int_{\mathbb{D}} \int_0^1 |zf''(tz)|^p (1 - |z|^2)^\alpha dA(z) \\ &\leq C \|f'\|_{\mathcal{B}_{p,\alpha}(\mathbb{D})}^p, \end{aligned}$$

which implies $\|f\|_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \leq C \|f\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})}$. □

As a result of the previous lemma, one can observe that for $p > \alpha + 2$,

$$\mathcal{Z}_{p,\alpha}(\mathbb{D}) \subset \mathcal{B}_{p,\alpha}(\mathbb{D}) \subset H^\infty(\mathbb{D}) \subset A_{p,\alpha}(\mathbb{D}). \tag{6}$$

Next, we obtain a characterization for the boundedness of the weighted composition operators $M_u C_\psi$ on analytic function space $\mathcal{Z}_{p,\alpha}(\mathbb{D})$ in terms of the Carleson measure.

Theorem 2.4. Assume $\alpha > -1$ and $2 + \alpha < p < \infty$. Let $u \in H(\mathbb{D})$ and $\psi \in \mathcal{S}$. Then $M_u C_\psi : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \rightarrow \mathcal{Z}_{p,\alpha}(\mathbb{D})$ is bounded if and only if $u, u\psi \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$ and $\gamma_{p,\alpha,u}$ is an (α, p) -Carleson measure.

Proof. Assume that $M_u C_\psi$ is bounded on $\mathcal{Z}_{p,\alpha}(\mathbb{D})$. For any $f \in A_{p,\alpha}(\mathbb{D})$ such that $f(0) = 0$ define the function $h(z) := \int_0^z f(w) dw$. Then we get $h'(z) = f(z)$ and $h(0) = 0$. Again set $g(z) = \int_0^z h(w) dw$, we obtain $g''(z) = f(z) \in A_{p,\alpha}(\mathbb{D})$, thus $g(z) \in \mathcal{Z}_{p,\alpha}$ and $g(0) = g'(0) = 0$. In addition, we have

$$(ug(\psi(z)))'' = u''(z)g(\psi(z)) + 2u'(z)\psi'(z)g'(\psi(z)) + u(z)\psi''(z)g'(\psi(z)) + u(z)\psi'^2(z)g''(\psi(z)).$$

For functions $f_1(z) = 1$ and $f_2(z) = z$, applying the boundedness of operator $M_u C_\psi$ on the space $\mathcal{Z}_{p,\alpha}(\mathbb{D})$ yields that

$$\|u''\|_{A_{p,\alpha}(\mathbb{D})} \leq \|u\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} = \|M_u C_\psi(f_1)\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} < \infty, \tag{7}$$

and

$$\begin{aligned} \|M_u C_\psi(f_2)\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} &= \|u\psi\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} = \|(u\psi)''\|_{A_{p,\alpha}(\mathbb{D})} + C \\ &= \|u''\psi + \psi''u + 2\psi'u'\|_{A_{p,\alpha}(\mathbb{D})} + C \\ &< \infty. \end{aligned} \tag{8}$$

Thus, it follows from relations (7) and (8) that

$$\|\psi''u + 2u'\psi'\|_{A_{p,\alpha}(\mathbb{D})} < \infty.$$

In addition, apply the change of variable formula (set $\psi(z) = w$) and Lemma 2.3 to yield that

$$\begin{aligned} \left(\int_{\mathbb{D}} |f(w)|^p d\gamma_{p,\alpha,u}(w) \right)^{1/p} &= \left(\int_{\mathbb{D}} |f(w)|^p E_{p,\alpha,u}(\psi, w) dA(w) \right)^{1/p} \\ &= \left(\int_{\mathbb{D}} |u(z)\psi'^2(z)|^p |f(\psi(z))|^p (1 - |z|^2)^\alpha dA(z) \right)^{1/p} \\ &= \|u\psi'^2 f \circ \psi\|_{A_{p,\alpha}(\mathbb{D})} = \|u\psi'^2 g'' \circ \psi\|_{A_{p,\alpha}(\mathbb{D})} \\ &= \|(ug \circ \psi)'' - (u\psi'' + 2u'\psi')g' \circ \psi - u''g \circ \psi\|_{A_{p,\alpha}(\mathbb{D})} \\ &\leq \|M_u C_\psi(g)\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} + \|g'\|_\infty \|u\psi'' + 2u'\psi'\|_{A_{p,\alpha}(\mathbb{D})} + \|g\|_\infty \|u''\|_{A_{p,\alpha}(\mathbb{D})} \\ &\leq C\|g\|_{\mathcal{Z}_{p,\alpha}} = C\|g''\|_{A_{p,\alpha}(\mathbb{D})} = C\|f\|_{A_{p,\alpha}(\mathbb{D})}. \end{aligned}$$

Hence, we get the boundedness of inclusion map $I : A_{p,\alpha}(\mathbb{D}) \rightarrow L^p(\gamma_{p,\alpha,u})$ and thus $\gamma_{p,\alpha,u}$ is an (α, p) -Carleson measure.

