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Original Article

On stochastic comparisons of finite α-mixture of additive hazard rate models

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ABSTRACT: This paper discusses stochastic comparisons on the finite α-mixture of additive hazard models. Sufficient conditions on the underlying distribution parameters and the mixing probabilities are established for the comparisons of different α-mixtures of survival or distribution functions of these models with respect to the usual stochastic order and the hazard rate order, respectively. Several examples are also presented to illustrate the theoretical findings.

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1. Introduction

Finite mixture models have found important applications in many areas, such as reliability theory, actuarial science, economics, etc. For example, manufactured engineering items are often heterogeneous due to different reasons such as the quality of resources and components used in the production process, operational history and human errors; see Finkelstein [7] and Cha and Finkelstein [5]. In this regard, the finite mixture model can be used for modelling lifetime data arising from a finite number of heterogeneous subpopulations.

large amount of research on different aspects of mixture models has been published in the past few years. For example, Finkelstein and Esaulova [8] considered a survival model that generalizes additive hazards models, proportional hazards models, and accelerated life models, and discussed the asymptotic behavior of the mixture failure rate. Finkelstein and Esaulova [9] discussed the problem of mixture failure rate ordering for the ordered mixing distributions. Franco et al. [10] studied the generalized mixture of Weibull components and presented

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E-mail addresses: `aqeel_alwaseti@yahoo.com` (A. Lazam Razzaq), `i.almasi@razi.ac.ir` (I. Almasi), `gh.saadatkia@kut.ac.ir` (Gh. Saadat Kia) some reliability properties. Navarro [15] established sufficient conditions for the hazard rate and likelihood ratio comparisons of generalized mixtures. Amini-Seresht and Zhang [1] investigated stochastic comparisons for two classical finite mixture models with different baseline random variables and different mixing proportions in the sense of some traditional stochastic orders. Barmalzan et al. [4] investigated the usual stochastic ordering and the reversed hazard rate ordering properties of finite mixture models with components having location-scale distributions. For more detailed treatment of the properties and applications of finite mixture models, interested readers are referred to the monographs by Everitt and Hand [6] and Titterington et al. [17].

The finite α-mixture model (with two possibly different underlying distributions) is originally used in Van Erven and Harremos [18] and Asadi et al. [2] to study different types of divergence problems in the area of information theory. Asadi et al. [3] presented its definition and then study the failure rate properties of this family and establish closures under monotone failure rates of the underlying components. It is of interest to note that the α-mixture model uniquely possesses a mathematical property known in economics as the constant elasticity of substitution, which provides an important interpretation for α in practice. Constant elasticity of substitution is widely used for modeling production functions and utility (consumption) functions with multiple inputs. This model includes the survival mixture model, the failure rate mixture model, models that are stochastically closer to each of these conventional mixtures, and many other useful models. Moreover, the α-mixture model also provides a flexible tool for modeling the lifetimes of heterogeneous units.

The finite α-mixture model includes many existing mixture models as special cases, to name a few, the survival/distribution mixture, the failure rate mixture, and many other mixture models. Among them, the finite α-mixture of survival functions is defined through the weighted α-th power mean of a finite number of proportional hazard rates (PHR) models with different baselines. In particular, if we index the baseline survival functions with some parameters of interest, i.e, the semi-parametric survival functions $\bar{F}(\cdot;\beta_i)$, for $i=1,\dots,n$, then its explicit expression can be written as

$$\bar{F}_\alpha(x)=\begin{cases}\left[\sum_{i=1}^n p_i\bar{F}^\alpha(x;\beta_i)\right]^{1/\alpha}, & \alpha\in\mathbb{R}\backslash\{0\},\\ \prod_{i=1}^n \bar{F}^{p_i}(x;\beta_i), & \alpha=0,\end{cases} \tag{1}$$

for $x\in\mathbb{R}$, where $p_i\geq 0$ and $\sum_{i=1}^n p_i=1$. As pointed out by Asadi et al. [3], the α-mixture model in (1) is a flexible family of mixture distributions containing the following several models as special cases:

(i) For $\alpha=0$, we have $-\log\bar{F}_0(x)=\sum_{i=1}^n p_i\left[-\log\bar{F}(x;\beta_i)\right]$. This results in the special mixture-type model on the cumulative hazard rate and hence the ordinary hazard scale model.

(ii) For $\alpha=1$, it reduces to the usual mixture distribution.

(iii) For $\alpha=-1$, it reduces to the harmonic mixture (mean) of the baseline survival functions.

(iv) For $n=2$ and $\alpha=m^{-1}$, it reduces to the binomial expansion mixture

$$\bar{F}_{\frac{1}{m}}(x)=\sum_{k=0}^m\binom{m}{k}p_1^{m-k}p_2^k\bar{F}^{1-\frac{k}{m}}(x;\beta_1)\bar{F}^{\frac{k}{m}}(x;\beta_2),$$

where $\binom{m}{k}$ is the binomial coefficient.

Furthermore, the α-mixture of distribution function can be defined as follows:

$$F_\alpha(x)=\begin{cases}\left[\sum_{i=1}^n p_iF^\alpha(x;\beta_i)\right]^{1/\alpha}, & \alpha\in\mathbb{R}\backslash\{0\},\\ \prod_{i=1}^n F^{p_i}(x;\beta_i), & \alpha=0,\end{cases} \tag{2}$$

for $x\in\mathbb{R}$. It should be mentioned that the α-mixture of densities, the α-mixture of survival functions and the α-mixture of distribution functions are completely different models unless $\alpha=1$. For $\alpha<0$ ($\alpha>0$), the α-mixture model has an increasing a decreasing) failure rate (IFR (DFR)) when all components of the mixture are IFR (DFR). A similar closure property holds in terms of the increasing (decreasing) failure rate average (IFRA (DFRA)). Interested readers are referred to Asadi et al. [3] for comprehensive discussions on the α-mixture model and its applications.

Comparison of important lifetime characteristics associated with lifetimes of technical systems is a topic of great interest in reliability theory. A convenient tool for this purpose is the theory of stochastic orderings; see the monographs by Müller and Stoyan [14] and Shaked and Shanthikumar [16]. Suppose we have n different subpopulations with an infinite number of components in the i-th subpopulation having common reliability function $\bar{F}_i$, for $i = 1, \dots, n$. Now, let us draw a unit with certain probabilities from each subpopulation, say $(p_1, \dots, p_n)$. The reliability of the drawn item will naturally be described by the corresponding mixed reliabilities, i.e., the mixture model (1) with $\alpha = 1$. Hazra and Finkelstein [11] established some results concerning stochastic comparisons of two finite mixtures of three semi-parametric survival/distribution functions including the PHR model, the proportional reversed hazard rates (PRHR) model, and the accelerated lifetime model (i.e., scale model).

The important semi-parametric family of distributions offered by the additive hazard (AH) model has found key applications in reliability theory. A random variable X is said to follow an AH model (denoted by $AH(\theta; \bar{F})$) if its survival function is expressed as

$$\bar{F}(x; \theta, \beta) = e^{-\theta x} \bar{F}(x; \beta), \qquad x > 0, \ \theta > 0.$$

It is then easy in this case to see that $r(x; \theta) = \theta + r(x; \beta)$, where $r(\cdot; \beta)$ is the baseline hazard rate function. Finkelstein [7] has studied the reliability properties of AH model, and has provided some applications of this model. See also Finkelstein and Esaulova [8] and Finkelstein and Esaulova [9], in which the heterogeneity of components (populations) follows AH model.

In this paper, we discuss stochastic comparisons on the finite α-mixture of additive hazard models. Sufficient conditions on the underlying distribution parameters and the mixing probabilities are established for the comparisons of different α-mixtures of survival or distribution functions of these models with respect to the usual stochastic order and the hazard rate order, respectively.

The rest of this paper is organized as follows. Section 2 recalls some basic concepts that will be used in the sequel. Section 3 presents stochastic comparisons of finite α-mixtures of survival functions of additive hazard models with respect to the usual stochastic order and the hazard rate order. Section 4 deals with sufficient conditions for comparing finite α-mixtures of distribution functions of additive hazard models in the sense of the usual stochastic order. Some concluding remarks are discussed in Section 5.

2. Preliminaries

In this section, we present some basic definitions and useful lemmas that are essential for subsequent developments. Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. We use '$\overset{\text{sgn}}{=}$' to denote that both sides of the equality have the same sign. We assume that all expectations exist wherever they are given.

Let X and Y be two random variables with density functions f_X and f_Y, distribution functions F_X and F_Y, survival functions $\bar{F}_X$ and $\bar{F}_Y$ and hazard rate functions r_X and r_Y, respectively. Then, some key stochastic orders are as defined below.

Definition 2.1. *X is said to be larger than Y in the*

(i) usual stochastic order (denoted by $X \geq_{\text{st}} Y$) if $\bar{F}_X(t) \geq \bar{F}_Y(t)$, for all $t \in \mathbb{R}$, or equivalently $\mathbb{E}[\phi(X)] \geq \mathbb{E}[\phi(Y)]$ for all increasing functions $\phi : \mathbb{R} \to \mathbb{R}$;

(ii) hazard rate order (denoted by $X \geq_{\text{hr}} Y$) if and only if $\bar{F}_X(t)/\bar{F}_Y(t)$ is increasing in $t \in \mathbb{R}$, or equivalently $r_Y(t) \geq r_X(t)$ for all $t \in \mathbb{R}$;

(iii) likelihood ratio order (denoted by $X \geq_{\text{lr}} Y$) if $f_X(t)/f_Y(t)$ is increasing in $t \in \mathbb{R}$.

