



Short Contribution

The undirected power graph on the conjugacy classes of a finite group

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ABSTRACT: Let G be a finite group. The undirected power graph on the conjugacy classes of G is the simple graph $\mathcal{P}_C(G)$ whose vertices are the conjugacy classes of G and two distinct vertices C and C' are adjacent if one is a subset of a power of the other. In this paper, we show that the graph $\mathcal{P}_C(G)$ is 2-connected whenever either $|\pi(G)| > 1$ or $Z(G)$ is cyclic. Moreover, we classify finite groups G whose associated graph $\mathcal{P}_C(G) - \{e\}$ are bipartite.

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1. Introduction

Throughout this paper, G is a finite group and $a^G = \{a^g | g \in G\}$ is the conjugacy class of a in G . Observe that the power of a conjugacy class

$$(a^G)^m = \{a^{h_1 a^{h_2} \dots a^{h_m}} | h_1, \dots, h_m \in G\},$$

is a union of conjugacy classes of G , see [2] for more information.

Some researchers have defined the graphs related to conjugacy classes of finite groups, see [3, 6]. In further research, various authors tried to find some connections between the algebraic properties of a group and the associated graph's properties.

Inspired by the results, in [10], we defined the undirected power graph on the conjugacy classes $\mathcal{P}_C(G)$ of a group G as a simple graph in which the vertices are the conjugacy classes of G and two distinct vertices C and C' are adjacent in $\mathcal{P}_C(G)$ if one is a subset of a power of the other. We can easily show that two distinct vertices C and

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C' are adjacent in $\mathcal{P}_C(G)$ if either $\langle C \rangle \subseteq \langle C' \rangle$ or $\langle C' \rangle \subseteq \langle C \rangle$. In other words, this graph illustrates the relationships among the elements of the partially ordered set containing all subgroups generated by a conjugacy class of G .

On the other hand, we can see that if G is abelian, $\mathcal{P}_C(G)$ coincides with the power graph $\mathcal{P}(G)$, defined by Chakrabarty et al. [4]. In [9], the authors discussed the 2-connectivity of power graphs. Motivated by this result, in this paper, we show that the graph $\mathcal{P}_C(G)$ is 2-connected, $\mathcal{P}_C(G)$ does not have a cut vertex, if $|\pi(G)| > 1$, where $\pi(G)$ is the set of prime numbers dividing the order of G . Furthermore, we classify finite groups G whose associated graphs are connected bipartite graphs.

The notations we use are standard. For every element a in G , $o(a)$ is its order. $\text{Soc}(G)$ denotes the subgroup generated by all minimal normal subgroups of G and $F(G)$ is the Fitting subgroup of G .

2. Main Results

We know that if $\mathcal{P}_C(G) - \{e\}$ is connected, then $\mathcal{P}_C(G)$ is 2-connected. In what follows, first, we investigate 2-connectivity of $\mathcal{P}_C(G)$.

Theorem 2.1. *Let G be a finite group and $|\pi(G)| > 1$. Then the graph $\mathcal{P}_C(G)$ is 2-connected and $\text{diam}(\mathcal{P}_C(G) - \{e\}) \leq 4$.*

Proof. Assume that a^G and b^G are two distinct conjugacy classes of G . If $\langle a^G \rangle \cap \langle b^G \rangle \neq \{1\}$, then there exists some element e such that

$$\langle e^G \rangle \subseteq \langle a^G \rangle \cap \langle b^G \rangle,$$

and so $d(a^G, b^G) \leq 2$. Otherwise, $\langle a^G \rangle \cap \langle b^G \rangle = \{1\}$, consider $A = \langle a^G \rangle \times \langle b^G \rangle$ and $o(a) \leq o(b)$. First, suppose that $o(a)$ does not divide $o(b)$. As $o(ab) = [o(a), o(b)]$, we deduce that

$$b^{o(a)} = (ab)^{o(a)} \in \langle ab \rangle \subseteq \langle (ab)^G \rangle,$$

and

$$b^{o(a)} \in \langle b \rangle \subseteq \langle (b)^G \rangle,$$

which implies that $(b^{o(a)})^G$ is adjacent to $(ab)^G$ and $(b)^G$. Similarly, we can show that $(a^{o(b)})^G$ is adjacent to $(ab)^G$ and $(a)^G$. The previous arguments guarantee that $(a^G, (a^{o(b)})^G, (ab)^G, (b^{o(a)})^G, b^G)$ is a path in $\mathcal{P}_C(G) - \{e\}$ and in this case $d(a^G, b^G) \leq 4$.

Now, consider $o(a)$ divide $o(b)$. Since $o(ab) = o(b)$, we can write that $o(a) = p^\alpha m$ with $(p, m) = 1$ and $o(b) = p^\beta n$ where $(p, n) = 1$, $\alpha \leq \beta$, and $m|n$ for some natural numbers m, n, α , and β and $p \in \pi(o(a)) \cap \pi(o(b))$. We shall break the proof into two cases.

