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Original Article

Group analysis and numerical approximation of proliferating and maturing cellular populations model

Mohammad Hadi Noori Skandari, S. Reza Hejazi*, Hamid Erfanian Oraei Dehrokhi

Faculty of Mathematical Sciences, Shahrood University of Technology, Shahrood, Iran

ABSTRACT: The present paper focuses on the symmetry analysis and numerical simulation of a type of delay PDEs called proliferating and maturing cellular population equations, which includes a delay $\tau > 0$ and a shrinked argument ax. The aim of this research is to establish the symmetry group of the considered equation by extending the Lie group analysis of differential equations to delay differential equations. Subsequently, an extended Jacobi-pseudo-spectral (JPS) method is applied to find numerical solutions to the equation.

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1. Introduction

Delay differential equations (DDEs) play an important role in several real world applications, like biology, medicine, control theory, climate models and many others. For example, there is an applied problem in this area called aftereffect. It is well known that, together with the increasing expectations of dynamic performances, engineers need their models to behave more like the real process. Many processes include aftereffect phenomena in their inner dynamics. In addition, actuators, sensors and communication networks that are now involved in feedback control loops introduce such delays. Finally, besides actual delays, time lags are frequently used to simplify very high order models. Then, the interest for DDEs keeps on growing in all scientific areas and especially, in control engineering. Delay systems are still resistant to many classical controllers: one could think that the simplest approach would consist in replacing them by some finite-dimensional approximations. Unfortunately, ignoring effects which are adequately represented by DDEs is not a general alternative: in the best situation (constant and known delays), it

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* *Corresponding author.*

E-mail addresses: `nouriskandari@shahroodut.ac.ir` (M. H. Noori Skandari), `r_hejazi@shahroodut.ac.ir` (S. R. Hejazi), `h-erfanian94@gmail.com` (H. Erfanian Oraei Dehrokhi) leads to the same degree of complexity in the control design. In worst cases (time-varying delays, for instance), it is potentially disastrous in terms of stability and oscillations.

Delay differential equations have another names such as time-delay systems, systems with aftereffect or dead-time, hereditary systems, equations with deviating argument or differential-difference equations. Because of the richness of the history of time-delay differential equations, they have found numerous applications in various fields. So, time-delay differential equations continue to be an active area of research, with new applications and theoretical results being discovered all the time. In economics, time-delay differential equations have been used to model the dynamics of markets, the effects of taxation and the behavior of consumers and firms. In engineering, they have been used to model the dynamics of control systems, the stability of mechanical structures and the behavior of materials. And finally, these type of equations are commonly used to model the dynamics of populations with time delays in their reproduction, the spread of infectious diseases with incubation periods, and the synchronization of coupled oscillators with delayed interactions.

Recently, trending in DDEs led to proliferation of studies that it became an increasingly interest in mathematics. Most studies in the field of DDEs have only focused on approximate analysis with some standard analytic methods. For example Senu et. al. [19], developed two-derivative Runge-Kutta type method with three-stage fifth-order for solving a special type of third-order DDEs with constant delay. In this work an algorithm based on Newton interpolation and hybrid with the TDRKT method is built to approximate the solution of third-order DDEs. Also Laplace transforms could be used for solving linear DDEs [12]. In this situation the form of the resulting solution is a non-harmonic Fourier series. A sufficient degree of accuracy can often be achieved by using a relatively small number of terms in the associated truncated series [11]. However, another Laplace based method is discussed in [12]. The existence of solutions in cones to delayed higher-order differential equations is established in [3]. The block methods consisting of generalized Störmer-Cowell methods are proposed by Li et. al. in [13]. In this article they deal with numerical solutions for a class of nonlinear second order delay differential equations.

Along with this growth in theory and applications of DDEs, however, there has been little quantitative analytical analysis on these type of equations. The paper seeks to apply Lie group theory of differential equations and its extension on a DPDE which models the proliferating and maturing cellular populations model. It is a first-order equation in the form of

$$\frac{\partial u(t,x)}{\partial t}+r\frac{\partial}{\partial x}(xu(t,x))=-\gamma u(t,x)+\lambda u(t-\tau,ax)\left(1-u(t-\tau,ax)\right), \tag{1}$$

which describes cell population dynamics, in which there is simultaneous proliferation and maturation. The term $u(t,x)$ commonly referred to total number of proliferating cells at time t and maturation level x. The last term in Eq. (1) is a delay term as it depends on the population of cells at an earlier time $t-\tau$ and a maturation level ax. In this equation, the age τ at cytokinesis is identical between cells. The parameter $\gamma>0$ is the rate of cells death which is independent of age or maturation level, and mature with a velocity, which is proportional to the maturity x. Also $r>0$ is a constant of proportionality.