Conversely, for any $f \in \mathcal{Z}_{p,\alpha}$ such that $f(0) = 0$, we have

$$\begin{aligned} \|(uf \circ \psi)''\|_{A_{p,\alpha}(\mathbb{D})} &= \|u''f \circ \psi + 2u'\psi'f' \circ \psi + u\psi''f' \circ \psi + u\psi'^2 f'' \circ \psi\|_{A_{p,\alpha}(\mathbb{D})} \\ &\leq \|u\psi''f' \circ \psi + 2u'\psi'f' \circ \psi + u''\psi f' \circ \psi - u''\psi f' \circ \psi\|_{A_{p,\alpha}(\mathbb{D})} + \|u''f \circ \psi + u\psi'^2 f'' \circ \psi\|_{A_{p,\alpha}(\mathbb{D})} \\ &\leq \|(u\psi)''f' \circ \psi\|_{A_{p,\alpha}(\mathbb{D})} + \|u''\psi f' \circ \psi\|_{A_{p,\alpha}(\mathbb{D})} + \|u''f \circ \psi\|_{A_{p,\alpha}(\mathbb{D})} + \|u\psi'^2 f'' \circ \psi\|_{A_{p,\alpha}(\mathbb{D})} \\ &\leq \|f'\|_\infty \|u\psi\|_{\mathcal{Z}_{p,\alpha}} + \|f'\|_\infty \|\psi\|_\infty \|u''\|_{A_{p,\alpha}(\mathbb{D})} + \|f\|_\infty \|u\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} + \|u\psi'^2 f'' \circ \psi\|_{A_{p,\alpha}(\mathbb{D})}. \end{aligned}$$

In addition, we have that

$$\begin{aligned} |(uf \circ \psi)'(0)| + |(uf \circ \psi)(0)| &= |u'(0)f \circ \psi(0) + u(0)\psi'(0)f' \circ \psi(0)| + |u(0)f(\psi(0))| \\ &\leq \|u\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} \|f\|_\infty + C\|u\psi\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} \|f\|_\infty + \|u\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} \|f\|_\infty. \end{aligned}$$

Now by applying the assumptions $u \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$, $u\psi \in \mathcal{Z}_{p,\alpha}$ and Lemma 2.3, we obtain

$$\begin{aligned} \|M_u C_\psi(f)\|_{\mathcal{Z}_{p,\alpha}} &= \|(uf \circ \psi)''\|_{A_{p,\alpha}(\mathbb{D})} + |(uf \circ \psi)'(0)| + |(uf \circ \psi)(0)| \\ &\leq C\|f\|_{\mathcal{Z}_{p,\alpha}} + \|u\psi'^2 f'' \circ \psi\|_{A_{p,\alpha}(\mathbb{D})} \\ &\leq C\|f\|_{\mathcal{Z}_{p,\alpha}} + \int_{\mathbb{D}} \frac{|f'' \circ \psi|^p |u|^p |\psi'^2| (1 - |z|^2)^\alpha}{|\psi'^2|^{1-p}} dA(z) \\ &\leq C\|f\|_{\mathcal{Z}_{p,\alpha}} + \int_{\mathbb{D}} |f''(w)|^p d\gamma_{p,\alpha,u}(w). \end{aligned}$$

Since $\gamma_{p,\alpha,u}$ is an (α, p) -Carleson measure, we get that

$$\|M_u C_\psi(f)\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} \leq C\|f\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})}.$$

So, we arrive at the boundedness of $M_u C_\psi$ on $\mathcal{Z}_{p,\alpha}(\mathbb{D})$. □