Case 1. If $n \neq 1$, then we observe that

$$a^{mn} = (a^m b^{p^\beta})^n \in \langle a^m b^{p^\beta} \rangle \subseteq \langle (a^m b^{p^\beta})^G \rangle.$$

It follows from $o(a^{mn}) = o(a^m) = p^\alpha$ that $\langle a^m \rangle = \langle a^{mn} \rangle \subseteq \langle (a^m b^{p^\beta})^G \rangle$ and so $(a^m)^G$ is adjacent to $(a^m b^{p^\beta})^G$ and $(a)^G$ in $\mathcal{P}_C(G)$. Similarly, we can obtain that $(b^{p^\beta})^G$ is adjacent to $(a^m b^{p^\beta})^G$ and $(b)^G$. Therefore,

$$(a^G, (a^m)^G, (a^m b^{p^\beta})^G, (b^{p^\beta})^G, b^G),$$

is a path in $\mathcal{P}_C(G) - \{e\}$ and in this case $d(a^G, b^G) \leq 4$.

Case 2. If $n = 1$, then there exists $q \in \pi(G) - \{p\}$, as $|\pi(G)| > 1$. Let x be an element of order q in G . A proof similar to the previous paragraphs, we conclude that $(a^G, C_1, x^G, C_2, b^G)$, where C_1 is either $(ax)^G$ or $e^G \subseteq \langle a^G \rangle \cap \langle x^G \rangle$, and C_2 is either $(bx)^G$ or $f^G \subseteq \langle x^G \rangle \cap \langle b^G \rangle$, is a path in $\mathcal{P}_C(G) - \{e\}$ and $d(a^G, b^G) \leq 4$ in this case also. \square

In the following example, we present a finite group G whose associated graph $\mathcal{P}_C(G)$ is not 2-connected.

Example 2.1. Let $G = \overbrace{(\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2)}^{r \geq 0} \times D_8$. We can check that the graph $\mathcal{P}_C(G) - \{e\}$ has $2^{r+1} - 1$ connected components, in which $2^{r+1} - 2$ of the components are isolated vertices and one of the components is a star graph with $3 \cdot 2^r$ edges.

Proposition 2.2. *Let G be a finite p -group. If $Z(G)$ is cyclic, then $\mathcal{P}_C(G)$ is 2-connected and $d(\mathcal{P}_C(G) - \{e\}) \leq 3$.*

Proof. First, suppose that G is abelian. In this case, the power graph on conjugacy classes $\mathcal{P}_C(G)$ coincides with the power graph $\mathcal{P}(G)$. It follows from Theorem 2.1 of [9] that $\mathcal{P}_C(G)$ is 2-connected if and only if G is a cyclic group. Consequently, $\mathcal{P}_C(G)$ is a complete graph.

Now, assume that G is non-abelian and $Z(G) = \langle z \rangle$. Let $a, b \in G$. As the subgroups generated by a^G and b^G are normal subgroups of the p -group G , both have non-trivial intersection with $Z(G)$. Therefore, there exists some positive integers i and j such that

$$\{e\} \neq z^i \in \langle a^G \rangle \cap Z(G) \text{ and } \{e\} \neq z^j \in \langle b^G \rangle \cap Z(G).$$

Hence, (a^G, z^i, z^j, b^G) is a path in $\mathcal{P}_C(G) - \{e\}$ and $d(a^G, b^G) \leq 3$. □

In the next example, we present a finite p -group G whose associated graph $\mathcal{P}_C(G)$ is 2-connected, however, the center $Z(G)$ is not cyclic.

Example 2.2. Consider the $G = \text{SmallGroup}(64, 59)$, that is isomorphic to $Z_4 \times (Z_4 \rtimes Z_4)$. Using GAP [1], we can see that $Z(G) \cong Z_4 \times Z_2 \times Z_2$ is not cyclic but $\mathcal{P}_C(G) - \{e\}$ is connected with $\text{diam}(\mathcal{P}_C(G) - \{e\}) = 5$. This argument verifies that the upper bound for $\text{diam}(\mathcal{P}_C(G) - \{e\})$ mentioned in Theorem 2.1 may not remain true for some p -groups.

For a finite group G , in Theorem 2.3 of [10], the authors considered the case where $\text{diam}(\mathcal{P}_C(G) - \{e\}) = 1$ and prove that such a group has a unique chief series each of whose members is a normal subgroup generated by a conjugacy class of G . Assume that $\text{diam}(\mathcal{P}_C(G) - \{e\}) = 2$. As a complete bipartite graph is an example of a graph with diameter two, it seems logical to discuss the case where $\mathcal{P}_C(G) - \{e\}$ is a complete bipartite graph.