In the future the term $u(t-\tau,ax)$ is replaced by u^{τ}. So, the Eq. (1) is rewritten to

$$u_t+rxu_x=\lambda u^{\tau}(1-u^{\tau})-(r+\gamma)u. \tag{2}$$

When one is confronted with a complicated system of PDEs arising from some physically important problem, the discovery of any explicit solutions whatsoever is of great interest. Explicit solutions can be used as a models for physical experiments, as benchmarks for testing numerical methods, etc., and often reflect the asymptotic or dominant behavior of more general types of solutions. There are several methods for finding solutions of a system of differential equations in both exact and approximate cases. The method of Lie transformations (Lie symmetries or group theory of differential equations) is a strange unlimited tool in order to find exact solutions for all kinds of systems containing both ODEs or PDEs in linear or nonlinear form. Some differential operators called *symmetries* are found for this purpose. These operators are the infinitesimal generators corresponding to the largest group transformations acting on the independent and dependent variables of the system with the property that they transform solutions of the system to other solutions. So, if we have the group of symmetries together with an initial solution, the wide range of solutions will be given by using this method.

Symmetry method for solving differential equations has powerful tools for this goal, specially when the purpose is based on finding exact solutions of a given system. One of the most important application of symmetry method is to reduce the systems of differential equations, i.e., finding equivalent systems of differential equations of simpler form, that is called reduction process. This method provides a systematic computational algorithm for determining a large classes of special solutions. The solutions of the obtained equivalent system will correspond to solutions of the original system. There is a lot of papers in the literature for this process and one can find the classical reduction method in [1, 2, 9, 10, 15, 16, 17, 18]. But, this method is rarely used in the field of DDEs. Thus, the method is implemented and extended on the considered model in order to find an exact solution for the Eq. (2). Also, the

JPS method as a strong method which gives suitable approximate solutions with acceptable errors, is applied for computing the approximate solutions for numerical simulation [4, 5, 6, 7, 8, 14].

The overall structure of the paper takes the form of three chapters, including this introductory section and includes a discussion of the implication of the findings to future researches into this area. After giving the mathematical formulation in first section, the second section is concerned with the methodology used for the symmetry analysis of DDEs. Then, the findings show that the Eq. (2) admits only a single symmetry. The exact solution of the equation given through the symmetry is computed in the sequel. The third section presents the achievements of the numerical results, focusing on the extended JPS method. For this goal, the shifted Jacobi polynomial of degree N and order (α, β) is defined via an expansion. The remaining part of this section is devoted for numerical results including approximate solutions and their residual errors with some plotted graphs.

2. Group analysis of proliferating and maturing cellular populations model

The aim of this section is to provide and review a conceptual theoretical framework based on the invariant transformations for the given equation (2). In the part that follows, the methodology is described then, it will be used to find the symmetry of the equation.

Definition 2.1. *Suppose $\mathfrak{M}$ is the underlying manifold of a system of differential equations by the coordinates (t, x, u) and $\mathcal{I}$ is a symmetric interval with respect to zero. Let $\Phi : \mathfrak{M} \times \mathcal{I} \to \mathfrak{M}$ be a transformation on $\mathfrak{M}$. Let $\Phi_\varepsilon(t, x, u)$ satisfies the conditions*

- $\Phi_0(t, x, u) = (t, x, u)$,
- $\Phi_\varepsilon^{-1}(t, x, u) = \Phi_{-\varepsilon}(t, x, u)$,
- $\Phi_{\varepsilon_1}(\Phi_{\varepsilon_2}) = \Phi_{\varepsilon_1+\varepsilon_2}(t, x, u)$,

for parameter $\varepsilon \in \mathcal{I}$. Then the set of transformation $\{\Phi_\varepsilon\}$ forms a one-parameter group for manifold $\mathfrak{M}$.

If this one-parameter group leaves invariant $\mathfrak{M}$, i.e. $\Phi_\varepsilon(t, x, u) \in \mathfrak{M}$, then, Φ_ε is called a symmetry for $\mathfrak{M}$.

Suppose $\bar{X}$ is an infinitesimal generator for the first order DDE

$$\Delta(x, t, u, u^\tau, u_x, u_t) = 0. \tag{3}$$

$\bar{X}$ corresponds to a symmetry transformation of Eq. (3) iff $\bar{X}$ leaves (3) invariant, i.e.

$$\bar{X}(\Delta(x, t, u, u^\tau, u_x, u_t)) \equiv 0, \quad (\text{mod } (3)), \tag{4}$$

where

$$\bar{X} = (\zeta - \xi u_x - \eta u_t)\frac{\partial}{\partial u} + \zeta^x \frac{\partial}{\partial u_x} + \zeta^t \frac{\partial}{\partial u_t} + (\zeta^\tau - u_x^\tau - u_t^\tau \eta^\tau)\frac{\partial}{\partial u^\tau}, \tag{5}$$

is the first prolongation of the infinitesimal generator

$$X = \xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial t} + \zeta\frac{\partial}{\partial u} + \zeta^\tau\frac{\partial}{\partial u^\tau}. \tag{6}$$