Lemma 2.5 ([11]). *Let $-1 < \alpha$, $\alpha + 2 < p$, $\psi \in \mathcal{S}$ and $u \in H(\mathbb{D})$. Then $M_u C_\psi : \mathcal{B}_{p,\alpha}(\mathbb{D}) \mapsto A_{p,\alpha}(\mathbb{D})$ is bounded if and only if $u \in A_{p,\alpha}(\mathbb{D})$.*

From the norm definition on $\mathcal{Z}_{p,\alpha}(\mathbb{D})$, for any $g \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$, $u \in H(\mathbb{D})$ and $\psi \in \mathcal{S}$, we have another equality as follows

$$\|M_u C_\psi g\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} = \|M_{u'} C_\psi g + M_u DC_\psi g\|_{\mathcal{B}_{p,\alpha}(\mathbb{D})} + C.$$

Now, by using Theorem 2.4 and Lemma 2.5, we get the following result which gives us another interesting characterization for the boundedness of weighted composition operator $M_u C_\psi$ on analytic function space $\mathcal{Z}_{p,\alpha}(\mathbb{D})$.

Theorem 2.6. *Let $-1 < \alpha$ and $2 + \alpha < p < \infty$, $u \in H(\mathbb{D})$ and $\psi \in \mathcal{S}$ such that $\psi' \in H^\infty(\mathbb{D})$. Then operator $M_u C_\psi : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \mapsto \mathcal{Z}_{p,\alpha}(\mathbb{D})$ is bounded if and only if both the two following operators are bounded*

$$\begin{aligned} M_{u'} C_\psi : \mathcal{Z}_{p,\alpha}(\mathbb{D}) &\mapsto \mathcal{B}_{p,\alpha}(\mathbb{D}), \\ M_u DC_\psi : \mathcal{Z}_{p,\alpha}(\mathbb{D}) &\mapsto \mathcal{B}_{p,\alpha}(\mathbb{D}). \end{aligned}$$

Proof. Suppose that $M_u C_\psi : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \mapsto \mathcal{Z}_{p,\alpha+p}(\mathbb{D})$ is bounded, then by applying Theorem 2.4, $u, u.\psi \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$. Then $u'', (u.\psi)'' \in A_{p,\alpha}(\mathbb{D})$ and $u' \in \mathcal{B}_{p,\alpha}(\mathbb{D}) \subset A_{p,\alpha}(\mathbb{D})$, so the boundedness of ψ, ψ' on \mathbb{D} give us

$$\begin{aligned} \|(u\psi)'\|_{A_{p,\alpha}(\mathbb{D})} &\leq \|(u.\psi)''\|_{A_{p,\alpha}(\mathbb{D})} + \|((u'\psi)')\|_{A_{p,\alpha}(\mathbb{D})} \\ &\leq \|(u.\psi)''\|_{A_{p,\alpha}(\mathbb{D})} + \|u''\psi + u'\psi'\|_{A_{p,\alpha}(\mathbb{D})} \\ &< \infty. \end{aligned}$$

Now by applying theorem 2.5, we obtain the boundedness of operator $M_{(u\psi)'} C_\psi : \mathcal{B}_{p,\alpha}(\mathbb{D}) \mapsto A_{p,\alpha}(\mathbb{D})$, which is equivalent to the boundedness of operator

$$M_{(u\psi)'} C_\psi D : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \mapsto A_{p,\alpha}(\mathbb{D}). \tag{9}$$

On the other hand, by applying our hypothesis and Theorem 2.4, $\gamma_{p,\alpha,u}$ is an (α, p) -Carleson measure. So we can easily check that $M_{u\psi'^2} C_\psi : A_{p,\alpha}(\mathbb{D}) \mapsto A_{p,\alpha}(\mathbb{D})$ is bounded. Hence, we get the boundedness of operator

$$M_{u\psi'^2} C_\psi D^2 : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \mapsto A_{p,\alpha}(\mathbb{D}). \tag{10}$$

Also the point evaluation maps on the Banach space $\mathcal{Z}_{p,\alpha}(\mathbb{D})$ are linear bounded functionals. Therefore, by applying relations (9) and (10), we get

$$\|M_u DC_\psi f\|_{\mathcal{B}_{p,\alpha}(\mathbb{D})} = \|M_{(u\psi)'} C_\psi Df\|_{A_{p,\alpha}(\mathbb{D})} + \|M_{u\psi'^2} C_\psi D^2 f\|_{A_{p,\alpha}(\mathbb{D})} + |M_{u\psi'} C_\psi Df(0)| \leq \|f\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})}, \tag{11}$$

for any $f \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$. Moreover, the triangle inequality, our assumption and (11) give us

$$\begin{aligned} \|M_{u'} C_\psi f\|_{\mathcal{B}_{p,\alpha}(\mathbb{D})} &\leq \|M_{u'} C_\psi f + M_u DC_\psi f - M_u DC_\psi f\|_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \\ &\leq \|M_u C_\psi f\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} + \|M_u DC_\psi f\|_{\mathcal{B}_{p,\alpha}(\mathbb{D})} \\ &\leq \|f\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})}. \end{aligned} \tag{12}$$

Relations (11) and (12) complete the proof.