Theorem 2.3. Let G be a finite group. The graph $\mathcal{P}_C(G) - \{e\}$ is a connected bipartite graph if and only if $\mathcal{P}_C(G) - \{e\}$ is a complete bipartite graph. This case holds if and only if one of the following situation occurs:

1. G is a 2-group, where $G' = Z(G) = \text{Soc}(G)$ and $G/Z(G)$ are both elementary abelian group. Moreover, $b^G = bZ(G)$ for each $b \in G - Z(G)$.
2. G is a Frobenius group with a cyclic complement of order 2 and an elementary abelian 3-group as kernel.

Proof. Assume that $\mathcal{P}_C(G) - \{e\}$ is a connected bipartite graph with bi-partition $(A|B)$, where $A = \{C_1, \dots, C_n\}$ and $B = \{C'_1, \dots, C'_m\}$ for some positive integers m and n . Without loss of generality, assume that there exists a vertex $C_t \in A$ such that $\langle C_t \rangle \subseteq \langle C'_l \rangle$ for some $l \in \{1, \dots, m\}$. One can observe that the structure of $\mathcal{P}_C(G) - \{e\}$ forces $\langle C_i \rangle = 1 \cup C_i$, for each $i \in \{1, \dots, n\}$. On the other hand, for each $j \in \{1, \dots, m\}$, let

$$I_j := \{r \in \{1, \dots, n\} | \langle C_r \rangle \subseteq \langle C'_j \rangle\}.$$

Henceforth,

$$\langle C'_j \rangle = (1 \cup C'_j) \cup_{i \in I_j} C_i.$$

for $1 \leq j \leq m$. We can check that $\langle C_i \rangle$ is an elementary abelian p_i -subgroup for some $p_i \in \pi(G)$ and $\langle C_i \rangle \subseteq G'$, by Corollary 1.7 of [11]. It follows that the order of each element of $\langle C'_j \rangle$ is either prime or the square of a prime number. Hence, we conclude that

$$\text{Soc}(G) = \langle C_1 \rangle \times \dots \times \langle C_n \rangle.$$

is an elementary abelian p -group for some $p \in \pi(G)$. Set $N = \text{Soc}(G)$. Since

$$\langle (bN)^{G/N} \rangle = 1 \cup (bN)^{G/N}.$$

for each $b \in G - N$, we obtain from Theorem 2.1 of [8] that G/N is an elementary abelian q -group for some $q \in \pi(G)$. Therefore, G is solvable.

On the other hand, we know that $\langle a^G \rangle = \langle (a^s)^G \rangle$ for positive number s with $(s, o(a)) = 1$. We yields form our assumption that a^s must be conjugate to a , for every $a \in G$. Thus, G is rational.

As G is a solvable group all of whose elements have prime power order, Theorem 1 of [7] guarantees that one of the following cases occurs:

Case 1. G is a p -group.

Since G is rational, we deduce that $p = 2$, $G' = Z(G) = \text{Soc}(G)$, and $cl(b) = bZ(G)$ for each $b \in G - Z(G)$. In this case, $\mathcal{P}_C(G) - \{e\}$ is a complete bipartite graph.

- Case 2.** G is a Frobenius group with kernel P_p and complement P_q , in which P_r is a Sylow r -subgroup of G . We deduce that P_p is an elementary abelian p -group and P_q is cyclic of order q . As G is rational, it can be derived from Main Theorem of [5] that $q = 2$ and $G \cong P_3 \rtimes Z_2$, where P_3 is an elementary abelian 3-group. Consequently, it can be seen that $\mathcal{P}_C(G) - \{e\} = K_{1, (|P_3|-1)/2}$ is a star graph.
- Case 3.** $\pi(G/F(G)) = \{p, q\}$ and each Sylow subgroup of $G/F(G)$ is cyclic. Since $G/\text{Soc}(G)$ is an elementary abelian q -group for some $q \in \pi(G)$, we get that $G \cong F(G) \rtimes H$ is a Frobenius group, in which $|H| = q$. It is similar to the previous paragraph and the proof is complete. \square

The next Corollary is a straightforward result from the previous theorem.

Corollary 2.4. *Let G be a finite group. The graph $\mathcal{P}_C(G) - \{e\}$ is a tree if and only if $\mathcal{P}_C(G) - \{e\}$ is a star graph if and only if G is a group satisfying Theorem 2.3 with $|Z(G)| \leq 2$.*

In the following example, we provide a finite group satisfying Theorem 2.3.

Example 2.3. *Consider $G = \text{SmallGroup}(64, 245)$, that is a central extension of $Z(G) = Z_2 \times Z_2$ by $Z_2 \times Z_2 \times Z_2 \times Z_2$. We can check that $\mathcal{P}_C(G) - \{e\} = K_{3, 15}$ is a complete bipartite graph whose vertices can be partitioned into two sets $Z(G) - \{1\}$ and 15 conjugacy classes contained in $G - Z(G)$. Moreover, $b^G = bZ(G)$ for each $b \in G - Z(G)$.*

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