The following implication between the above orders is well known:

$$X \leq_{\text{lr}} Y \Longrightarrow X \leq_{\text{hr}} Y \Longrightarrow X \leq_{\text{st}} Y.$$

Interested readers may refer to Müller and Stoyan [14] and Shaked and Shanthikumar [16] for comprehensive discussions on various stochastic orderings and their relationships.

Majorization is a very helpful tool in establishing various inequalities arising from many research areas.

Definition 2.2. *Let $a_{(1)} \leq \cdots \leq a_{(n)}$ and $b_{(1)} \leq \cdots \leq b_{(n)}$ be the increasing arrangements of $\boldsymbol{a} = (a_1, \dots, a_n)$ and $\boldsymbol{b} = (b_1, \dots, b_n)$, respectively. Then,*

(i) $\boldsymbol{a}$ is said to majorize $\boldsymbol{b}$, denoted by $\boldsymbol{a} \overset{\text{m}}{\succeq} \boldsymbol{b}$, if $\sum_{j=1}^{i} a_{(j)} \leq \sum_{j=1}^{i} b_{(j)}$ for $i = 1, \dots, n-1$, and $\sum_{j=1}^{n} a_{(j)} = \sum_{j=1}^{n} b_{(j)}$;

(ii) $\boldsymbol{a}$ is said to weakly supermajorize $\boldsymbol{b}$, denoted by $\boldsymbol{a} \overset{\text{w}}{\succeq} \boldsymbol{b}$, if $\sum_{j=1}^{i} a_{(j)} \leq \sum_{j=1}^{i} b_{(j)}$ for $i = 1, \dots, n$;

(iii) $\boldsymbol{a}$ is said to weakly submajorize $\boldsymbol{b}$, denoted by $\boldsymbol{a} \succeq_{\text{w}} \boldsymbol{b}$, if $\sum_{j=i}^{n} a_{(j)} \geq \sum_{j=i}^{n} b_{(j)}$ for $i = 1, \dots, n$.

The majorization order implies both weak submajorization and supermajorization orders. Interested readers may refer to Marshall et al. [13] for comprehensive discussions on their properties and applications.

Definition 2.3. *A real-valued function ϕ defined on a set $I^n \subseteq \mathbb{R}^n$ is said to be Schur-convex (Schur-concave) on I^n if $\boldsymbol{u} \overset{\text{m}}{\succeq} \boldsymbol{v}$ implies $\phi(\boldsymbol{u}) \geq (\leq) \phi(\boldsymbol{v})$ for any $\boldsymbol{u}, \boldsymbol{v} \in I^n$.*

Lemma 2.4 (Marshall et al. [13]). *Consider the real-valued continuously differentiable function ϕ on J^n, where $J \subseteq \mathbb{R}$ is an open interval. Then, ϕ is Schur-convex (Schur-concave) on J^n if and only if ϕ is symmetric on J^n, and for all $i \neq j$ and all $\boldsymbol{u} \in J^n$, $(u_i - u_j)\left(\phi_{(i)}(\boldsymbol{u}) - \phi_{(j)}(\boldsymbol{u})\right) \geq 0 (\leq 0)$, where $\phi_{(k)}(\boldsymbol{x})$ to denote the partial derivative of $\phi(\boldsymbol{x})$ with respect to its k-th component, for $k = 1, \ldots, n$.*

Lemma 2.5 (Marshall et al. [13]). *A real-valued function ϕ on $\mathbb{R}^n$ has the property $\phi(\boldsymbol{x}) \geq \phi(\boldsymbol{y})$ whenever $\boldsymbol{x} \succeq_{\text{w}} (\overset{\text{w}}{\succeq}) \boldsymbol{y}$ if and only if ϕ is increasing (decreasing) and Schur-convex on $\mathbb{R}^n$.*

3. α-mixtures of additive hazard survival functions

In this section, we discuss stochastic comparisons of the α-mixture of additive hazard survival functions in the sense of the usual stochastic order as well as the hazard rate order. Let us set

$$\mathcal{E}_n^+ = \{(x_1, x_2, \ldots, x_n) : x_1 \geq x_2 \cdots \geq x_n > 0\}$$
$$\mathcal{D}_n^+ = \{(x_1, x_2, \ldots, x_n) : 0 < x_1 \leq x_2 \cdots \leq x_n\}.$$

Theorem 3.1. *Let*

$$\bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x) = \left[\sum_{i=1}^n p_i \left\{e^{-\theta_i x} \bar{F}(x; \beta_i)\right\}^\alpha\right]^{1/\alpha} \quad \text{and} \quad \bar{F}_{W_n(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})}(x) = \left[\sum_{i=1}^n q_i \left\{e^{-\lambda_i x} \bar{F}(x; \gamma_i)\right\}^\alpha\right]^{1/\alpha}$$

be the survival functions of two α-mixtures of additive hazard survival functions corresponding to $V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})$ and $W_n(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})$, respectively.

- *(i) Suppose that $\bar{F}(x;\beta)$ is decreasing and log-convex in $\beta > 0$, for all $x \in \mathbb{R}$. Then, for $\boldsymbol{p} \succeq_{\text{w}} \boldsymbol{q}$, $\boldsymbol{\beta} \overset{\text{w}}{\succeq} \boldsymbol{\gamma}$, $\boldsymbol{\theta} \overset{\text{w}}{\succeq} \boldsymbol{\lambda}$, $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_n^+$, $\boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_n^+$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathcal{D}_n^+$ and $\alpha > 0$, we have $V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta}) \geq_{\text{st}} W_n(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})$.*
- *(ii) Suppose that $\bar{F}(x;\beta)$ is increasing and log-concave in $\beta > 0$, for all $x \in \mathbb{R}$. Then, for $\boldsymbol{p} \succeq_{\text{w}} \boldsymbol{q}$, $\boldsymbol{\beta} \overset{\text{w}}{\succeq} \boldsymbol{\gamma}$, $\boldsymbol{\theta} \succeq_{\text{w}} \boldsymbol{\lambda}$, $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_n^+$, $\boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{E}_n^+$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathcal{D}_n^+$ and $\alpha < 0$, we have $V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta}) \leq_{\text{st}} W_n(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})$.*

Proof. (i) Suppose $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_n^+$, $\boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_n^+$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathcal{D}_n^+$ and $\alpha > 0$. Let $\bar{F}_{M_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\beta})} = \left[\sum_{i=1}^n q_i \left\{e^{-\theta_i x} \bar{F}(x;\beta_i)\right\}^\alpha\right]^{1/\alpha}$ be the survival function of the α-mixture of additive hazard survival functions corresponding to $M_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\beta})$. First, we establish that $\bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x) \geq \bar{F}_{M_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\beta})}(x)$, for all $x \in \mathbb{R}$. Note that

$$\frac{\partial \bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial p_i} = \frac{1}{\alpha} \left(e^{-\theta_i x} \bar{F}(x;\beta_i)\right)^\alpha \left[\sum_{i=1}^n p_i \left\{e^{-\theta_i x} \bar{F}(x;\beta_i)\right\}^\alpha\right]^{\frac{1}{\alpha}-1} \geq 0,$$

and

$$\frac{\partial \bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial p_1} - \frac{\partial \bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial p_2} \overset{\text{sgn}}{=} \left(e^{-\theta_1 x} \bar{F}(x;\beta_1)\right)^\alpha - \left(e^{-\theta_2 x} \bar{F}(x;\beta_2)\right)^\alpha \geq 0,$$

and then

$$(p_1 - p_2) \left(\frac{\partial \bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial p_1} - \frac{\partial \bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial p_2}\right) \geq 0.$$

Then, the desired result follows from Lemma 2.5.

Let $\bar{F}_{U_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\gamma})} = \left[\sum_{i=1}^n q_i \left\{e^{-\theta_i x} \bar{F}(x;\gamma_i)\right\}^\alpha\right]^{1/\alpha}$ be the survival function of the α-mixture of additive hazard survival functions corresponding to $U_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\gamma})$.

Now, we prove that $\bar{F}_{M_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\beta})}(x) \geq \bar{F}_{U_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\gamma})}(x)$ for all $x \in \mathbb{R}$. Since $\bar{F}(x;\beta_i)$ is decreasing in β_i, we can observe that

$$\frac{\partial \bar{F}_{M_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial \beta_i} = q_i e^{-\alpha\,\theta_i x}\left(\frac{\frac{\partial \bar{F}(x;\beta_i)}{\partial \beta_i}}{\bar{F}(x;\beta_i)}\right)\bar{F}^{\alpha}(x;\beta_i)\left[\sum_{i=1}^{n} q_i \left\{e^{-\theta_i x}\bar{F}(x;\beta_i)\right\}^{\alpha}\right]^{\frac{1}{\alpha}-1} \leq 0.$$

Further,

$$\frac{\partial \bar{F}_{M_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial \beta_1} - \frac{\partial \bar{F}_{M_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial \beta_2} \overset{\text{sgn}}{=} \left[q_1 e^{-\alpha\,\theta_1 x}\left(\frac{\frac{\partial \bar{F}(x;\beta_1)}{\partial \beta_1}}{\bar{F}(x;\beta_1)}\right)\bar{F}^{\alpha}(x;\beta_1) - q_2 e^{-\alpha\,\theta_2 x}\left(\frac{\frac{\partial \bar{F}(x;\beta_2)}{\partial \beta_2}}{\bar{F}(x;\beta_2)}\right)\bar{F}^{\alpha}(x;\beta_2)\right]$$
$$\leq 0,$$

which follows from the fact that, for $q_1 \geq q_2$, $\beta_2 \geq \beta_1$ and $\alpha > 0$, $\bar{F}^{\alpha}(x;\beta_1) \geq \bar{F}^{\alpha}(x;\beta_2)$ and $-\frac{\partial \log(\bar{F}(x;\beta_1))}{\partial \beta_1} \geq -\frac{\partial \log(\bar{F}(x;\beta_2))}{\partial \beta_2} \geq 0$.