The converse part is obvious by using the triangle inequality, the boundedness of the point evaluation maps on $\mathcal{Z}_{p,\alpha}(\mathbb{D})$ and the fact that

$$\|M_u C_\psi f\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} = \|M_{u'} C_\psi f + M_u DC_\psi f\|_{\mathcal{B}_{p,\alpha}(\mathbb{D})} + |M_u C_\psi f(0)|. \quad \square$$

In the following theorem, we are giving some equivalent statements to characterize the boundedness of the Stević-Sharma operator $T_{u_1, u_2, \psi}$ on analytic function spaces $\mathcal{Z}_{p,\alpha}(\mathbb{D})$. The statements of Theorem 2.4 and Lemma 2.3, will be used here to receive our desired results.

Theorem 2.7. Assume $-1 < \alpha$ and $2 + \alpha < p < \infty$. Let $u_1, u_2 \in H(\mathbb{D})$ and $\psi \in \mathcal{S}$, such that $\psi, u_2 \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$. Then the following statements are equivalent:

- (a) $T_{u_1, u_2, \psi} : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \rightarrow \mathcal{Z}_{p,\alpha}(\mathbb{D})$ is bounded.
- (b) The both following operators are bounded

$$M_{u_1} C_\psi : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \mapsto \mathcal{Z}_{p,\alpha}(\mathbb{D}), \quad M_{u_2} C_\psi D : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \mapsto \mathcal{Z}_{p,\alpha}(\mathbb{D}).$$

- (c) $u_1, u_1\psi, u_2\psi \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$, γ_{p,α,u_1} is a measure of (α, p) -Carleson and $\gamma_{p,\alpha+p,u_2}$ is a measure of $(\alpha + p, p)$ -Carleson.

Proof. (a) \implies (b). Let $T_{u_1, u_2, \psi} : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \rightarrow \mathcal{Z}_{p,\alpha}(\mathbb{D})$ is bounded. Then for any $f \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$, $\|u_1 f \circ \psi + u_2 f' \circ \psi\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} < \infty$. So by assuming $f = 1$, we have that

$$u_1 \in \mathcal{Z}_{p,\alpha}(\mathbb{D}). \tag{13}$$

Also by choosing $f = z$ we get that

$$\|u_1\psi + u_2\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} < \infty. \tag{14}$$

But we have assumed that $u_2 \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$ then by the triangle inequality,

$$\|u_1\psi\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} \leq \|u_1\psi + u_2\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} + \|u_2\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} < \infty.$$

On the other hand, relations (13), (14) and $u_2 \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$, give us

$$\|(u_1\psi + u_2)''\|_{A_{p,\alpha}(\mathbb{D})} = \|u_1'\psi' + (u_1\psi')'\|_{A_{p,\alpha}(\mathbb{D})} < \infty. \tag{15}$$

Again by putting $f = z^2$, we obtain

$$\|(u_1\psi^2 + 2u_2\psi)''\|_{A_{p,\alpha}(\mathbb{D})} = \|u_1''\psi^2 + 2\psi(u_1'\psi' + (u_1\psi')') + 2(u_1\psi'^2) + (2u_2\psi)''\|_{A_{p,\alpha}(\mathbb{D})} < \infty. \tag{16}$$

Also, by applying Lemma 2.3 and $u_2 \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$, boundedness of ψ on \mathbb{D} , the assumption $\psi \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$ and relation (6), we have that

$$\begin{aligned} \|(u_2\psi)''\|_{A_{p,\alpha}} &\leq \|u_2''\|_{A_{p,\alpha}} \|\psi\|_\infty + \|2u_2'\|_\infty \|\psi'\|_{A_{p,\alpha}} + \|u_2\|_\infty \|\psi''\|_{A_{p,\alpha}} \\ &\leq \|u_2\|_{\mathcal{Z}_{p,\alpha}} \|\psi\|_\infty + 2\|u_2\|_{\mathcal{Z}_{p,\alpha}} \|\psi'\|_{A_{p,\alpha}} + \|u_2\|_{\mathcal{Z}_{p,\alpha}} \|\psi''\|_{A_{p,\alpha}} < \infty. \end{aligned} \tag{17}$$