Here

$$\xi(x, t, u) = \frac{\partial \bar{x}}{\partial \varepsilon}\Big|_{\varepsilon=0}, \qquad \eta(x, t, u) = \frac{\partial \bar{t}}{\partial \varepsilon}\Big|_{\varepsilon=0}, \qquad \zeta(x, t, u) = \frac{\partial \bar{u}}{\partial \varepsilon}\Big|_{\varepsilon=0},$$

where

$$\bar{x} = \Phi_\varepsilon^x(x, t, u), \quad \bar{t} = \Phi_\varepsilon^t(x, t, u), \quad \bar{u} = \Phi_\varepsilon^u(x, t, u),$$

and

$$\begin{aligned} \xi^\tau &= \xi(x, t-\tau, u^\tau), & \eta^\tau &= \eta(x, t-\tau, u^\tau), & \zeta^\tau &= \zeta(x, t-\tau, u^\tau), \\ \zeta^u &= \zeta - u_x\xi - u_t\eta, & \zeta^x &= D_x(\zeta - u_x\xi - u_t\eta), & \zeta^t &= D_t(\zeta - u_x\xi - u_t\eta), \end{aligned}$$

for total derivative operators

$$\begin{aligned} D_x &= \frac{\partial}{\partial x} + u_x\frac{\partial}{\partial u} + u_x^\tau\frac{\partial}{\partial u^\tau} + u_{xx}\frac{\partial}{\partial u_x} + \cdots, \\ D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + u_t^\tau\frac{\partial}{\partial u^\tau} + u_{xt}\frac{\partial}{\partial u_x} + \cdots. \end{aligned}$$

2.1. Determining system for Eq. (1)

By the theory of existence of a solution of a DDE, the initial value problem has a particular solution corresponding to a particular initial value. Because initial values are arbitrary, variables u, u^τ and theirderivatives can be considered as arbitrary elements. Since every transformed-solution $\bar{u}(\bar{x},\bar{t})$ is a solution of Eq. (2), the determining equation must be identical to zero. Thus, if determining Eq. (4) is written as a polynomial of variables and their derivatives, the coefficients of these variables in the equations must vanish. In order to solve a determining equation, one solves the several equations of these coefficients. This method is called splitting the determining equation. Unknown functions ξ, η andζ can be obtained from this process.

The determining system for Eq. (2) is constructed by substituting

$$\Delta = u_t + rxu_x - \lambda u^\tau(1-u^\tau) + (r+\gamma)u,$$

into Eq. (4). So, if we expand the identity

$$\tilde{X}(\Delta) \equiv 0\Big|_{\Delta=0},$$

and solve the concluded determining system, the single symmetry of the Eq. (3) is derived by

$$X = (1+4e^{-t})\frac{\partial}{\partial u} + 2\frac{\partial}{\partial u^\tau}. \tag{7}$$

2.2. Exact solution of Eq. (2)

The method of finding exact solutions from reduced form of a given system of DDEs is same as the other system of differential equations. The characteristic system extracted from operator (7) is

$$\frac{dx}{d\epsilon} = 0, \quad \frac{dt}{d\epsilon} = 0, \quad \frac{du}{d\epsilon} = (1+4e^{-t}), \quad \frac{du^\tau}{d\epsilon} = 2. \tag{8}$$

Thus, solving system (8) in order to find the reduced form of the Eq. (2) with respect to the corresponding invariant function, gives

$$2u - u^\tau(1+4e^{-t}) = 0, \tag{9}$$

as an implicit exact solution.

3. Numerical experiments

In this section the JPS method is extended in order to find numerical solutions of the considered DDE.

3.1. Approximate solution of hyperbolic delay partial differential equation with shrinked argument

Consider the hyperbolic DDE (1) on $[0,T]\times[0,L]$ with condition

$$u(t,x) = \phi(t,x), \quad -\tau \le t \le 0, \;\; 0 \le x \le L, \tag{10}$$

where $u(t,x)$ is an unknown function, $\phi(t,x)$ is the given delay function, $0<a<1$ and $r,\gamma,\lambda,T,L>0$, are given constants, $0<\tau<T$ is the delay time, and ax is the shrinked argument. It is trivial that

$$\frac{\partial}{\partial x}\big(xu(t,x)\big) = u(t,x) + x\frac{\partial u}{\partial x}(t,x), \tag{11}$$

and

$$u(t-\tau,x) = \phi(t-\tau,x), \quad 0 \le t \le \tau, \; 0 \le x \le L. \tag{12}$$

By relations (11) and (12), we can rewrite equations (1) and (10) as the following equivalent system

$$\begin{cases} \frac{\partial u}{\partial t}(t,x) = \begin{cases} -r\frac{\partial u}{\partial x}(t,x) - (r+\gamma)u(t,x) + \lambda\phi(t-\tau,ax)\big(1-\phi(t-\tau,ax)\big), & 0<t\le\tau, \;\; 0\le x\le L, \\ -r\frac{\partial u}{\partial x}(t,x) - (r+\gamma)u(t,x) + \lambda u(t-\tau,ax)\big(1-u(t-\tau,ax)\big), & \tau<t\le T, \;\; 0\le x\le L, \end{cases} \\ u(0,x) = \phi(0,x), & 0\le x\le L. \end{cases} \tag{13}$$