Now, we prove that $\bar{F}_{U_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\gamma})}(x) \geq \bar{F}_{W_n(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})}(x)$ for all $x \in \mathbb{R}$. Since $\bar{F}(x;\gamma_i)$ is decreasing in γ_i, we can observe that

$$\frac{\partial \bar{F}_{U_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\gamma})}(x)}{\partial \theta_i} = -q_i x \left\{e^{-\theta_i x}\bar{F}(x;\gamma_i)\right\}^{\alpha}\left[\sum_{i=1}^{n} q_i \left\{e^{-\theta_i x}\bar{F}(x;\gamma_i)\right\}^{\alpha}\right]^{\frac{1}{\alpha}-1} \leq 0.$$

Further,

$$\frac{\partial \bar{F}_{U_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\gamma})}(x)}{\partial \theta_1} - \frac{\partial \bar{F}_{U_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\gamma})}(x)}{\partial \theta_2} \overset{\text{sgn}}{=} \left[q_2 \left\{e^{-\theta_2 x}\bar{F}(x;\gamma_2)\right\}^{\alpha} - q_1 \left\{e^{-\theta_1 x}\bar{F}(x;\gamma_1)\right\}^{\alpha}\right] \leq 0,$$

which follows from the fact that, for $q_1 \geq q_2$, $\beta_2 \geq \beta_1$ and $\alpha > 0$, $\bar{F}^{\alpha}(x;\beta_1) \geq \bar{F}^{\alpha}(x;\beta_2)$. Thus, the proof is finished.

(ii) The proof is similar to that of Part (i) and is therefore omitted here for the sake of brevity. □

In light of Theorem 3.1, a lower or upper bound for the survival function of α-mixture of survival functions can be established as stated in the following corollary.

Corollary 3.2. *Set* $(q_1,\ldots,q_n) = (\bar{p},\ldots,\bar{p})$ *and* $(\gamma_1,\ldots,\gamma_n) = (\bar{\beta},\ldots,\bar{\beta})$, *where* $\bar{p} = n^{-1}\sum_{i=1}^{n} p_i$ *and* $\bar{\beta} = n^{-1}\sum_{i=1}^{n}\beta_i$. *It is easy to observe that* $\boldsymbol{p} \overset{\text{w}}{\succeq} \boldsymbol{q}$, $\boldsymbol{p} \succeq_{\text{w}} \boldsymbol{q}$ *and* $\boldsymbol{\beta} \overset{\text{w}}{\succeq} \boldsymbol{\gamma}$.

(i) Suppose that $\bar{F}(x;\beta)$ *is increasing and concave in* $\beta > 0$, *for all* $x \in \mathbb{R}$. *Then, for* $(\boldsymbol{p},\boldsymbol{\beta}) \in \mathcal{U}_n$ *and* $\alpha \geq 1$, *we have* $\bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\beta})}(x) \geq (n\,\bar{p})^{1/\alpha}\,\bar{F}(x;\bar{\beta})$, *for all* $x \in \mathbb{R}$.

(ii) Suppose that $\bar{F}(x;\beta)$ *is decreasing and convex in* $\beta > 0$, *for all* $x \in \mathbb{R}$. *Then, for* $(\boldsymbol{p},\boldsymbol{\beta}) \in \mathcal{U}_n$ *and* $\alpha < 1$, *it follows that* $\bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\beta})}(x) \leq (n\,\bar{p})^{1/\alpha}\,\bar{F}(x;\bar{\beta})$, *for all* $x \in \mathbb{R}$.

The following three corollaries can be obtained from Theorem 3.1 directly. The proofs are simple and omitted here for brevity.

Corollary 3.3. *Let* $\bar{F}(x;\beta) = \bar{F}^{\beta}(x)$ *and* $\bar{F}(x;\gamma) = \bar{F}^{\gamma}(x)$, *for* $x \in \mathbb{R}$. *Then, for* $\boldsymbol{p} \succeq_{\text{w}} \boldsymbol{q}$, $\boldsymbol{\beta} \overset{\text{w}}{\succeq} \boldsymbol{\gamma}$, $\boldsymbol{\theta} \overset{\text{w}}{\succeq} \boldsymbol{\lambda}$, $\boldsymbol{p},\boldsymbol{q} \in \mathcal{E}_n^{+}$, $\boldsymbol{\theta},\boldsymbol{\lambda} \in \mathcal{D}_n^{+}$, $\boldsymbol{\beta},\boldsymbol{\gamma} \in \mathcal{D}_n^{+}$ *and* $\alpha > 0$, *we have* $V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta}) \geq_{\text{st}} W_n(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})$.

Corollary 3.4. *Let* $\bar{F}(x;\beta) = \bar{F}(\beta x)$ *and* $\bar{F}(x;\gamma) = \bar{F}(\gamma x)$, *for* $x \in \mathbb{R}$. *Suppose* $r(t;\beta)$ *is decreasing in* t. *Then, for* $\boldsymbol{p} \succeq_{\text{w}} \boldsymbol{q}$, $\boldsymbol{\beta} \overset{\text{w}}{\succeq} \boldsymbol{\gamma}$, $\boldsymbol{\theta} \overset{\text{w}}{\succeq} \boldsymbol{\lambda}$, $\boldsymbol{p},\boldsymbol{q} \in \mathcal{E}_n^{+}$, $\boldsymbol{\theta},\boldsymbol{\lambda} \in \mathcal{D}_n^{+}$, $\boldsymbol{\beta},\boldsymbol{\gamma} \in \mathcal{D}_n^{+}$ *and* $\alpha > 0$, *we have* $V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta}) \geq_{\text{st}} W_n(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})$.

Corollary 3.5. *Let* $\bar{F}(x;\beta) = 1 - F^{\beta}(x)$ *and* $\bar{F}(x;\gamma) = 1 - F^{\gamma}(x)$, *for* $x \in \mathbb{R}$. *Then, for* $\boldsymbol{p} \succeq_{\text{w}} \boldsymbol{q}$, $\boldsymbol{\beta} \overset{\text{w}}{\succeq} \boldsymbol{\gamma}$, $\boldsymbol{\theta} \succeq_{\text{w}} \boldsymbol{\lambda}$, $\boldsymbol{p},\boldsymbol{q} \in \mathcal{E}_n^{+}$, $\boldsymbol{\theta},\boldsymbol{\lambda} \in \mathcal{E}_n^{+}$, $\boldsymbol{\beta},\boldsymbol{\gamma} \in \mathcal{D}_n^{+}$ *and* $\alpha < 0$, *we have* $V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta}) \leq_{\text{st}} W_n(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})$.

The following counterexample explains that the result in Theorem 3.1 may be not true any more if the conditions $\boldsymbol{p},\boldsymbol{q} \in \mathcal{E}_n^{+}$, $\boldsymbol{\theta},\boldsymbol{\lambda} \in \mathcal{D}_n^{+}$, $\boldsymbol{\beta},\boldsymbol{\gamma} \in \mathcal{D}_n^{+}$ are removed.

Proof. (i) Suppose $\boldsymbol{p},\boldsymbol{q}\in\mathcal{E}_n^+$, $\boldsymbol{\theta},\boldsymbol{\lambda}\in\mathcal{D}_n^+$, $\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathcal{D}_n^+$ and $\alpha>0$. Let $\bar{F}_{M_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\beta})}=\left[\sum_{i=1}^n q_i\left\{e^{-\theta_i x}\bar{F}(x;\beta_i)\right\}^\alpha\right]^{1/\alpha}$ be the survival function of the α-mixture of additive hazard survival functions corresponding to $M_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\beta})$. First, we establish that $\bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x)\geq\bar{F}_{M_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\beta})}(x)$, for all $x\in\mathbb{R}$. Note that

$$\frac{\partial\bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial p_i}=\frac{1}{\alpha}\left(e^{-\theta_i x}\bar{F}(x;\beta_i)\right)^\alpha\left[\sum_{i=1}^n p_i\left\{e^{-\theta_i x}\bar{F}(x;\beta_i)\right\}^\alpha\right]^{\frac{1}{\alpha}-1}\geq 0,$$

and

$$\frac{\partial\bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial p_1}-\frac{\partial\bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial p_2}\overset{\text{sgn}}{=}\left(e^{-\theta_1 x}\bar{F}(x;\beta_1)\right)^\alpha-\left(e^{-\theta_2 x}\bar{F}(x;\beta_2)\right)^\alpha\geq 0,$$

and then

$$(p_1-p_2)\left(\frac{\partial\bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial p_1}-\frac{\partial\bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial p_2}\right)\geq 0.$$

Then, the desired result follows from Lemma 2.5.