Therefore, by relations (13), (14), (15), (16) and (17), we get that

$$2u_1\psi'^2 \in A_{p,\alpha}(\mathbb{D}). \tag{18}$$

Then by applying the boundedness of $T_{u_1,u_2,\psi}$ on $\mathcal{Z}_{p,\alpha}(\mathbb{D})$ and the triangle inequality, for any $f \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$, we have that

$$\begin{aligned} \|M_{u_1}C_\psi f\|_{\mathcal{Z}_{p,\alpha}} &= \|(u_1f \circ \psi)''\|_{A_{p,\alpha}} + \underbrace{|(u_1f \circ \psi)(0)| + |(u_1f \circ \psi)'(0)|}_L \\ &= \|u_1''f \circ \psi + u_1'\psi'f' \circ \psi + (u_1\psi')'f' \circ \psi + u_1\psi'^2f'' \circ \psi\|_{A_{p,\alpha}} + L \\ &\leq \|u_1''\|_{A_{p,\alpha}} \|f\|_\infty + \|u_1'\psi' + (u_1\psi')'\|_{A_{p,\alpha}} \|f'\|_\infty + \|u_1\psi'^2\|_{A_{p,\alpha}} \|f''\|_\infty + L. \end{aligned}$$

Now by applying lemma 2.3, relations (13), (15), (18) and since evaluations at $\psi(0)$ are bounded linear operators on $\mathcal{Z}_{p,\alpha}$ and $\mathcal{B}_{p,\alpha}$, we obtain

$$\|M_{u_1}C_\psi f\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} \preceq \|f\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})}.$$

Hence, $M_{u_1}C_\psi : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \rightarrow \mathcal{Z}_{p,\alpha}(\mathbb{D})$ is bounded.

Now the boundedness of $T_{u_1,u_2,\psi}$ on $\mathcal{Z}_{p,\alpha}(\mathbb{D})$ and the triangle inequality, give us

$$\|M_{u_2}C_\psi Df\|_{\mathcal{Z}_{p,\alpha}} \leq \|T_{u_1,u_2,\psi}(f)\|_{\mathcal{Z}_{p,\alpha}} + \|M_{u_1}C_\psi f\|_{\mathcal{Z}_{p,\alpha}} \preceq \|f\|_{\mathcal{Z}_{p,\alpha}},$$

which completes the proof.

(b) \iff (c). As an application of Lemma 2.1, we have that $M_{u_2}C_\psi D : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \rightarrow \mathcal{Z}_{p,\alpha}(\mathbb{D})$ is bounded if and only if $M_uC_\psi : \mathcal{Z}_{p,\alpha+p}(\mathbb{D}) \rightarrow \mathcal{Z}_{p,\alpha}(\mathbb{D})$ is bounded and according to Theorem 2.4, this happens if and only if $u_2, u_2\psi \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$ and $\gamma_{p,\alpha+p,u_2}$ be a measure of $(\alpha+p, p)$ -Carleson. Now using Theorem 2.4 about the boundedness of $M_{u_1}C_\psi$ on $\mathcal{Z}_{p,\alpha}(\mathbb{D})$ completes the proof.

(b) \implies (a). It is clear by using the triangle inequality. □

3. Compactness

In this section, we provide characterization for the compactness of product type operator M_uC_ψ between (α, p) -Besov-Zygmund spaces. Also we aim to consider the compactness of the Stević-Sharma operator, then as results we have the necessary and sufficient conditions for the compactness of other product type operators between Besov-Zygmund spaces.

By applying the similar method as in [3, Proposition 3.11], we get the following characterization for the compactness of the bounded linear operators between Besov-Zygmund spaces.

