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Original Article

On the adjacency dimension of some star related trees

Elham Hardani, Ali Behtoei*

Department of Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin, Iran

ABSTRACT: Locating or resolving sets are introduced as a graph-theoretic model of robot navigation and has different applications in diverse areas like network discovery, computer science and chemistry. These applications leads to some graph parameters, like the metric dimension and the adjacency dimension. A subset S of the vertices of a graph G is an adjacency resolving set for G if for each pair of distinct vertices $x, y \in V(G) \setminus S$, there exists $s \in S$ which is adjacent to exactly one of these two vertices. An adjacency resolving set with the minimum cardinality is called an adjacency basis and its cardinality is the adjacency dimension of G. Since the problem of computing the adjacency dimension of a graph is NP-hard, finding the adjacency dimension of special classes of graphs or obtaining good bounds on this invariant is valuable. In this paper we determine the adjacency dimension of some famous star related trees.

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1. Introduction

Throughout this paper, we only consider simple graphs. At first, we collect some standard graph-theoretic terminologies and notations in this section, see [13]. Let $\mathbb{N}$ denotes the set of all positive integers. Given a connected graph $G = (V, E)$ with vertex set V and edge set E, consider the function $d_G : V \times V \to \mathbb{N} \cup \{0\}$ where $d_G(x, y)$ is the length of a shortest path between two vertices x and y in G. Clearly, (V, d_G) is a metric space and the diameter of a G is understood in this metric. An ordered vertex set $S \subseteq V$ is said to be a metric generator for G if it is a generator of the metric space (V, d_G), i.e. each point of the space is uniquely determined by its distances from the elements of S. A minimum metric generator is called a metric basis, and its cardinality is the metric dimension of G, denoted by $\dim(G)$. Motivated by the problem of uniquely determining the location of an intruder in a network, the concept of metric dimension of a graph was introduced by Slater in [23], where the metric generators were called locating sets. The concept of metric dimension of a graph was also introduced independently by Harary and Melter in [14], where metric generators were called resolving sets. Applications of this invariant to the navigation of robots in networks are discussed in [19] and applications to chemistry in [18]. It is straightforward to see that when $n \geq 2$, for the complete graph K_n and the path P_n we have $\dim(K_n) = n - 1$ and $\dim(P_n) = 1$, respectively. In [3],

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*Corresponding author.

E-mail addresses: `e.hardany@yahoo.com` (E. Hardani), `a.behtoei@sci.ikiu.ac.ir` (A. Behtoei)

it is shown that for the wheel graph we have $\dim(W_n) = \lfloor \frac{2n+2}{5} \rfloor$. This graph parameter was studied further in a number of other papers including [4, 6, 21]. Several variations of metric generators including resolving dominating sets [2], independent resolving sets [7], local resolving sets [20], 1-locating dominating sets [5], and strong resolving sets [22]. have since been introduced and studied. Now consider the distance function $d_2 : V \times V \to \mathbb{N} \cup \{0\}$, where $d_2(x, y) = min\{d_G(x, y), 2\}$. Let $S = \{s_1, s_2, \dots, s_k\}$ be a non empty ordered subset of $V = V(G)$. For each $v \in V(G)$, the k-tuple $r_a(v|S) = (d_2(v, s_1), d_2(v, s_2), \dots, d_2(v, s_k))$ is called the adjacency representation of v with respect to S, and S is an adjacency resolving set (or an adjacency generator) for G if for each pair of distinct vertices $v_1, v_2 \in V(G)$ we have $r_a(v_1|S) \neq r_a(v_2|S)$. An adjacency resolving set with the minimum cardinality is called an adjacency basis and its cardinality is the adjacency dimension of G which is denoted by $\operatorname{adim}(G)$, see [17]. It is easy to show that S is an adjacency resolving set for G if for each pair of different vertices $x, y \in V(G) \setminus S$ we have $N[x] \cap S \neq N[y] \cap S$, or equivalently there exists $s_i \in S$ which is adjacent to exactly one of these two vertices, that is $|N(s_i) \cap \{x, y\}| = 1$, where $N(s_i)$ and $N[s_i]$ denote the open neighborhood and the closed neighborhood of the vertex s_i in G, respectively. Note that we have $r_a(v|S) = (2, 2, \dots, 2)$ just when $N[v] \cap S = \emptyset$. Specially, S is an adjacency resolving set for G if and only if it is an adjacency resolving set for its complement $\overline{G}$, and consequently $\operatorname{adim}(G) = \operatorname{adim}(\overline{G})$. Also, note that for each graph G of diameter at most two, we have $d_2(x, y) = d_G(x, y)$ and hence, $\operatorname{adim}(G) = \dim(G)$. Moreover, $\operatorname{adim}(K_n) = \dim(K_n)$ and $\operatorname{adim}(W_n) = \dim(W_n)$. This concept has been studied further by many scientists. Fernau and Rodriguez-Velazquez in [10] and [11] have shown that the metric dimension of the corona product of a graph of order n and some nontrivial graph H equals n times the adjacency dimension of H, and they proved that the problem of computing the adjacency dimension is an NP-hard problem. This suggests finding the adjacency dimension for special classes of graphs or obtaining good bounds on this invariant. To see more results in this subject or related subjects, the reader is referred to [1, 9, 15, 16].