Let $\bar{F}_{U_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\gamma})}=\left[\sum_{i=1}^n q_i\left\{e^{-\theta_i x}\bar{F}(x;\gamma_i)\right\}^\alpha\right]^{1/\alpha}$ be the survival function of the α-mixture of additive hazard survival functions corresponding to $U_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\gamma})$.

Now, we prove that $\bar{F}_{M_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\beta})}(x)\geq\bar{F}_{U_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\gamma})}(x)$ for all $x\in\mathbb{R}$. Since $\bar{F}(x;\beta_i)$ is decreasing in β_i, we can observe that

$$\frac{\partial\bar{F}_{M_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial\beta_i}=q_i e^{-\alpha\,\theta_i x}\left(\frac{\frac{\partial\bar{F}(x;\beta_i)}{\partial\beta_i}}{\bar{F}(x;\beta_i)}\right)\bar{F}^\alpha(x;\beta_i)\left[\sum_{i=1}^n q_i\left\{e^{-\theta_i x}\bar{F}(x;\beta_i)\right\}^\alpha\right]^{\frac{1}{\alpha}-1}\leq 0.$$

Further,

$$\begin{aligned}\frac{\partial\bar{F}_{M_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial\beta_1}-\frac{\partial\bar{F}_{M_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial\beta_2}&\overset{\text{sgn}}{=}\left[q_1 e^{-\alpha\,\theta_1 x}\left(\frac{\frac{\partial\bar{F}(x;\beta_1)}{\partial\beta_1}}{\bar{F}(x;\beta_1)}\right)\bar{F}^\alpha(x;\beta_1)-q_2 e^{-\alpha\,\theta_2 x}\left(\frac{\frac{\partial\bar{F}(x;\beta_2)}{\partial\beta_2}}{\bar{F}(x;\beta_2)}\right)\bar{F}^\alpha(x;\beta_2)\right]\\&\leq 0,\end{aligned}$$

which follows from the fact that, for $q_1\geq q_2$, $\beta_2\geq\beta_1$ and $\alpha>0$, $\bar{F}^\alpha(x;\beta_1)\geq\bar{F}^\alpha(x;\beta_2)$ and $-\frac{\partial\log(\bar{F}(x;\beta_1))}{\partial\beta_1}\geq-\frac{\partial\log(\bar{F}(x;\beta_2))}{\partial\beta_2}\geq 0$.

Now, we prove that $\bar{F}_{U_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\gamma})}(x)\geq\bar{F}_{W_n(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})}(x)$ for all $x\in\mathbb{R}$. Since $\bar{F}(x;\gamma_i)$ is decreasing in γ_i, we can observe that

$$\frac{\partial\bar{F}_{U_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\gamma})}(x)}{\partial\theta_i}=-q_i x\left\{e^{-\theta_i x}\bar{F}(x;\gamma_i)\right\}^\alpha\left[\sum_{i=1}^n q_i\left\{e^{-\theta_i x}\bar{F}(x;\gamma_i)\right\}^\alpha\right]^{\frac{1}{\alpha}-1}\leq 0.$$

Further,

$$\frac{\partial\bar{F}_{U_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\gamma})}(x)}{\partial\theta_1}-\frac{\partial\bar{F}_{U_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\gamma})}(x)}{\partial\theta_2}\overset{\text{sgn}}{=}\left[q_2\left\{e^{-\theta_2 x}\bar{F}(x;\gamma_2)\right\}^\alpha-q_1\left\{e^{-\theta_1 x}\bar{F}(x;\gamma_1)\right\}^\alpha\right]\leq 0,$$

which follows from the fact that, for $q_1\geq q_2$, $\beta_2\geq\beta_1$ and $\alpha>0$, $\bar{F}^\alpha(x;\beta_1)\geq\bar{F}^\alpha(x;\beta_2)$. Thus, the proof is finished.

(ii) The proof is similar to that of Part (i) and is therefore omitted here for the sake of brevity.

□

In light of Theorem 3.1, a lower or upper bound for the survival function of α-mixture of survival functions can be established as stated in the following corollary.

Corollary 3.6. *Set* $(q_1,\ldots,q_n)=(\bar{p},\ldots,\bar{p})$ *and* $(\gamma_1,\ldots,\gamma_n)=(\bar{\beta},\ldots,\bar{\beta})$, *where* $\bar{p}=n^{-1}\sum_{i=1}^n p_i$ *and* $\bar{\beta}=n^{-1}\sum_{i=1}^n\beta_i$. *It is easy to observe that* $\boldsymbol{p}\overset{\text{w}}{\succeq}\boldsymbol{q}$, $\boldsymbol{p}\succeq_{\text{w}}\boldsymbol{q}$ *and* $\boldsymbol{\beta}\overset{\text{w}}{\succeq}\boldsymbol{\gamma}$.

(i) Suppose that $\bar{F}(x;\beta)$ is increasing and concave in $\beta>0$, for all $x\in\mathbb{R}$. Then, for $(\boldsymbol{p},\boldsymbol{\beta})\in\mathcal{U}_n$ and $\alpha\geq 1$, we have $\bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\beta})}(x)\geq (n\,\bar{p})^{1/\alpha}\,\bar{F}(x;\bar{\beta})$, for all $x\in\mathbb{R}$.

(ii) Suppose that $\bar{F}(x;\beta)$ is decreasing and convex in $\beta>0$, for all $x\in\mathbb{R}$. Then, for $(\boldsymbol{p},\boldsymbol{\beta})\in\mathcal{U}_n$ and $\alpha<1$, it follows that $\bar{F}_{V_n(\boldsymbol{p},\boldsymbol{\beta})}(x)\leq (n\,\bar{p})^{1/\alpha}\,\bar{F}(x;\bar{\beta})$, for all $x\in\mathbb{R}$.

The following three corollaries can be obtained from Theorem 3.1 directly. The proofs are simple and omitted here for brevity.

Corollary 3.7. *Let $\bar{F}(x;\beta)=\bar{F}^{\beta}(x)$ and $\bar{F}(x;\gamma)=\bar{F}^{\gamma}(x)$, for $x\in\mathbb{R}$. Then, for $\boldsymbol{p}\succeq_{\mathrm{w}}\boldsymbol{q}$, $\boldsymbol{\beta}\overset{\mathrm{w}}{\succeq}\boldsymbol{\gamma}$, $\boldsymbol{\theta}\overset{\mathrm{w}}{\succeq}\boldsymbol{\lambda}$, $\boldsymbol{p},\boldsymbol{q}\in\mathcal{E}_n^+$, $\boldsymbol{\theta},\boldsymbol{\lambda}\in\mathcal{D}_n^+$, $\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathcal{D}_n^+$ and $\alpha>0$, we have $V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})\geq_{\mathrm{st}} W_n(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})$.*

Corollary 3.8. *Let $\bar{F}(x;\beta)=\bar{F}(\beta x)$ and $\bar{F}(x;\gamma)=\bar{F}(\gamma x)$, for $x\in\mathbb{R}$. Suppose $r(t;\beta)$ is decreasing in t. Then, for $\boldsymbol{p}\succeq_{\mathrm{w}}\boldsymbol{q}$, $\boldsymbol{\beta}\overset{\mathrm{w}}{\succeq}\boldsymbol{\gamma}$, $\boldsymbol{\theta}\overset{\mathrm{w}}{\succeq}\boldsymbol{\lambda}$, $\boldsymbol{p},\boldsymbol{q}\in\mathcal{E}_n^+$, $\boldsymbol{\theta},\boldsymbol{\lambda}\in\mathcal{D}_n^+$, $\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathcal{D}_n^+$ and $\alpha>0$, we have $V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})\geq_{\mathrm{st}} W_n(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})$.*

Corollary 3.9. *Let $\bar{F}(x;\beta)=1-F^{\beta}(x)$ and $\bar{F}(x;\gamma)=1-F^{\gamma}(x)$, for $x\in\mathbb{R}$. Then, for $\boldsymbol{p}\succeq_{\mathrm{w}}\boldsymbol{q}$, $\boldsymbol{\beta}\overset{\mathrm{w}}{\succeq}\boldsymbol{\gamma}$, $\boldsymbol{\theta}\succeq_{\mathrm{w}}\boldsymbol{\lambda}$, $\boldsymbol{p},\boldsymbol{q}\in\mathcal{E}_n^+$, $\boldsymbol{\theta},\boldsymbol{\lambda}\in\mathcal{E}_n^+$, $\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathcal{D}_n^+$ and $\alpha<0$, we have $V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})\leq_{\mathrm{st}} W_n(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})$.*

The following counterexample explains that the result in Theorem 3.1 may be not true any more if the conditions $\boldsymbol{p},\boldsymbol{q}\in\mathcal{E}_n^+$, $\boldsymbol{\theta},\boldsymbol{\lambda}\in\mathcal{D}_n^+$, $\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathcal{D}_n^+$ are removed.