The main goal is to present an extended JPS method to numerically solve the above system. In extended JPS method, we approximate the solution of (10) as follows

$$u(t,x) \approx u_N(t,x) = \begin{cases} u_N^{(1)}(t,x) = \sum_{i=0}^{N}\sum_{j=0}^{N} \bar{u}_{ij}\bar{\psi}_i^N(t)\psi_j^N(x), & 0\le t\le\tau, \; 0\le x\le L, \\ u_N^{(2)}(t,x) = \sum_{i=0}^{N}\sum_{j=0}^{N} \hat{u}_{ij}\hat{\psi}_i^N(t)\psi_j^N(x), & \tau<t\le T, \; 0\le x\le L, \end{cases} \tag{14}$$

where $(\bar{u}_{ij}, \hat{u}_{ij})$ are unknown coefficients, functions $\psi_j(.)^N, \bar{\psi}_i(.)^N$ and $\hat{\psi}_i(.)^N$ are the Lagrange interpolating polynomials defined as

$$\psi_j^N(x) = \prod_{l=0, l\neq j}^{N} \frac{x - x_l}{x_j - x_l}, \quad 0 \leq x \leq L,$$

$$\bar{\psi}_i^N(t) = \prod_{l=0, l\neq i}^{N} \frac{t - \bar{t}_l}{\bar{t}_i - \bar{t}_l}, \quad 0 \leq t \leq \tau,$$

$$\hat{\psi}_i^N(t) = \prod_{l=0, l\neq i}^{N} \frac{t - \hat{t}_l}{\hat{t}_i - \hat{t}_l}, \quad \tau \leq t \leq T,$$

where $\{x_k\}_{k=0}^N$, $\{\bar{t}_k\}_{k=0}^N$ and $\{\hat{t}_k\}_{k=0}^N$ are the shifted JGL nodes defined on intervals $[0, L]$, $[0, \tau]$ and $[\tau, T]$, respectively.

Generally, the shifted JGL nodes $\{z_k\}_{k=0}^N$ on interval $[R_1, R_2]$ are defined as the roots of polynomial

$$Q_{N+1}(z) = \left[1 - \left(\frac{2z}{R_2 - R_1} - \frac{R_2 + R1}{R_2 - R_1}\right)^2\right] \frac{d}{dz} \tilde{J}_N^{\alpha,\beta}(z).$$

where shifted Jacobi polynomial $\tilde{J}_N^{\alpha,\beta}(.)$ of degree N and order (α, β) is defined by the following expansion on interval $[R_1, R_2]$,

$$\tilde{J}_N^{\alpha,\beta}(z) = \frac{\Gamma(N + \alpha + 1)}{N!\,\Gamma(N + \alpha + \beta + 1)} \sum_{k=0}^{N} \binom{n}{k} \frac{\Gamma(N + k + \alpha + \beta + 1)}{\Gamma(k + \alpha + 1)} \left(\frac{z}{R_2 - R_1} - \frac{R_2}{R_2 - R_1}\right)^k,$$

where $\alpha, \beta > -1$ and $R_2 > R_1$. Note that the Lagrange polynomials satisfy

$$\psi_j^N(x_k) = \bar{\psi}_j^N(\bar{t}_k) = \hat{\psi}_j^N(\hat{t}_k) = \begin{cases} 1, & k = j, \\ 0, & k \neq j, \end{cases} \tag{15}$$

for $j, k = 0, 1, \ldots, N$. On the other hand, as a direct result of relation (14), if $\tau < \frac{T}{2}$ then for all $(t, x) \in [\tau, T] \times [0, L]$

$$u_N(t - \tau, x) = \begin{cases} u_N^{(1)}(t - \tau, x), & \tau \leq t < 2\tau, \\ u_N^{(2)}(t - \tau, x), & 2\tau \leq t \leq T, \end{cases} \tag{16}$$

also, if $\tau \geq \frac{T}{2}$ then for all $(t, x) \in [\tau, T] \times [0, L]$

$$u_N(t - \tau, x) = u_N^{(1)}(t - \tau, x). \tag{17}$$

By applying (14) and inserting (16) and (17) in system (13), we get the following equivalent system

$$\begin{cases} \dfrac{\partial u_N^{(1)}}{\partial t}(t, x) = -r\dfrac{\partial u_N^{(1)}}{\partial x}(t, x) - (r + \gamma)u_N^{(1)}(t, x) & \\ \qquad\qquad + \lambda\phi(t - \tau, ax)\big(1 - \phi(t - \tau, ax)\big), & 0 < t \leq \tau, \ \ 0 \leq x \leq L, \\ \dfrac{\partial u_N^{(2)}}{\partial t}(t, x) = \begin{cases} -r\dfrac{\partial u_N^{(2)}}{\partial x}(t, x) - (r + \gamma)u_N^{(2)}(t, x) \\ \qquad + \lambda u_N^{(1)}(t - \tau, ax)\big(1 - u_N^{(1)}(t - \tau, ax)\big), & \tau < t < \zeta_\tau, \ \ 0 \leq x \leq L, \\ -r\dfrac{\partial u_N^{(2)}}{\partial x}(t, x) - (r + \gamma)u_N^{(2)}(t, x) \\ \qquad + \lambda u_N^{(2)}(t - \tau, ax)\big(1 - u_N^{(2)}(t - \tau, ax)\big), & \zeta_\tau \leq t \leq T, \ \ 0 \leq x \leq L, \end{cases} \\ u_N^{(1)}(0, x) = \phi(0, x), & 0 \leq x \leq L, \\ u_N^{(2)}(0, x) = u_N^{(1)}(\tau, x), & 0 \leq x \leq L, \end{cases} \tag{18}$$

where $[0, T] = [0, \tau] \cup [\tau, \zeta_\tau] \cup [\zeta_\tau, T]$ and

$$\zeta_\tau = \begin{cases} 2\tau, & \tau \leq \frac{T}{2}, \\ T, & \tau > \frac{T}{2}. \end{cases} \tag{19}$$