2. Main results

In this section we determine the adjacency dimension of the following star related trees.

Definition 2.1 ([8]). *An $(n, k)-$banana tree denoted by $B_{n,k}$ is a tree obtained by connecting one leaf of n copies of an k-star graph with a single root vertex (that is distinct from all stars). For example $B_{6,5}$ is shown in Figure 1.*

Definition 2.2 ([8]). *An (n, k)-firecracker graph is a tree denoted by $F_{n,k}$ and is obtained by the connection of n k-stars by linking one leaf of each, see Figure 2.*

Definition 2.3 ([12]). *Consider the integers $l_1, l_2, \dots, l_n$ in which $n \geq 2$ and $l_i \geq 2$ for each $i \in \{1, 2, \dots, n\}$. The graph obtained from the n stars $K_{1,l_1}, K_{1,l_2}, \dots, K_{1,l_n}$ by joining the central vertices of K_{1,l_j} and $K_{1,l_{j+1}}$ to a new vertex w_j for $j = 1, 2, 3, \dots, n-1$ is denoted by $T(K_{1,l_1} : K_{1,l_2} : \cdots : K_{1,l_n})$, see Figure 3.*

Definition 2.4 ([12]). *Consider the integers $l_1, l_2, \dots, l_n$ in which $n \geq 2$ and $l_i \geq 3$ for each $i \in \{1, 2, \dots, n\}$. The graph obtained from the n stars $K_{1,l_1}, K_{1,l_2}, \dots, K_{1,l_n}$ by joining a leaf of K_{1,l_j} and a leaf of $K_{1,l_{j+1}}$ to a new vertex w_j by an edge, for each $j = 1, 2, 3, \dots, n-1$, is denoted by $T(K_{1,l_1} \circ K_{1,l_2} \circ \cdots \circ K_{1,l_n})$. The graph $T(K_{1,5} \circ K_{1,5} \circ K_{1,6} \circ K_{1,4})$ is shown in Figure 4.*

Definition 2.5 ([12]). *Consider the integers $l_1, l_2, \dots, l_n$ in which $n \geq 2$ and $l_i \geq 2$ for each $i \in \{1, 2, \dots, n\}$. The graph obtained by connecting a new vertex v_0 to the central vertices of stars $K_{1,l_1}, K_{1,l_2}, \dots, K_{1,l_n}$ is called the shrub graph and it is denoted by $St(l_1, l_2, \dots, l_n)$. For example $St(4, 3, 5)$ is shown in Figure 5.*

In the following theorem, we determine the exact value of the adjacency dimension of a banana tree.

Theorem 2.6. *Let n, k be two positive integers with $k \geq 4$ and $n \geq 2$. Then the adjacency dimension of the banana tree $B_{n,k}$ is given by $\operatorname{adim}(B_{n,k}) = n(k-2)$.*