Example 3.1. *Let $\bar{F}(x;\beta)=e^{-\beta x}$ and $\bar{F}(x;\gamma)=e^{-\gamma x}$, for $x\in\mathbb{R}_+$. Take $(p_1,p_2,p_3)=(0.59,0.3,0.11)$, $(q_1,q_2,q_3)=(0.6,0.3,0.1)$, $(\theta_1,\theta_2,\theta_3)=(2,10.1,3.9)$, $(\lambda_1,\lambda_2,\lambda_3)=(2,10,4)$, $(\beta_1,\beta_2,\beta_3)=(5,3,2)$, $(\gamma_1,\gamma_2,\gamma_3)=(6,3,2)$ and $\alpha=3$. It is easy to check that $\boldsymbol{p}\succeq_{\mathrm{w}}\boldsymbol{q}$, $\boldsymbol{\beta}\overset{\mathrm{w}}{\succeq}\boldsymbol{\gamma}$ and $\boldsymbol{\theta}\overset{\mathrm{w}}{\succeq}\boldsymbol{\lambda}$, but $\boldsymbol{\theta}\notin\mathcal{D}_n^+$ and $\boldsymbol{\lambda}\notin\mathcal{D}_n^+$. Figure 1 plots the difference of the survival functions $\bar{F}_{V_3(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x)-\bar{F}_{W_3(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})}(x)$ on $x\in[0,2]$, which indicates that the usual stochastic ordering does not hold between $V_3(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})$ and $W_3(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})$.*

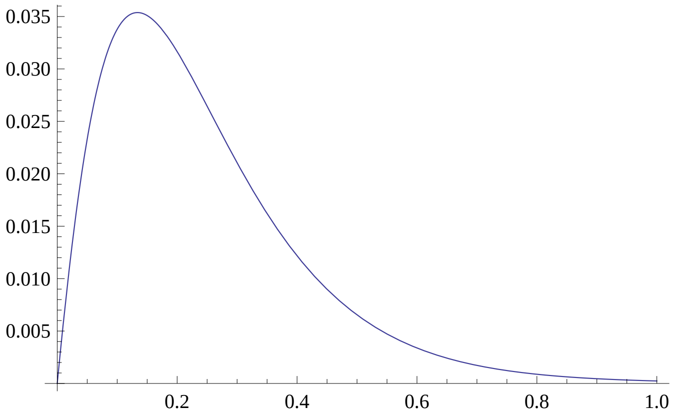

Figure 1: Plot of $\bar{F}_{V_3(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x)-\bar{F}_{W_3(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})}(x)$ on $x\in[0,1]$.

Next, we establish sufficient conditions for comparing α-mixtures of additive hazard survival functions in the sense of the hazard rate ordering.

Theorem 3.10. *Let*

$$\bar{F}_{V_2(\boldsymbol{p},\boldsymbol{\theta})}(x)=\left[\sum_{i=1}^{2}p_i\left\{e^{-\theta_i x}\bar{F}(x;\beta_i)\right\}^{\alpha}\right]^{1/\alpha}\quad\text{and}\quad\bar{F}_{W_2(\boldsymbol{q},\boldsymbol{\lambda})}(x)=\left[\sum_{i=1}^{2}q_i\left\{e^{-\lambda_i x}\bar{F}(x;\gamma_i)\right\}^{\alpha}\right]^{1/\alpha}$$

be the survival functions of two α-mixtures of survival functions of additive hazard models corresponding to $V_2(\boldsymbol{p},\boldsymbol{\theta})$ and $W_2(\boldsymbol{q},\boldsymbol{\lambda})$, respectively.

(i) Suppose that $\bar{F}(x;\beta)$ is decreasing with respect to $\beta>0$, and $r(x;\beta)$ is increasing with respect to β, for all $x\in\mathbb{R}$. Then, for $\boldsymbol{q}\overset{\mathrm{m}}{\succeq}\boldsymbol{p}$, $\boldsymbol{\lambda}\overset{\mathrm{m}}{\succeq}\boldsymbol{\theta}$, $\boldsymbol{p},\boldsymbol{q}\in\mathcal{E}_2^+$, $\boldsymbol{\theta},\boldsymbol{\lambda}\in\mathcal{D}_2^+$, $\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathcal{D}_2^+$ and $\alpha>0$, we have $V_2(\boldsymbol{p},\boldsymbol{\theta})\geq_{\mathrm{hr}} W_2(\boldsymbol{q},\boldsymbol{\lambda})$.

(ii) Suppose that $\bar{F}(x;\beta)$ is increasing with respect to $\beta>0$, and $r(x;\beta)$ is decreasing with respect to β, for all $x\in\mathbb{R}$. Then, for $\boldsymbol{q}\overset{\mathrm{m}}{\succeq}\boldsymbol{p}$, $\boldsymbol{\lambda}\overset{\mathrm{m}}{\succeq}\boldsymbol{\theta}$, $\boldsymbol{p},\boldsymbol{q}\in\mathcal{E}_2^+$, $\boldsymbol{\theta},\boldsymbol{\lambda}\in\mathcal{D}_2^+$, $\boldsymbol{\beta},\boldsymbol{\gamma}\in\mathcal{E}_2^+$ and $\alpha<0$, we have $V_2(\boldsymbol{p},\boldsymbol{\theta})\geq_{\mathrm{hr}} W_2(\boldsymbol{q},\boldsymbol{\lambda})$.

Proof. (i) Assume that $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_2^+$, $\boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_2^+$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathcal{D}_2^+$ and $\alpha > 0$. The hazard rate function of $V_2(\boldsymbol{p}, \boldsymbol{\theta})$ is

$$r(x; \boldsymbol{p}, \boldsymbol{\theta}) = \frac{\sum_{i=1}^{2} p_i \left(\theta_i + r(x; \beta_i)\right) \{e^{-\theta_i x} \bar{F}(x; \beta_i)\}^{\alpha}}{\sum_{i=1}^{2} p_i \{e^{-\theta_i x} \bar{F}(x; \beta_i)\}^{\alpha}}.$$

Note that

$$\frac{\partial r(x; \boldsymbol{p}, \boldsymbol{\theta})}{\partial p_1} \overset{\text{sgn}}{=} p_2 \{e^{-\theta_1 x} \bar{F}(x; \beta_1)\}^{\alpha} \{e^{-\theta_2 x} \bar{F}(x; \beta_2)\}^{\alpha} \left[(\theta_1 + r(x; \beta_1)) - (\theta_2 + r(x; \beta_2))\right].$$

Then, we have

$$\frac{\partial r(x; \boldsymbol{p}, \boldsymbol{\theta})}{\partial p_1} - \frac{\partial r(x; \boldsymbol{p}, \boldsymbol{\theta})}{\partial p_2} \overset{\text{sgn}}{=} -(p_1 - p_2) \left[(\theta_1 + r(x; \beta_1)) - (\theta_2 + r(x; \beta_2))\right].$$

For $\beta_1 \leq \beta_2$, $\theta_1 \leq \theta_2$ and the increasing property of $r(x; \beta)$, we conclude that

$$(p_1 - p_2) \left(\frac{\partial r(x; \boldsymbol{p}, \boldsymbol{\theta})}{\partial p_1} - \frac{\partial r(x; \boldsymbol{p}, \boldsymbol{\theta})}{\partial p_2} \right) \geq 0, \tag{3}$$

which implies $r(x; \boldsymbol{p}, \boldsymbol{\theta}) \leq r(x; \boldsymbol{q}, \boldsymbol{\theta})$ for all $x \in \mathbb{R}$ by applying Lemma 2.4.

Let $M_2(\boldsymbol{q}, \boldsymbol{\theta})$ be the α-mixture of survival functions of additive hazard models with parameters $\boldsymbol{q}$ and $\boldsymbol{\theta}$. The hazard rate function of $M_2(\boldsymbol{q}, \boldsymbol{\theta})$ is

$$r(x; \boldsymbol{q}, \boldsymbol{\theta}) = \frac{\sum_{i=1}^{2} q_i \left(\theta_i + r(x; \beta_i)\right) \{e^{-\theta_i x} \bar{F}(x; \beta_i)\}^{\alpha}}{\sum_{i=1}^{2} q_i \{e^{-\theta_i x} \bar{F}(x; \beta_i)\}^{\alpha}} = \frac{A_1(\theta_1 + r(x; \beta_1)) + A_2(\theta_2 + r(x; \beta_2))}{A_1 + A_2},$$

where $A_1 = q_1 \{e^{-\theta_1 x} \bar{F}(x; \beta_1)\}^{\alpha}$ and $A_2 = q_2 \{e^{-\theta_2 x} \bar{F}(x; \beta_2)\}^{\alpha}$. Observe that

$$\frac{\partial r(x; \boldsymbol{q}, \boldsymbol{\theta})}{\partial \theta_1} \overset{\text{sgn}}{=} \frac{\partial A_1}{\partial \theta_1} A_2 \left[(\theta_1 + r(x; \beta_1)) - (\theta_2 + r(x; \beta_2))\right] + \frac{\partial A_1}{\partial \theta_1} + A_1 A_2.$$

Now, by using decreasing property of $\bar{F}(x; \beta)$ with respect to β, $q_1 \geq q_2$, $\theta_1 \leq \theta_2$, $\beta_1 \leq \beta_2$ and $\alpha > 0$, we get

$$\frac{\partial r(x; \boldsymbol{q}, \boldsymbol{\theta})}{\partial \theta_1} - \frac{\partial r(x; \boldsymbol{q}, \boldsymbol{\theta})}{\partial \theta_2} \overset{\text{sgn}}{=} \alpha x \left(q_2 \{e^{-\theta_2 x} \bar{F}(x; \beta_2)\}^{\alpha} - q_1 \{e^{-\theta_1 x} \bar{F}(x; \beta_1)\}^{\alpha} \right) \leq 0,$$

and then

$$(\theta_1 - \theta_2) \left(\frac{\partial r(x; \boldsymbol{q}, \boldsymbol{\theta})}{\partial \theta_1} - \frac{\partial r(x; \boldsymbol{q}, \boldsymbol{\theta})}{\partial \theta_2} \right) \geq 0, \tag{4}$$

which implies $r(x; \boldsymbol{q}, \boldsymbol{\theta}) \leq r(x; \boldsymbol{q}, \boldsymbol{\lambda})$ for all $x \in \mathbb{R}$ by applying Lemma 2.4.