Notice that, the condition $u_N^{(2)}(0,x) = u_N^{(1)}(\tau,x), \quad 0 \le x \le L$ is added to guarantee the continuity of approximation $u_N(.,.)$ defined by (14). Now, by relations (14) and (15), we achieve

$$u(\bar{t}_k, x_p) \approx u_N^{(1)}(\bar{t}_k, x_p) = \bar{u}_{kp}, \quad u(\hat{t}_k, x_p) \approx u_N^{(2)}(\hat{t}_k, x_p) = \hat{u}_{kp}, \tag{20}$$

for $k,p = 0,1,\dots,N$. Also, by (15), to approximate the partial differentiations of $u(.,.)$ at the collocation points we have

$$\frac{\partial u}{\partial t}(\bar{t}_k, x_p) \approx \frac{\partial u_N^{(1)}}{\partial t}(\bar{t}_k, x_p) = \sum_{i=0}^{N} \bar{u}_{ip} \bar{D}_{ki}, \tag{21}$$

$$\frac{\partial u}{\partial t}(\hat{t}_k, x_p) \approx \frac{\partial u_N^{(2)}}{\partial t}(\hat{t}_k, x_p) = \sum_{i=0}^{N} \hat{u}_{ip} \hat{D}_{ki}, \tag{22}$$

$$\frac{\partial u}{\partial x}(\bar{t}_k, x_p) \approx \frac{\partial u_N^{(1)}}{\partial x}(\bar{t}_k, x_p) = \sum_{j=0}^{N} \bar{u}_{kj} D_{pj}, \tag{23}$$

$$\frac{\partial u}{\partial x}(\hat{t}_k, x_p) \approx \frac{\partial u_N^{(2)}}{\partial x}(\hat{t}_k, x_p) = \sum_{j=0}^{N} \hat{u}_{kj} D_{pj}, \tag{24}$$

where

$$\bar{D}_{ki} = \bar{\psi}_i'(\bar{t}_k), \ \hat{D}_{ki} = \hat{\psi}_i'(\hat{t}_k), \ D_{pj} = \psi_j'(x_p), \tag{25}$$

for $i,j,k,p = 0,1,\dots,N$. Now, by Eqs. (20)–(24) the following system of algebraic equations is achieved:

$$\begin{cases}
\begin{aligned}\sum_{i=0}^{N} \bar{u}_{ip}\bar{D}_{ki} = &- r x_p \sum_{j=0}^{N} \bar{u}_{kj} D_{pj} - (r+\gamma)\bar{u}_{kp} \\ &+ \lambda \phi(\bar{t}_k - \tau, a x_p)\big(1 - \phi(\bar{t}_k - \tau, a x_p)\big),\end{aligned} & k = 1,2,\dots,N, \ p = 0,1,\dots,N \\
\sum_{i=0}^{N} \hat{u}_{ip}\hat{D}_{ki} = \begin{cases} \begin{aligned}&- r x_p \sum_{j=0}^{N} \hat{u}_{kj} D_{pj} - (r+\gamma)\hat{u}_{kp} \\ &\quad + \lambda u_N^{(1)}(\hat{t}_k - \tau, a x_p)\big(1 - u_N^{(1)}(\hat{t}_k - \tau, a x_p)\big),\end{aligned} & k = 1,2,\dots,m_\tau - 1, \ p = 0,1,\dots,N \\ \begin{aligned}&- r x_p \sum_{j=0}^{N} \hat{u}_{kj} D_{pj} - (r+\gamma)\hat{u}_{kp} \\ &\quad + \lambda u_N^{(2)}(\hat{t}_k - \tau, a x_p)\big(1 - u_N^{(2)}(\hat{t}_k - \tau, a x_p)\big),\end{aligned} & k = m_\tau,\dots,N, \ p = 0,1,\dots,N \end{cases} \\
\bar{u}_{0p} = \phi(0, x_p), & p = 0,1,\dots,N, \\
\hat{u}_{0p} = \bar{u}_{Np}, & p = 0,1,\dots,N,
\end{cases} \tag{26}$$

where $u_N^{(l)}(.,.)$ for $l = 1,2$ are defined in (14). Also,

$$m_\tau = \begin{cases} \min\left\{0 < k \le N : \hat{t}_k \ge 2\tau\right\}, & \tau \le \frac{T}{2}, \\ N, & \tau > \frac{T}{2}. \end{cases} \tag{27}$$

Note that system (26) has $2(N+1)^2$ equations and $2(N+1)^2$ unknown variables $(\bar{u}_{kp}, \hat{u}_{kp}), \ k,p = 0,1,\dots,N$.

3.2. Numerical simulation

In this section, the behavior of delay partial differential equation (1) with condition (10) for different parameters and delay functions is illustrated by solving algebraic system (26). We utilize Levenberg-Marquardt algorithm and FSOLVE command in MATLAB software to solve this system.