Proof. Let the vertex set of $B_{n,k}$ be

$$V(B_{n,k}) = \{r\} \cup \left(\bigcup_{i=1}^{n} \{x_1^i, x_2^i, \dots, x_k^i\} \right),$$

and its edge set be

$$E(B_{n,k}) = \{rx_k^i : 1 \leq i \leq n\} \cup \left\{x_1^i x_2^i, x_1^i x_3^i, \dots, x_1^i x_k^i \ : \ 1 \leq i \leq n\right\},$$

see Figure 1. At first, we show that $\operatorname{adim}(B_{n,k}) \leq n(k-2)$. Let

$$W_0 = \{r, x_1^1, x_1^2, x_1^3, \dots, x_1^{n-1}\} \cup \left(\bigcup_{i=1}^{n} \{x_2^i, x_3^i, \dots, x_{k-2}^i\} \right).$$

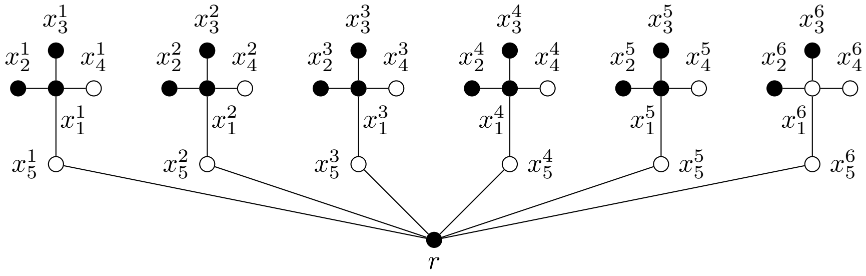

Figure 1: The banana tree $B_{6,5}$.

Note that for each $1 \le i \le n$, vertices $x_2^i, x_3^i, \ldots, x_{k-1}^i$ are twins. Also, we have

$$V(B_{n,k}) \setminus W_0 = \{x_1^n\} \cup \{x_{k-1}^1, x_{k-1}^2, x_{k-1}^3, \ldots, x_{k-1}^n\} \cup \{x_k^1, x_k^2, x_k^3, \ldots, x_k^n\}.$$

Note that $k-2 \ge 2$, $x_2^n \in W_0$ and x_1^n is the unique vertex adjacent to x_2^n. Thus, x_1^n is adjacently resolved from all other vertices. For each $1 \le i \le n-1$, x_{k-1}^i is adjacent just to $x_1^i \in W_0$ and x_k^i is adjacent to both $x_1^i \in W_0$ and $r \in W_0$. The vertex x_{k-1}^n is the unique vertex with adjacency representation $(2,2,\ldots,2)$, and x_k^n is adjacent just to r. Therefore, W_0 is an adjacency resolving set for $B_{n,k}$ and hence, $\operatorname{adim}(B_{n,k}) \le |W_0| = n(k-2)$.

Now let B be an adjacency basis for the banana tree $B_{n,k}$. We show that $\operatorname{adim}(B_{n,k}) \ge n(k-2)$, which will complete the proof. Note that for each $1 \le i \le n$, vertices $\{x_2^i, x_3^i, \ldots, x_{k-1}^i\}$ are twins and hence,

$$|B \cap \{x_2^i, x_3^i, \ldots, x_{k-1}^i\}| \ge (k-3).$$

This implies that

$$\left| B \cap \left(\bigcup_{i=1}^{n} \{x_2^i, x_3^i, \ldots, x_{k-1}^i\} \right) \right| \ge n(k-3).$$

Without loss of generality, we can assume that $\{x_2^i, x_3^i, \ldots, x_{k-2}^i\} \subseteq B$ for each $1 \le i \le n$. Now consider the following two cases.

Case 1. $r \notin B$.
In this case, if $B \cap \{x_1^i, x_{k-1}^i, x_k^i\} = \emptyset$ for some $i \in \{1, 2, \ldots, n\}$, then we have

$$r_a(x_{k-1}^i|B) = (2,2,\ldots,2) = r_a(x_k^i|B),$$

which is a contradiction. Hence, $|B \cap \{x_1^i, x_{k-1}^i, x_k^i\}| \ge 1$ for each $i \in \{1, 2, \ldots, n\}$, and this implies that

$$|B| \ge n(k-3) + n = n(k-2).$$

Case 2. $r \in B$.
In this case, if $B \cap \{x_k^i, x_1^i, x_k^j, x_1^j\} = \emptyset$ for some $i \ne j$, then $r_a(x_k^i|B) = r_a(x_k^j|B)$, which is a contradiction. Thus, for $i \ne j$ we have $|B \cap \{x_k^i, x_1^i, x_k^j, x_1^j\}| \ge 1$ and this implies that

$$\left| B \cap \left(\bigcup_{i=1}^{n} \{x_k^i, x_1^i\} \right) \right| \ge n-1.$$