(ii) The proof is similar to that of Part (i) and is therefore omitted here for the sake of brevity.

□

The following corollaries are direct results of Part (i) of Theorem 3.10, for which the proofs are simple and thus omitted here.

Corollary 3.11. *Let* $\bar{F}(x; \beta) = \bar{F}^{\beta}(x)$ *and* $\bar{F}(x; \gamma) = \bar{F}^{\gamma}(x)$, *for* $x \in \mathbb{R}$. *Then, for* $\boldsymbol{q} \overset{\text{m}}{\succeq} \boldsymbol{p}$, $\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\theta}$, $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_n^+$, $\boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_n^+$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathcal{D}_n^+$ *and* $\alpha > 0$, *we have* $V_2(\boldsymbol{p}, \boldsymbol{\theta}) \geq_{\text{hr}} W_2(\boldsymbol{q}, \boldsymbol{\lambda})$.

Corollary 3.12. *Let* $\bar{F}(x; \beta) = \bar{F}(\beta x)$, *for* $x \in \mathbb{R}$. *Further, suppose that* $r(x)$ *is increasing in* $x \in \mathbb{R}$. *Then, for* $\boldsymbol{q} \overset{\text{m}}{\succeq} \boldsymbol{p}$, $\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\theta}$, $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_n^+$, $\boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_n^+$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathcal{D}_n^+$ *and* $\alpha > 0$, *we have* $V_2(\boldsymbol{p}, \boldsymbol{\theta}) \geq_{\text{hr}} W_2(\boldsymbol{q}, \boldsymbol{\lambda})$.

The following corollaries are direct results of Part (ii) of Theorem 3.10, for which the proofs are simple and thus omitted here.

Corollary 3.13. *Let* $\bar{F}(x; \beta) = 1 - F^{\beta}(x)$ *and* $\bar{F}(x; \beta) = 1 - F^{\gamma}(x)$, *for* $x \in \mathbb{R}$. *Further, suppose that* $r(x)$ *is increasing in* $x \in \mathbb{R}$. *Then, for* $\boldsymbol{q} \overset{\text{m}}{\succeq} \boldsymbol{p}$, $\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\theta}$, $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_n^+$, $\boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_n^+$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathcal{D}_n^+$ *and* $\alpha > 0$, *we have* $V_2(\boldsymbol{p}, \boldsymbol{\theta}) \geq_{\text{hr}} W_2(\boldsymbol{q}, \boldsymbol{\lambda})$.

Proof. It is easy to observe that $\bar{F}(x;\beta) = 1 - F^{\beta}(x)$ is decreasing in β. Note that $r(x;\beta) = \beta_i F^{\beta}(x)/(1-F^{\beta}(x))$. Then, we obtain

$$r'(x;\beta) \overset{\text{sgn}}{=} 1 + \beta \ln F(x) - F^{\beta}(x) = \Delta(\beta), \quad \textit{say},$$

It can be readily seen that $\Delta'(\beta) = (1-F^{\beta}(x)) \ln F(x)$ is non-positive. Then, for $\beta > 0$, we have $\Delta(\beta) \le \Delta(0) = 0$, which shows that $r(x;\beta)$ is decreasing in β. $\square$

The following counterexample shows that the result of Theorem 3.10 cannot be extended for $n \ge 3$.

Example 3.2. *Let us consider the exponential distribution with survival function* $\bar{F}(x;\beta) = e^{-\beta x}$. *Take* $(p_1,p_2,p_3) = (0.59, 0.3, 0.11)$, $(q_1,q_2,q_3) = (0.6, 0.3, 0.1)$, $(\theta_1,\theta_2,\theta_3) = (2,4,10)$, $(\lambda_1,\lambda_2,\lambda_3) = (2, 3.9, 10.1)$, $(\beta_1,\beta_2,\beta_3) = (5,3,2)$ *and* $(\gamma_1,\gamma_2,\gamma_3) = (6,3,2)$. *It is easy to observe that* $\boldsymbol{q} \overset{\text{m}}{\succeq} \boldsymbol{p}$, $\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\theta}$, $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_3^+$, $\boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_3^+$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathcal{D}_3^+$. *The hazard rates functions of* $V_3(\boldsymbol{p},\boldsymbol{\theta})$ *and* $W_3(\boldsymbol{q},\boldsymbol{\lambda})$ *are given by*

$$r(x;\boldsymbol{p},\boldsymbol{\theta}) = \frac{\sum_{i=1}^{3} p_i\,(\theta_i + \beta_i)\, e^{-\alpha\,(\theta_i+\beta_i)x}}{\sum_{i=1}^{3} p_i\, e^{-\alpha\,(\theta_i+\beta_i)x}} \quad \textit{and} \quad r(x;\boldsymbol{q},\boldsymbol{\lambda}) = \frac{\sum_{i=1}^{3} q_i\,(\lambda_i + \gamma_i)\, e^{-\alpha\,(\lambda_i+\gamma_i)x}}{\sum_{i=1}^{3} q_i\, e^{-\alpha\,(\lambda_i+\gamma_i)x}}, \qquad x > 0.$$

Now, for $\alpha = 0.1$, *we have* $r(10;\boldsymbol{p},\boldsymbol{\theta}) \approx 7.0041 < 7.3448 \approx r(10;\boldsymbol{q},\boldsymbol{\lambda})$, *but for* $\alpha = 0.4$, *we have* $r(10;\boldsymbol{p},\boldsymbol{\theta}) \approx 7.0000 > 6.9263 \approx r(10;\boldsymbol{q},\boldsymbol{\lambda})$, *which means that* $V_3(\boldsymbol{p},\boldsymbol{\beta})$ *and* $W_3(\boldsymbol{q},\boldsymbol{\gamma})$ *cannot be compared in terms of the hazard rate order for* $n \ge 3$.

The following counterexample proves that the result of Theorem 3.10 cannot be extended to the likelihood ratio ordering for $n = 2$.

Let us consider the standard exponential as a baseline distribution for the α-mixture model. Take $(p_1,p_2) = (0.56, 0.44)$, $(q_1,q_2) = (0.6, 0.4)$, $(\theta_1,\theta_2) = (7,13)$, $(\lambda_1,\lambda_2) = (5,15)$, $(\beta_1,\beta_2) = (3,6)$, $(\gamma_1,\gamma_2) = (4,8)$, and $\alpha = 5$. It is easy to observe that
$\boldsymbol{q} \overset{\text{m}}{\succeq} \boldsymbol{p}$, $\boldsymbol{\lambda} \overset{\text{m}}{\succeq} \boldsymbol{\theta}$, $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_2^+$, $\boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_2^+$ and $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathcal{D}_2^+$, we have $V_2(\boldsymbol{p},\boldsymbol{\theta}) \ge_{\text{hr}} W_2(\boldsymbol{q},\boldsymbol{\lambda})$.
The ratio of the density functions of $V_2(\boldsymbol{p},\boldsymbol{\theta})$ and $W_2(\boldsymbol{q},\boldsymbol{\lambda})$ is given by

$$\frac{f_{V_2(\boldsymbol{p},\boldsymbol{\theta})}(x)}{f_{W_2(\boldsymbol{q},\boldsymbol{\lambda})}(x)} = \frac{\left(5.6e^{-50x} + 8.36e^{-95x}\right)\left(0.56e^{-50x} + 0.44e^{-95x}\right)^{-0.8}}{\left(5.4e^{-45x} + 9.2e^{-115x}\right)\left(0.6e^{-45x} + 0.4e^{-115x}\right)^{-0.8}}, \quad x \in \mathbb{R}_+.$$

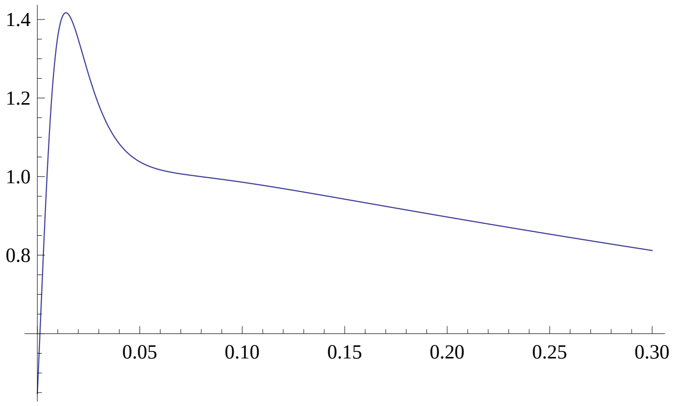

Figure 2: The ratio of density functions of the $V_2(\boldsymbol{p},\boldsymbol{\theta})$ and $W_2(\boldsymbol{q},\boldsymbol{\lambda})$

As plotted in Figure 2, it can be seen that the ratio function is not monotone for $x \in \mathbb{R}_+$, which means that $V_2(\boldsymbol{p},\boldsymbol{\theta}) \not\ge_{\text{lr}} W_2(\boldsymbol{q},\boldsymbol{\lambda})$ and $V_2(\boldsymbol{p},\boldsymbol{\theta}) \not\le_{\text{lr}} W_2(\boldsymbol{q},\boldsymbol{\lambda})$.