Note that the Levenberg-Marquardt algorithm is an effective method for solving systems of algebraic nonlinear equations. While it's widely known for its application in curve-fitting problems, it can also be adapted for finding solutions to systems of nonlinear equations. In fact the algorithm essentially modifies the Gauss-Newton method to handle cases where the problem is ill-conditioned or the initial guess is far from the solution. It introduces a damping parameter to switch between the Gauss-Newton method and gradient descent. To more clarification, define system (26) as $F(V) = 0$ where

$$V = (v_1, v_2, \dots, v_{2(N+1)^2}) = \big((\bar{u}_{kp}, \hat{u}_{kp}) : \ k,p = 0,1,\dots,N\big), \ \ F(V) = (F_1(V), F_2(V), \dots, F_{2(N+1)^2}(V)).$$

Start with an initial guess $V^{(0)}$ for the unknown vector V. Compute the Jacobian matrix $J(V)$ which contains the first derivatives of each equation with respect to each unknown, that is $J_{ij} = \frac{\partial F_i}{\partial v_j}$. Update the solution vector V using the following iterative formula

$$V^{(r+1)} = V^{(r)} - \left(J(V^{(r)})^T J(V^{(r)}) + \mu I\right)^{-1} J(V^{(r)})^T F(V^{(r)}), \quad r = 0, 1, 2, \ldots$$

Here, μ is the damping parameter and $\mathbf{I}$ is the identity matrix. The parameter μ is adjusted dynamically at each iteration. The algorithm is robust and can handle cases where the initial guess is not close to the final solution. It combines the fast convergence of the Gauss-Newton method with the stability of gradient descent. The convergence can be checked by evaluating the norm of residual vector $F(V^{(r+1)})$. If the norm is below a predefined threshold, the algorithm has converged to a solution.

The approximate solution of Eq. (1) with condition (10) in the following Cases 1-6 are not available, generally. So after calculating the approximate solutions, we compute the residual error (RE) with the following relation

$$RE_N(t,x) = \left|\frac{\partial u_N}{\partial t} + r\frac{\partial}{\partial x}\left(x u_N(t,x)\right) + \gamma u_N(t,x) - \lambda u_N(t-\tau, ax)\left(1 - u_N(t-\tau, ax)\right)\right|.$$

Moreover, we define the following errors to illustrate the stability of presented method and show that they convergent to zero by increasing N:

$$E_\infty^N = \|u_N(.,.) - u_{N-1}(.,.)\|_\infty, \;\; E_2^N = \|u_N(.,.) - u_{N-1}(.,.)\|_2.$$

Case 1: Assume

$$r = 1, \;\; \gamma = 1.5, \;\; \lambda = 2.5, \;\; T = L = 1, \;\; \tau = 0.25, \;\; a = 0.5, \;\; N = 6, \;\; \alpha = 0.25, \;\; \beta = 0, \quad \phi = \sin(\pi x)\, \cos(\pi t x).$$

The obtained approximate solution and its residual error are illustrated in Figure 1.

Case 2: Assume

$$r = 1, \;\; \gamma = 4, \;\; \lambda = 1, \;\; T = L = 1, \;\; \tau = 0.25, \;\; a = 0.5, \;\; N = 6, \;\; \alpha = 0.25, \;\; \beta = 0, \;\; \phi = \sin(\pi x)\, \cos(\pi t x).$$

The gained approximate solution and its residual error are displayed in Figure 2.

Case 3: Assume

$$r = 1, \;\; \gamma = 6, \;\; \lambda = 2, \;\; T = L = 1, \;\; \tau = 0.25, a = 0.25, \;\; N = 6, \;\; \alpha = 0.25, \;\; \beta = 0, \;\; \phi(t,x) = 4x^2 + \sin(\pi t)$$

The obtained approximate solution and its residual error are presented in Figure 3.

Case 4: Assume

$$r = 1, \;\; \gamma = 6, \;\; \lambda = 2, \;\; T = L = 1, \;\; \tau = 0.4, \;\; a = 0.8, \;\; N = 6, \;\; \alpha = 0.25, \;\; \beta = 0, \;\; \phi(t,x) = 4x^2 + \sin(\pi t).$$

Figure 4 illustrates the obtained approximate solution along with its residual error.

Case 5: Assume

$$r = 1, \;\; \gamma = 3, \;\; \lambda = 0.5, \;\; T = L = 1, \;\; \tau = 0.25, \;\; a = 0.75, \;\; N = 6, \;\; \alpha = 0.25, \;\; \beta = 0, \;\; \phi(t,x) = 1.$$

Figure 5 presents the obtained approximate solution along with its residual error.

Case 6: Assume

$$r = 1, \;\; \gamma = 3, \;\; \lambda = 0.5, \;\; T = L = 1, \;\; \tau = 0.25, \;\; a = 0.75, \;\; N = 6, \;\; \alpha = 0.25, \;\; \beta = 0, \;\; \phi(t,x) = 1 + x^2.$$

Figure 6 displays the obtained approximate solution and its residual error.