Lemma 3.1. *Suppose that $0 < \alpha < \infty$ and T be a bounded linear operator from $\mathcal{Z}_{p,\alpha}(\mathbb{D})$ into $\mathcal{Z}_{p,\alpha}(\mathbb{D})$. Then T is compact if and only if for any bounded sequence $\{h_n\}_0^\infty$ in $\mathcal{Z}_{p,\alpha}(\mathbb{D})$, which converges to 0 uniformly on any compact subset of \mathbb{D} , we have that*

$$\lim_{n \rightarrow \infty} \|T(h_n)\|_{\mathcal{Z}_{p,\alpha}} = 0.$$

Now by using Lemma 3.1 and the key Theorem 2.4, we arrive at the following theorem which states an interesting characterization for compactness of the weighted composition operator M_uC_ψ between Besov-Zygmund spaces, in terms of the Carleson measure.

Theorem 3.2. Assume $-1 < \alpha$ and $2 + \alpha < p < \infty$. Let $\psi \in \mathcal{S}$ and $u \in H(\mathbb{D})$. Then $M_u C_\psi : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \rightarrow \mathcal{Z}_{p,\alpha}(\mathbb{D})$ is compact if and only if $\psi, u \cdot \psi \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$ and $\gamma_{p,\alpha,\psi}$ is a compact (α, p) -Carleson measure.

Proof. Suppose that $M_u C_\psi : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \rightarrow \mathcal{Z}_{p,\alpha}(\mathbb{D})$ be compact and the bounded sequence $\{F_n\}_{n=0}^\infty \subset A_{p,\alpha}(\mathbb{D})$ converges uniformly to 0 on compact subsets of \mathbb{D} . Let $h_n(z) = \int_0^z F_n(w)dw$, then we get that $h'_n \in A_{p,\alpha}(\mathbb{D})$ and $h_n(0) = 0$. Now by taking $g_n(z) = \int_0^z h_n(\xi)d\xi$, we have that $g_n(0) = g'_n(0) = 0$ and $g''_n(z) \in A_{p,\alpha}(\mathbb{D})$. Thus $\{g_n\}_{n=0}^\infty$ is a sequence in $\mathcal{Z}_{p,\alpha}(\mathbb{D})$ that converges to 0 uniformly on compact subsets of \mathbb{D} . Since $M_u C_\psi$ is compact on $\mathcal{Z}_{p,\alpha}(\mathbb{D})$, $\|M_u C_\psi g_n\|_{\mathcal{Z}_{p,\alpha}} \rightarrow 0$, as $n \rightarrow \infty$. On the other hand, $M_u C_\psi$ is bounded on $\mathcal{Z}_{p,\alpha}(\mathbb{D})$ then by Theorem 2.4,

$$\|u\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} < \infty, \quad \|u \cdot \psi\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} < \infty. \tag{19}$$

Also similar to relation (8), we have that

$$\|2u'\psi' + u\psi''\|_{A_{p,\alpha}(\mathbb{D})} \leq \|(u \cdot \psi)''\|_{A_{p,\alpha}(\mathbb{D})} + \|u''\psi\|_{A_{p,\alpha}(\mathbb{D})} < \infty. \tag{20}$$

So, by using relations (19) and (20) we obtain

$$\begin{aligned} \|F_n\|_{L^p(\gamma_{p,\alpha,u})}^p &= \left(\int_{\mathbb{D}} |F_n(w)|^p d\gamma_{p,\alpha,u}(w) \right)^{1/p} = \left(\int_{\mathbb{D}} |u\psi'^2(z)|^p |F_n(\psi(z))|^p (1 - |z|^2)^\alpha dA(z) \right)^{1/p} \\ &= \|M_{u\psi'^2} C_\psi(F_n)\|_{A_{p,\alpha}} \\ &\leq \|u''g'_n \circ \psi\|_{A_{p,\alpha}(\mathbb{D})} + \|(ug_n \circ \psi)''\|_{A_{p,\alpha}} + \|(u\psi'' + 2u'\psi')g_n \circ \psi\|_{A_{p,\alpha}} \\ &\leq C\|g_n\|_{\mathcal{Z}_{p,\alpha}} = C\|F_n\|_{A_{p,\alpha}}. \end{aligned} \tag{21}$$

Thus $\|F_n\|_{L^p(\gamma_{p,\alpha,u})} \rightarrow 0$ as $n \rightarrow \infty$. It follows that the embedding $A_{p,\alpha}(\mathbb{D}) \subset L^p(\gamma_{p,\alpha,u})$ is compact. Therefore, according to relation (3), $\gamma_{p,\alpha,u}$ is a compact (α, p) -Carleson measure.