Thus,

$$|B| \ge n(k-3) + 1 + (n-1) = n(k-2).$$

Therefore, in each case we have $|B| \ge n(k-2)$ which completes the proof. □

Corollary 2.7. *If B is an adjacency basis for the banana tree $B_{n,k}$, then the root vertex is in B.*

Proof. By using the same notations applied in the proof of Theorem 2.6, assume on the contrary that there exists an adjacency basis B for $B_{n,k}$ such that $r \notin B$. We know that for each $1 \le i \le n$, vertices $x_2^i, x_3^i, \ldots, x_{k-1}^i$ are twins. Hence,

$$|B \cap \{x_2^i, x_3^i, \ldots, x_{k-1}^i\}| \ge (k-3),$$

and without loss of generality, we can assume that $\{x_2^i, x_3^i, \ldots, x_{k-2}^i\} \subseteq B$. Specially, we see that

$$\left| B \cap \left(\bigcup_{i=1}^{n} \{x_2^i, x_3^i, \ldots, x_{k-1}^i\} \right) \right| \geq n(k-3).$$

If $\{x_1^i, x_{k-1}^i, x_k^i\} \cap B = \emptyset$ for some i, then

$$N[x_{k-1}^i] \cap B = \emptyset = N[x_k^i] \cap B,$$

which is a contradiction. Thus, for each $1 \leq i \leq n$ we have $|\{x_1^i, x_{k-1}^i, x_k^i\} \cap B| \geq 1$. Hence,

$$\left| B \cap \left(\bigcup_{i=1}^{n} \{x_1^i, x_{k-1}^i, x_k^i\} \right) \right| \geq n,$$

which implies that $|B| \geq n(k-3) + n = n(k-2)$. Since $\operatorname{adim}(B_{n,k}) = n(k-2)$ by Theorem 2.6, for each $1 \leq i \leq n$ we must have $|\{x_1^i, x_{k-1}^i, x_k^i\} \cap B| = 1$.
If $\{x_1^i, x_{k-1}^i, x_k^i\} \cap B = \{x_1^i\}$ for some i, then

$$N[x_{k-1}^i] \cap B = \{x_1^i\} = N[x_k^i] \cap B,$$

which is a contradiction. This means that for each $1 \leq i \leq n$ we have $B \cap \{x_1^i, x_{k-1}^i, x_k^i\} = \{x_{k-1}^i\}$ or $B \cap \{x_1^i, x_{k-1}^i, x_k^i\} = \{x_{k-1}^i\}$, and hence $r_a(x_k^j|B) = (2, 2, \ldots, 2)$ or $r_a(x_{k-1}^j|B) = (2, 2, \ldots, 2)$, respectively. Therefore, there exists at least $n \geq 2$ vertices whose representations with respect to B is $(2, 2, \ldots, 2)$, a contradiction. Thus, we must have $r \in B$. ☐

The adjacency dimension of firecracker graphs is investigated and determined in the following result.

Theorem 2.8. *Let n,k be two positive integers with $k \geq 4$ and $n \geq 2$. Then the adjacency dimension of the firecracker graph is given by*

$$\operatorname{adim}(F_{n,k}) = n(k-2) + \left\lceil \frac{n}{3} \right\rceil - 1.$$

Proof. Let the vertex set of $F_{n,k}$ be

$$V(F_{n,k}) = \bigcup_{i=1}^{n} \{x_1^i, x_2^i, \ldots, x_k^i\},$$

and its edge set be as (see Figure 2)

$$E(F_{n,k}) = \{x_k^i x_k^{i+1} : 1 \leq i \leq n-1\} \cup \left\{ x_1^i x_2^i, x_1^i x_3^i, \ldots, x_1^i x_k^i \ : \ 1 \leq i \leq n \right\}.$$