Theorem 3.14. *Let*

$$\bar{F}_{V_2(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x) = \left[\sum_{i=1}^{2} p_i \left\{e^{-\theta_i x}\bar{F}(x;\beta_i)\right\}^{\alpha}\right]^{1/\alpha} \quad \textit{and} \quad \bar{F}_{W_2(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})}(x) = \left[\sum_{i=1}^{2} q_i \left\{e^{-\lambda_i x}\bar{F}(x;\gamma_i)\right\}^{\alpha}\right]^{1/\alpha}$$

be the survival functions of two α*-mixtures of survival functions of additive hazard models corresponding to* $V_2(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})$ *and* $W_2(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})$, *respectively. Suppose that* $\bar{F}(x;\beta)$ *is decreasing and log-convex in* β *and* $r(x;\beta)$ *is increasing and concave in* β, *for all* $x \in \mathbb{R}$. *Then, for* $\boldsymbol{q} \overset{\text{m}}{\succeq} \boldsymbol{p}$, $\boldsymbol{\beta} \overset{\text{m}}{\succeq} \boldsymbol{\gamma}$, $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_2^+$, $\boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_2^+$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathcal{D}_2^+$ *and* $\alpha > 0$, *we have* $V_2(\boldsymbol{p},\boldsymbol{\beta}) \le_{\text{hr}} W_2(\boldsymbol{q},\boldsymbol{\gamma})$.

Proof. Assume that $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_2^+$, $\boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_2^+$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathcal{D}_2^+$ and $\alpha > 0$. The hazard rate function of $V_2(\boldsymbol{p}, \boldsymbol{\beta})$ is

$$r(x; \boldsymbol{p}, \boldsymbol{\beta}) = \frac{\sum_{i=1}^2 p_i \left(\theta_i + r(x; \beta_i)\right) \{e^{-\theta_i x} \bar{F}(x; \beta_i)\}^\alpha}{\sum_{i=1}^2 p_i \{e^{-\theta_i x} \bar{F}(x; \beta_i)\}^\alpha}.$$

Note that

$$\frac{\partial r(x; \boldsymbol{p}, \boldsymbol{\beta})}{\partial p_1} \overset{\text{sgn}}{=} p_2 \{e^{-\theta_1 x} \bar{F}(x; \beta_1)\}^\alpha \{e^{-\theta_2 x} \bar{F}(x; \beta_2)\}^\alpha \left[(\theta_1 + r(x; \beta_1)) - (\theta_2 + r(x; \beta_2))\right].$$

Then, we have

$$\frac{\partial r(x; \boldsymbol{p}, \boldsymbol{\beta})}{\partial p_1} - \frac{\partial r(x; \boldsymbol{p}, \boldsymbol{\beta})}{\partial p_2} \overset{\text{sgn}}{=} -(p_1 - p_2) \left[(\theta_1 + r(x; \beta_1)) - (\theta_2 + r(x; \beta_2))\right].$$

For $\beta_1 \geq \beta_2$, $\theta_1 \geq \theta_2$ and from the increasing property of $r(x; \beta)$, we conclude that

$$(p_1 - p_2) \left(\frac{\partial r(x; \boldsymbol{p}, \boldsymbol{\beta})}{\partial p_1} - \frac{\partial r(x; \boldsymbol{p}, \boldsymbol{\beta})}{\partial p_2} \right) \leq 0, \tag{5}$$

which implies $r(x; \boldsymbol{p}, \boldsymbol{\beta}) \geq r(x; \boldsymbol{q}, \boldsymbol{\beta})$ for all $x \in \mathbb{R}$ by applying Lemma 2.4.

Let $M_2(\boldsymbol{q}, \boldsymbol{\beta})$ be the α-mixture of survival functions of additive hazard models with parameters $\boldsymbol{q}$ and $\boldsymbol{\beta}$. The hazard rate function of $M_2(\boldsymbol{q}, \boldsymbol{\beta})$ is

$$r(x; \boldsymbol{q}, \boldsymbol{\beta}) = \frac{\sum_{i=1}^2 q_i \left(\theta_i + r(x; \beta_i)\right) \{e^{-\theta_i x} \bar{F}(x; \beta_i)\}^\alpha}{\sum_{i=1}^2 q_i \{e^{-\theta_i x} \bar{F}(x; \beta_i)\}^\alpha} = \frac{A_1(\theta_1 + r(x; \beta_1)) + A_2(\theta_2 + r(x; \beta_2))}{A_1 + A_2},$$

where $A_1 = q_1 \{e^{-\theta_1 x} \bar{F}(x; \beta_1)\}^\alpha$ and $A_2 = q_2 \{e^{-\theta_2 x} \bar{F}(x; \beta_2)\}^\alpha$.

We also observe that

$$\frac{\partial r(x; \boldsymbol{q}, \boldsymbol{\beta})}{\partial \beta_1} = \frac{\partial A_1}{\partial \beta_1} A_2 \left[(\theta_1 + r(x; \beta_1)) - (\theta_2 + r(x; \beta_2))\right] + \frac{\partial r(x; \beta_1)}{\partial \beta_1} A_1 (A_1 + A_2).$$

and then for $\alpha > 0$, $q_1 \geq q_2$, $\theta_1 \leq \theta_2$, $\beta_1 \leq \beta_2$, $\bar{F}(x; \beta)$ is decreasing and log-convex in β and $r(x; \beta)$ is increasing and concave in β, we have

$$\begin{aligned}
\frac{\partial r(x; \boldsymbol{q}, \boldsymbol{\beta})}{\partial \beta_1} - \frac{\partial r(x; \boldsymbol{q}, \boldsymbol{\beta})}{\partial \beta_2} &= \alpha q_1 q_2 \left(\frac{\partial \log \bar{F}(x; \beta_1)}{\partial \beta_1} - \frac{\partial \log \bar{F}(x; \beta_2)}{\partial \beta_2} \right) \left(e^{-(\theta_1 + \theta_2) x} \bar{F}(x; \beta_1) \bar{F}(x; \beta_2) \right)^\alpha \\
&\quad \times \left[(\theta_1 + r(x; \beta_1)) - (\theta_2 + r(x; \beta_2))\right] + \left(q_1 \{e^{-\theta_1 x} \bar{F}(x; \beta_1)\}^\alpha + q_2 \left\{e^{-\theta_2 x} \bar{F}(x; \beta_2)\right\}^\alpha \right) \\
&\quad \times \left[q_1 \left\{e^{-\theta_1 x} \bar{F}(x; \beta_1)\right\}^\alpha \frac{\partial r(x; \beta_1)}{\partial \beta_1} - q_2 \left\{e^{-\theta_2 x} \bar{F}(x; \beta_2)\right\}^\alpha \frac{\partial r(x; \beta_2)}{\partial \beta_2} \right] \\
&\geq 0,
\end{aligned}$$

and then

$$(\beta_1 - \beta_2) \left(\frac{\partial r(x; \boldsymbol{q}, \boldsymbol{\beta})}{\partial \beta_1} - \frac{\partial r(x; \boldsymbol{q}, \boldsymbol{\beta})}{\partial \beta_2} \right) \leq 0,$$

which completes the proof of theorem. □

The following corollaries are direct results of Theorem 3.10, for which the proofs are simple and thus omitted here.

Corollary 3.15. *Let $\bar{F}(x; \beta) = \bar{F}^\beta(x)$ and $\bar{F}(x; \gamma) = \bar{F}^\gamma(x)$, for $x \in \mathbb{R}$. Then, for $\boldsymbol{q} \overset{\text{m}}{\succeq} \boldsymbol{p}$, $\boldsymbol{\beta} \overset{\text{m}}{\succeq} \boldsymbol{\gamma}$, $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_n^+$, $\boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_n^+$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathcal{D}_n^+$ and $\alpha > 0$, we have $V_2(\boldsymbol{p}, \boldsymbol{\beta}) \leq_{\text{hr}} W_2(\boldsymbol{q}, \boldsymbol{\gamma})$.*

4. α-mixtures of additive hazard distribution functions

This section discusses stochastic comparisons of α-mixtures of additive hazard distribution functions with respect to the usual stochastic order.

Theorem 4.1. *Let*

$$F_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x) = \left[\sum_{i=1}^{n} p_i \left\{1 - e^{-\theta_i x}\bar{F}(x;\beta_i)\right\}^{\alpha}\right]^{1/\alpha} \quad \text{and} \quad F_{W_n(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})}(x) = \left[\sum_{i=1}^{n} q_i \left\{1 - e^{-\lambda_i x}\bar{F}(x;\gamma_i)\right\}^{\alpha}\right]^{1/\alpha}$$

be the distribution functions of two α-mixtures of distribution functions of additive hazard models corresponding to $V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})$ and $W_n(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})$, respectively. Suppose that $\bar{F}(x;\beta)$ is decreasing and log-convex with respect to $\beta > 0$, for all $x \in \mathbb{R}$. Then, for $\boldsymbol{p} \overset{\mathrm{w}}{\succeq} \boldsymbol{q}$, $\boldsymbol{\beta} \overset{\mathrm{w}}{\succeq} \boldsymbol{\gamma}$, $\boldsymbol{\theta} \overset{\mathrm{w}}{\succeq} \boldsymbol{\lambda}$, $\boldsymbol{p},\boldsymbol{q} \in \mathcal{E}_n^+$, $\boldsymbol{\theta},\boldsymbol{\lambda} \in \mathcal{D}_n^+$, $\boldsymbol{\beta},\boldsymbol{\gamma} \in \mathcal{D}_n^+$ and $0 < \alpha \le 1$, we have $V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta}) \ge_{\mathrm{st}} W_n(\boldsymbol{q},\boldsymbol{\lambda},\boldsymbol{\gamma})$.