The results from Cases 1-8 demonstrate the presented method's strong performance, achieving a low residual error when solving equation (1) with the initial condition (10). Moreover, we show the numerical convergence of method by increasing N. For this goal, we consider Case 2 and show the errors E_∞^N and E_2^N in Figure 7. It can be seen that they go to zero by increasing N. Further, the obtained approximate solutions $u_N(.,.)$ for $N = 3, 4, \ldots, 11$ are illustrated in Figures 8, 9, 10 and 11 which they confirm the stable behavior in the results.

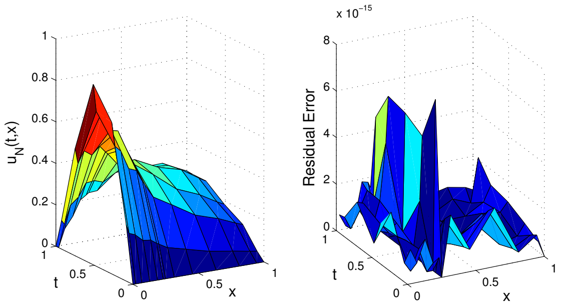

Figure 1: The obtained approximate solution and its residual error for Case 1.

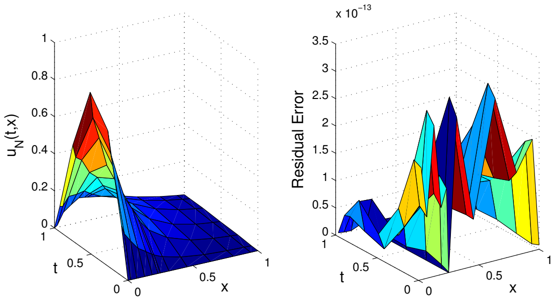

Figure 2: The obtained approximate solution and its residual error for Case 2.

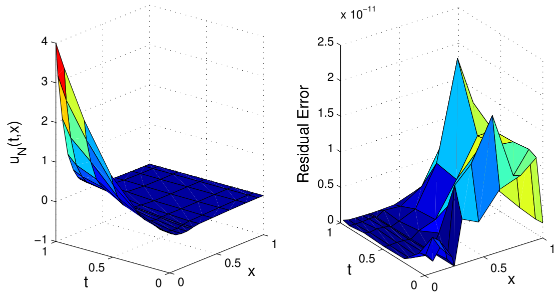

Figure 3: The obtained approximate solution and its residual error for Case 3.

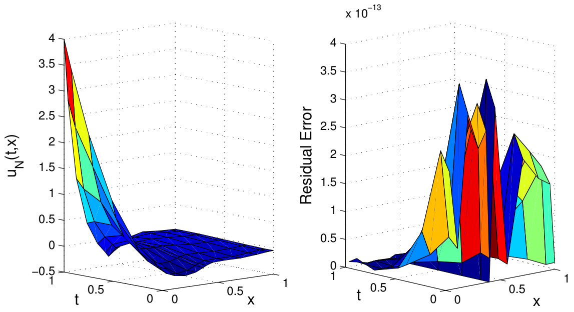

Figure 4: The obtained approximate solution and its residual error for Case 4.

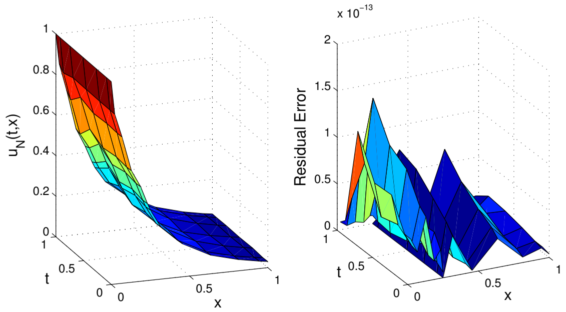

Figure 5: The obtained approximate solution and its residual error for Case 5.

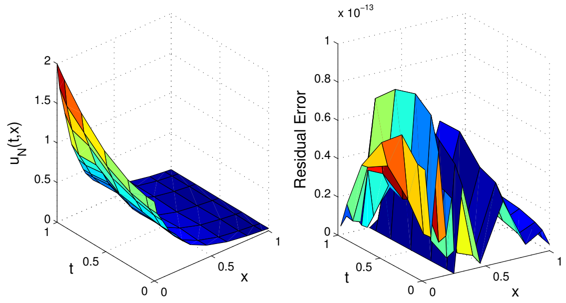

Figure 6: The obtained approximate solution and its residual error for Case 6.

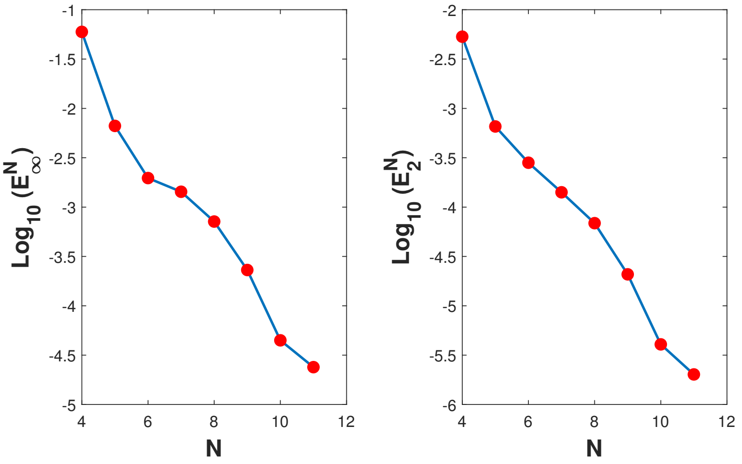

Figure 7: The errors E_∞^N and E_2^N with different values of N for Case 2.