For $n = 3t+1$ *or* $n = 3t+2$, let

$$W_0 = \{x_1^1, x_1^2, x_1^3, \ldots, x_1^{n-1}\} \cup \left(\bigcup_{i=1}^{n} \{x_2^i, x_3^i, \ldots, x_{k-2}^i\} \right) \cup \{x_k^{3i+1};\ 0 \leq i \leq t\},$$

and for $n = 3t$, let

$$W_0 = \{x_1^1, x_1^2, x_1^3, \ldots, x_1^{n-1}\} \cup \left(\bigcup_{i=1}^{n} \{x_2^i, x_3^i, \ldots, x_{k-2}^i\} \right) \cup \{x_k^{3i+2};\ 0 \leq i \leq t-1\}.$$

It is easy to check that W_0 is an adjacency resolving set for $F_{n,k}$ and hence,

$$\operatorname{adim}(F_{n,k}) \leq |W_0| = (n-1) + n(k-3) + \left\lceil \frac{n}{3} \right\rceil = n(k-2) + \left\lceil \frac{n}{3} \right\rceil - 1.$$

In what follows, we show that $\operatorname{adim}(F_{n,k}) \geq n(k-2) + \lceil \frac{n}{3} \rceil - 1$, which will complete the proof. Let B be an adjacency basis for $F_{n,k}$ and hence, $\operatorname{adim}(F_{n,k}) = |B|$. Note that for each $i \in \{1, 2, \ldots, n\}$, vertices $\{x_2^i, x_3^i, \ldots, x_{k-1}^i\}$ are twins and hence $|B \cap \{x_2^i, x_3^i, \ldots, x_{k-1}^i\}| \geq k-3$. Without loss of generality, we may assume that $\{x_2^i, x_3^i, \ldots, x_{k-2}^i\} \subseteq B$. Specially we have

$$\left| B \cap \left(\bigcup_{i=1}^{n} \{x_2^i, x_3^i, \ldots, x_{k-1}^i\} \right) \right| \geq n(k-3). \tag{1}$$

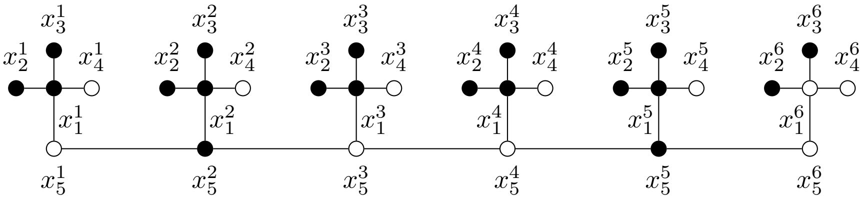

Figure 2: The firecracker graph $F_{6,5}$.

If $\{x_1^i, x_{k-1}^i, x_k^i\} \subseteq B$ for some i, then $B \backslash \{x_1^i\}$ is an adjacency resolving set for $F_{n,k}$ which contradicts the minimality of B. Thus, $|B \cap \{x_1^i, x_{k-1}^i, x_k^i\}| \leq 2$ for each $i \in \{1, 2, \ldots, n\}$. Moreover, if we have $\{x_1^i, x_{k-1}^i\} \subseteq B$ for some i, then we can replace B by $(B \setminus \{x_1^i\}) \cup \{x_k^i\}$. Hence, we may assume that $|B \cap \{x_1^i, x_{k-1}^i\}| \leq 1$ for each $i \in \{1, 2, \ldots, n\}$. If $B \cap \{x_1^i, x_{k-1}^i, x_1^j, x_{k-1}^j\} = \emptyset$ for some $i \neq j$, then we obtain

$$N[x_{k-1}^i] \cap B = \emptyset = N[x_{k-1}^j] \cap B,$$

which is a contradiction. Thus, for each $i \neq j$ we have

$$|B \cap \{x_1^i, x_{k-1}^i, x_1^j, x_{k-1}^j\}| \geq 1.$$

Therefore, there exists at most one $i \in \{1, 2, \ldots, n\}$ such that $|B \cap \{x_1^i, x_{k-1}^i\}| = 0$. Thus, one of the following two cases may occur.