Proof. Assume that $\boldsymbol{p},\boldsymbol{q} \in \mathcal{E}_n^+$, $\boldsymbol{\theta},\boldsymbol{\lambda} \in \mathcal{D}_n^+$, $\boldsymbol{\beta},\boldsymbol{\gamma} \in \mathcal{D}_n^+$ and $0 < \alpha \le 1$. Note that

$$\frac{\partial F_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial p_i} = \frac{1}{\alpha}\left(1 - e^{-\theta_i x}\bar{F}(x;\beta_i)\right)^{\alpha}\left[\sum_{i=1}^{n} p_i \left\{1 - e^{-\theta_i x}\bar{F}(x;\beta_i)\right\}^{\alpha}\right]^{\frac{1}{\alpha}-1} \ge 0,$$

and then

$$\frac{\partial F_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial p_1} - \frac{\partial F_{V_n(\boldsymbol{p},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial p_2} \overset{\mathrm{sgn}}{=} \left(1 - e^{-\theta_1 x}\bar{F}(x;\beta_1)\right)^{\alpha} - \left(1 - e^{-\theta_2 x}\bar{F}(x;\beta_2)\right)^{\alpha} \le 0.$$

Then, $F_{V_n(\boldsymbol{p},\boldsymbol{\beta})}(x) \ge F_{M_n(\boldsymbol{q},\boldsymbol{\beta})}(x)$ for all $x \in \mathbb{R}$ by applying Lemma 2.5.

Now, we prove that $F_{M_n(\boldsymbol{q},\boldsymbol{\beta})}(x) \ge F_{W_n(\boldsymbol{q},\boldsymbol{\gamma})}(x)$ for all $x \in \mathbb{R}$. Observe that

$$\begin{aligned}\frac{\partial F_{V_n(\boldsymbol{q},\boldsymbol{\beta})}(x)}{\partial \beta_i} &= -q_i\left(\frac{\partial \log \bar{F}(x;\beta_i)}{\partial \beta_i}\right) e^{-\theta_i x}\bar{F}(x;\beta_i)\ \left\{1 - e^{-\theta_i x}\bar{F}(x;\beta_i)\right\}^{\alpha-1} \times \left[\sum_{i=1}^{n} p_i \left\{1 - e^{-\theta_i x}\bar{F}(x;\beta_i)\right\}^{\alpha}\right]^{\frac{1}{\alpha}-1}\\ &\ge 0,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial F_{V_n(\boldsymbol{q},\boldsymbol{\beta})}(x)}{\partial \beta_1} - \frac{\partial F_{V_n(\boldsymbol{q},\boldsymbol{\beta})}(x)}{\partial \beta_2} \overset{\mathrm{sgn}}{=}\ & q_2\left(\frac{\partial \log \bar{F}(x;\beta_2)}{\partial \beta_2}\right) e^{-\theta_2 x}\bar{F}(x;\beta_2)\ \left\{1 - e^{-\theta_2 x}\bar{F}(x;\beta_2)\right\}^{\alpha-1}\\ & - q_1\left(\frac{\partial \log \bar{F}(x;\beta_1)}{\partial \beta_1}\right) e^{-\theta_1 x}\bar{F}(x;\beta_1)\ \left\{1 - e^{-\theta_1 x}\bar{F}(x;\beta_1)\right\}^{\alpha-1} \ge 0,\end{aligned}$$

since $q_1 \ge q_2$, $\beta_2 \ge \beta_1$, $\theta_2 \ge \theta_1$, $0 < \alpha \le 1$ and $\bar{F}(x;\beta)$ is decreasing and log-convex with respect to β. and then

$$(\beta_1 - \beta_2)\left(\frac{\partial F_{V_n(\boldsymbol{q},\boldsymbol{\beta})}(x)}{\partial \beta_1} - \frac{\partial F_{V_n(\boldsymbol{q},\boldsymbol{\beta})}(x)}{\partial \beta_2}\right) \le 0,$$

Now, we prove that $F_{M_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\beta})}(x) \ge F_{W_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\gamma})}(x)$ for all $x \in \mathbb{R}$. Observe that

$$\frac{\partial F_{M_n(\boldsymbol{q},\boldsymbol{\theta},\boldsymbol{\beta})}(x)}{\partial \theta_i} = q_i x\, e^{-\theta_i x}\bar{F}(x;\beta_i)\ \left\{1 - e^{-\theta_i x}\bar{F}(x;\beta_i)\right\}^{\alpha-1} \times \left[\sum_{i=1}^{n} p_i \left\{1 - e^{-\theta_i x}\bar{F}(x;\beta_i)\right\}^{\alpha}\right]^{\frac{1}{\alpha}-1} \ge 0,$$

and

$$\begin{aligned}\frac{\partial F_{V_n(\boldsymbol{q},\boldsymbol{\beta})}(x)}{\partial \theta_1} - \frac{\partial F_{V_n(\boldsymbol{q},\boldsymbol{\beta})}(x)}{\partial \theta_2} \overset{\mathrm{sgn}}{=}\ & q_1\, e^{-\theta_1 x}\bar{F}(x;\beta_1)\ \left\{1 - e^{-\theta_1 x}\bar{F}(x;\beta_1)\right\}^{\alpha-1}\\ & - q_2\, e^{-\theta_2 x}\bar{F}(x;\beta_2)\ \left\{1 - e^{-\theta_2 x}\bar{F}(x;\beta_2)\right\}^{\alpha-1} \ge 0,\end{aligned}$$

since $q_1 \ge q_2$, $\beta_2 \ge \beta_1$, $\theta_2 \ge \theta_1$, $0 < \alpha \le 1$ and $\bar{F}(x;\beta)$ is decreasing with respect to β. and then

$$(\theta_1 - \theta_2)\left(\frac{\partial F_{V_n(\boldsymbol{q},\boldsymbol{\beta})}(x)}{\partial \theta_1} - \frac{\partial F_{V_n(\boldsymbol{q},\boldsymbol{\beta})}(x)}{\partial \theta_2}\right) \le 0,$$

Thus, the proof is finished by using Lemma 2.5. □

With the help of Theorem 4.1, the following three corollaries can be obtained immediately.

Corollary 4.2. *Let* $\bar{F}(x;\beta) = \bar{F}^{\beta}(x)$ *and* $\bar{F}(x;\gamma) = \bar{F}^{\gamma}(x)$, *for* $x \in \mathbb{R}$. *Then, for* $\boldsymbol{p} \overset{\mathrm{w}}{\succeq} \boldsymbol{q}$, $\boldsymbol{\beta} \overset{\mathrm{w}}{\succeq} \boldsymbol{\gamma}$, $\boldsymbol{\theta} \overset{\mathrm{w}}{\succeq} \boldsymbol{\lambda}$, $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_n^+$, $\boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_n^+$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathcal{D}_n^+$ *and* $0 < \alpha \leq 1$, *we have* $V_n(\boldsymbol{p}, \boldsymbol{\theta}, \boldsymbol{\beta}) \geq_{\mathrm{st}} W_n(\boldsymbol{q}, \boldsymbol{\lambda}, \boldsymbol{\gamma})$.

Corollary 4.3. *Let* $F(x;\beta) = \bar{F}(\beta x)$ *and* $F(x;\gamma) = \bar{F}(\gamma x)$, *for* $x \in \mathbb{R}$. *Further, suppose* $r(t)$ *is decreasing in* t. *Then, for* $\boldsymbol{p} \overset{\mathrm{w}}{\succeq} \boldsymbol{q}$, $\boldsymbol{\beta} \overset{\mathrm{w}}{\succeq} \boldsymbol{\gamma}$, $\boldsymbol{\theta} \overset{\mathrm{w}}{\succeq} \boldsymbol{\lambda}$, $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{E}_n^+$, $\boldsymbol{\theta}, \boldsymbol{\lambda} \in \mathcal{D}_n^+$, $\boldsymbol{\beta}, \boldsymbol{\gamma} \in \mathcal{D}_n^+$ *and* $0 < \alpha \leq 1$, *we have* $V_n(\boldsymbol{p}, \boldsymbol{\theta}, \boldsymbol{\beta}) \geq_{\mathrm{st}} W_n(\boldsymbol{q}, \boldsymbol{\lambda}, \boldsymbol{\gamma})$.

5. Concluding Remarks

We have discussed some comparisons concerning finite α-mixture models, which may have important applications in many research areas. Some natural generalizations of this paper should be considered in future work. First, it is interesting to relax the assumption of Theorem 3.1 by means of the p-larger order [12]. Another possible extension is to consider the results of Theorem 3.10 and Theorem 3.14 to the case when the size of underlying distributions is arbitrary and greater than 2. Moreover, the extension of the finite α-mixture model to the case of the infinite α-mixture model and its related model properties and comparisons also remain as an interesting problem. We leave these ideas as open problems for further investigation.

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