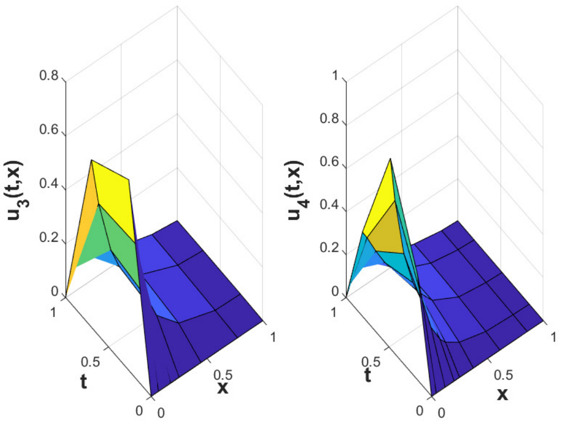

Figure 8: The obtained approximate solutions $u_3(.,.)$ and $u_4(.,.)$ for Case 2.

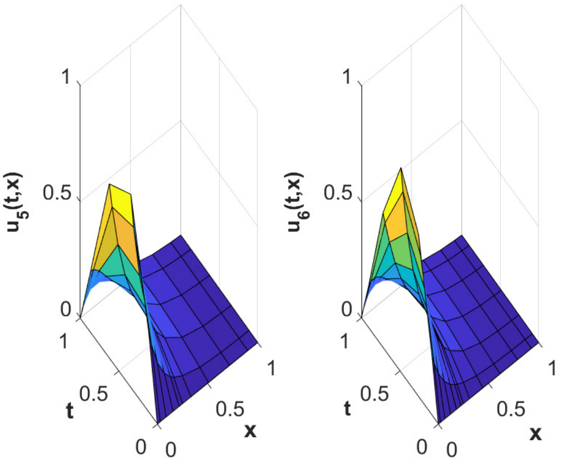

Figure 9: The obtained approximate solutions $u_5(.,.)$ and $u_6(.,.)$ for Case 2.

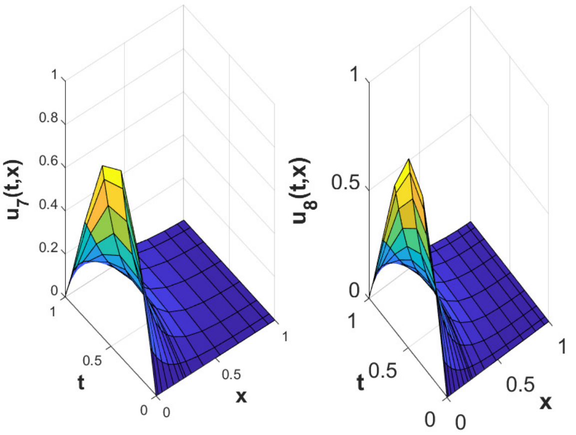

Figure 10: The obtained approximate solutions $u_7(.,.)$ and $u_8(.,.)$ for Case 2.

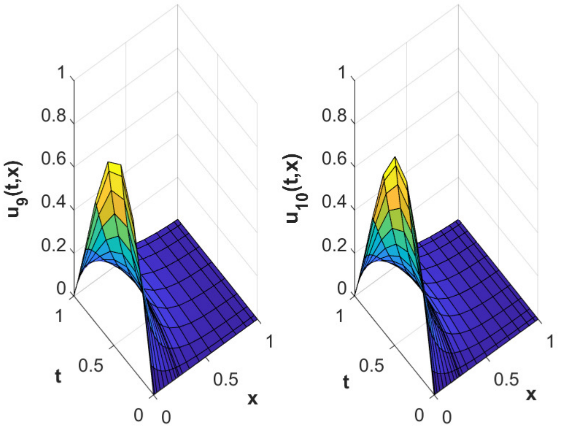

Figure 11: The obtained approximate solutions $u_9(.,.)$ and $u_{10}(.,.)$ for Case 2.

4. Concluding Remark

In this article we provided two approaches on a time-delay differential equation that models proliferating and maturing cellular populations. The exact method was based on the symmetry theory of differential equations using the invariant functions arised from the symmetry of the equation. The symmetry method for PDEs is extended to DDEs. Lie symmetries showed that the equation admits a single symmetry which is computed by standard Lie algorithm for finding symmetries. This symmetry gave an exact solution (9), called similarity solution in the literature. In the next section the JPS method is intended and some numerical solutions are given. The part of the paper devoted to the numerical simulation of the equation by JPS method which is a useful and operational method for solving PDEs including delay time. A potential problem is that the scope of this paper was to analyze a DDE, what about the combinations of the other equations such as fractional differential equations, delay-fractional differential equation and perturbed-delay differential equation. There are several important areas where this study makes some significant issues. In the future work we will try on the above types of differential equations.

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