Conversely, Let $\psi, u \cdot \psi \in \mathcal{Z}_{p,\alpha}$ and $\gamma_{p,\alpha,u}$ be a compact (α, p) -Carleson measure. Then according to Theorem 2.4, we have the boundedness of operator $M_u C_\psi$ on $\mathcal{Z}_{p,\alpha}(\mathbb{D})$. Also, similar to relation (20), $\|2u'\psi' + u\psi''\|_{A_{p,\alpha}(\mathbb{D})} < \infty$. Now for an arbitrary sequence $\{f_n\}_{n=0}^\infty \subset \mathcal{Z}_{p,\alpha}(\mathbb{D})$ such that f_n converges to zero uniformly on compact subsets of \mathbb{D} , we have the bounded sequence $\{f''_n\}_{n=0}^\infty \subset A_{p,\alpha}(\mathbb{D})$, such that f''_n converges to zero uniformly on compact subsets of \mathbb{D} and so $\int_{\mathbb{D}} |f''_n|^p d\gamma_{p,\alpha,u} = \|f''_n\|_{L^p(\gamma_{p,\alpha,u})}^p \rightarrow 0$, as $n \rightarrow \infty$. On the other hand, by using the assumptions, relation (21) and Lemma 2.3, we obtain

$$\begin{aligned} \|M_u C_\psi(f_n)\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} &= \|(u \cdot f_n \circ \psi)''\|_{A_{p,\alpha}(\mathbb{D})} + |(u \cdot f_n \circ \psi)'(0)| + |u \cdot f_n \circ \psi(0)| \\ &\leq \|u\psi'^2 f''_n \circ \psi\|_{A_{p,\alpha}(\mathbb{D})} + \|u'' f_n \circ \psi\|_{A_{p,\alpha}(\mathbb{D})} \\ &\quad + \|(2u'\psi' + u\psi'')f'_n \circ \psi\|_{A_{p,\alpha}(\mathbb{D})} + |(f_n \circ \psi)'(0)| + |f_n \circ \psi(0)| \\ &\leq \int_{\mathbb{D}} |f''_n(w)|^p d\gamma_{p,\alpha,u}(w) + \|u''\|_{A_{p,\alpha}} \|f'_n\|_\infty + \|2u'\psi' + u\psi''\|_{A_{p,\alpha}} \|f'\|_\infty + C\|f'_n\|_\infty + \|f_n\|_\infty \\ &\leq \|f''_n\|_{L^p(\gamma_{p,\alpha,u})}^p + C\|u\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} \|f'_n\|_\infty + C\|f'_n\|_\infty + \|f_n\|_\infty \\ &\leq \|f''_n\|_{L^p(\gamma_{p,\alpha,u})}^p + C\|f\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})}. \end{aligned}$$

Thus, we get that $\|M_u C_\psi(f_n)\|_{\mathcal{Z}_{p,\alpha}(\mathbb{D})} \rightarrow 0$, as $n \rightarrow 0$ and this completes the proof. □

Theorem 3.3. Let $-1 < \alpha$ and $2 + \alpha < p < \infty$, $u \in H(\mathbb{D})$ and $\psi \in \mathcal{S}$ such that $\psi' \in H^\infty(\mathbb{D})$. Then operator $M_u C_\psi : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \rightarrow \mathcal{Z}_{p,\alpha}(\mathbb{D})$ is compact if and only if both the two following operators are compact

$$M_{u'} C_\psi : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \rightarrow \mathcal{B}_{p,\alpha}(\mathbb{D}), \quad M_u DC_\psi : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \rightarrow \mathcal{B}_{p,\alpha}(\mathbb{D}).$$

Proof. By applying Theorem 2.6, relation (11) and using Lemma 3.1, the proof is straightforward. □

In the next theorem, we are interested to provide equivalent statements to characterize the compactness of the Stević-Sharma operator $T_{u_1, u_2, \psi} = M_{u_1} C_\psi + M_{u_2} C_\psi D$ between the Besov-Zygmund spaces. To achieve this result we will get help from Theorem 3.2, Lemma 3.1 and Lemma 2.3.

Theorem 3.4. Assume $-1 < \alpha$ and $2 + \alpha < p < \infty$. Let $u_1, u_2 \in H(\mathbb{D})$ and $\psi \in \mathcal{S}$ such that $u_2, \psi \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$. Then the following statements are equivalent:

(a) $T_{u_1, u_2, \psi} : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \rightarrow \mathcal{Z}_{p,\alpha}(\mathbb{D})$ is compact.

(b) The both following operators are compact

$$M_{u_1}C_\psi : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \longrightarrow \mathcal{Z}_{p,\alpha}(\mathbb{D}), \quad M_{u_2}C_\psi D : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \longrightarrow \mathcal{Z}_{p,\alpha}(\mathbb{D}).$$

(c) $u_1, u_1\psi, u_2\psi \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$, γ_{p,α,u_1} is a compact (α, p) -Carleson measure and $\gamma_{p,\alpha+p,u_2}$ is a compact $(\alpha + p, p)$ -Carleson measure.

Proof. (a) \Rightarrow (b). Suppose that $T_{u_1,u_2,\psi} : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \rightarrow \mathcal{Z}_{p,\alpha}(\mathbb{D})$ is compact, then $T_{u_1,u_2,\psi} : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \rightarrow \mathcal{Z}_{p,\alpha}(\mathbb{D})$ is bounded. So, for any bounded sequence $f_n \in A_{p,\alpha}(\mathbb{D})$ such that $f(0) = 0$ and f_n converges to zero uniformly on compact subsets of \mathbb{D} , let $h_n(z) = \int_0^z f_n(w)dw$ and $g_n(z) = \int_0^z h_n(w)dw$, then $g_n''(z) = f_n(z) \in A_{p,\alpha}(\mathbb{D})$ and $g(0) = g'(0) = 0$. Hence g_n is a bounded sequence in $\mathcal{Z}_{p,\alpha}(\mathbb{D})$ which converges to zero uniformly on compact subsets of \mathbb{D} . Then by applying Lemma 2.3 and relation (18), we get that

$$\begin{aligned} \left(\int_{\mathbb{D}} |f_n(w)|^p d\gamma_{p,\alpha,u_1}(w) \right)^{1/p} &= \left(\int_{\mathbb{D}} |u_1\psi'^2(z)|^p |f_n(\psi(z))|^p (1 - |z|^2)^\alpha dA(z) \right)^{1/p} \\ &= \|u_1\psi'^2 g_n'' \circ \psi\|_{A_{p,\alpha}} \\ &\leq \|u_1\psi'^2\|_{A_{p,\alpha}} \|g_n''\|_\infty \preceq \|u_1\psi'^2\|_{A_{p,\alpha}} \|g_n\|_{\mathcal{Z}_{p,\alpha}} \rightarrow 0. \end{aligned}$$

Therefore, relation (2) and Theorem 3.2 give us the compactness of operator $M_{u_1}C_\psi$ on $\mathcal{Z}_{p,\alpha}(\mathbb{D})$. Now by the compactness of $T_{u_1,u_2,\psi}$, for any bounded sequence $g_n \in \mathcal{Z}_{p,\alpha}(\mathbb{D})$ such that g_n converges to zero uniformly on compact subsets of \mathbb{D} , we get that

$$\|M_{u_2}C_\psi Dg_n\|_{\mathcal{Z}_{p,\alpha}} \leq \|T_{u_1,u_2,\psi}g_n\|_{\mathcal{Z}_{p,\alpha}} + \|M_{u_1}C_\psi g_n\|_{\mathcal{Z}_{p,\alpha}} \rightarrow 0,$$

which gives us the compactness of $M_{u_2}C_\psi D$ on $\mathcal{Z}_{p,\alpha}(\mathbb{D})$.

(b) \Leftrightarrow (c). By applying Lemma 2.3 and the same method as in Theorem 3.2 and Theorem 2.4, we see that $M_{u_2}C_\psi D : \mathcal{Z}_{p,\alpha}(\mathbb{D}) \longrightarrow \mathcal{Z}_{p,\alpha}(\mathbb{D})$ is compact if and only if $M_{u_1}C_\psi : \mathcal{Z}_{p,\alpha+p}(\mathbb{D}) \longrightarrow \mathcal{Z}_{p,\alpha}(\mathbb{D})$ is compact. Now using Theorem 3.2, the proof is complete.

(b) \Rightarrow (a). It is clear by using the triangle inequality. □

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Please cite this article using:

Sepideh Nasresfahani, Mostafa Hassanlou, Daryoush Molaei, Ebrahim Abbasi, Product type operators on (α, p) -Besov-Zygmund spaces, AUT J. Math. Comput., 7(3) (2026) 293-302
<https://doi.org/10.22060/AJMC.2025.22758.1191>