Case 1. There exists unique $1 \leq i \leq n$ such that $|B \cap \{x_1^i, x_{k-1}^i\}| = 0$.
In this case, we have $N[x_{k-1}^i] \cap B = \emptyset$ and

$$\left| B \cap \left(\bigcup_{1 \leq j \leq n,\ j \neq i} \{x_1^j, x_{k-1}^j\} \right) \right| = n - 1. \tag{2}$$

Since B is an adjacency basis, we must have $N[x_k^i] \cap B \neq N[x_{k-1}^i] \cap B$ and hence, $|B \cap N[x_k^i]| \geq 1$. Since $x_1^i \notin B$, $|B \cap (N[x_k^i] \backslash \{x_1^i\})| \geq 1$. If $x_1^j \in B$ for some $j \neq i$, then $x_{k-1}^j \notin B$ and the inequality $N[x_k^j] \cap B \neq N[x_{k-1}^j] \cap B$ implies that $|B \cap (N[x_k^j] \backslash \{x_1^j\})| \geq 1$. If $x_{k-1}^j \in B$ for some $j \neq i$, then $x_1^j \notin B$ and the inequality $N[x_k^j] \cap B \neq N[x_{k-1}^i] \cap B = \emptyset$ implies that $|B \cap (N[x_k^j] \setminus \{x_1^j\})| \geq 1$.
Therefore, for each $j \in \{1, 2, \ldots, n\}$ we have $|B \cap (N[x_k^j] \setminus \{x_1^j\})| \geq 1$. Hence, for every $2 \leq j \leq n-1$ we must have $|B \cap \{x_k^{j-1}, x_k^j, x_k^{j+1}\}| \geq 1$ which implies that

$$|B \cap \{x_k^1, x_k^2, \ldots, x_k^n\}| \geq \left\lceil \frac{n}{3} \right\rceil. \tag{3}$$

Now from (1), (2) and (3) we get

$$|B| \geq (n-1) + n(k-3) + \left\lceil \frac{n}{3} \right\rceil,$$

which completes the proof in this case.

Case 2. For each $1 \leq i \leq n$, $|B \cap \{x_1^i, x_{k-1}^i\}| = 1$.
In this case we see that

$$\left| B \cap \left(\bigcup_{i=1}^{n} \{x_1^i, x_{k-1}^i\} \right) \right| = n.$$

Let $\Omega = \{i\ : 1 \leq i \leq n,\ x_1^i \in B\}$. If $i \in \Omega$, then $x_{k-1}^i \notin B$ and the inequality $N[x_k^i] \cap B \neq N[x_{k-1}^i] \cap B$ implies that $|B \cap (N[x_k^i] \setminus \{x_1^i\})| \geq 1$.
If there exist $i \neq j$ such that $\{i, j\} \subseteq \{1, 2, \ldots, n\}$, $\{i, j\} \cap \Omega = \emptyset$ and

$$B \cap \big((N[x_k^i] \setminus \{x_1^i\}) \cup (N[x_k^j] \setminus \{x_1^j\}) \big) = \emptyset,$$

then we have $N[x_k^i] \cap B = \emptyset = N[x_k^j] \cap B$ which is a contradiction. Thus there exists at most one integer $i \in \{1, 2, \ldots, n\} \setminus \Omega$ such that $B \cap (N[x_k^i] \setminus \{x_1^i\}) = \emptyset$.

Therefore, there exists at most one integer $i \in \{1, 2, \ldots, n\}$, such that $B \cap (N[x_k^i] \setminus \{x_1^i\}) = \emptyset$. This means that

$$|B \cap \{x_k^1, x_k^2, \ldots, x_k^n\}| \geq \left\lceil \frac{n}{3} \right\rceil - 1,$$

and hence,

$$|B| \geq n(k-3) + n + \left\lceil \frac{n}{3} \right\rceil - 1,$$

as desired. The proof is complete. □

In the following theorem, the adjacency dimension of $T(K_{1,l_1} : K_{1,l_2} : \cdots : K_{1,l_n})$ is determined based on the number of stars and the number of leaves of each star.

Theorem 2.9. *The adjacency dimension of the graph $G = T(K_{1,l_1}:K_{1,l_2}:\cdots:K_{1,l_n})$ (see Figure 3) is*

$$\mathrm{adim}(G) = n + \Sigma_{i=1}^{n}(l_i - 1).$$

Proof. Assume that

$$V(G) = \{w_1, w_2, \ldots, w_{n-1}\} \cup \left(\bigcup_{i=1}^{n} \{r_i, x_1^i, x_2^i, \ldots, x_{l_i}^i\} \right),$$

and

$$E(G) = \left(\bigcup_{i=1}^{n} \{r_i x_1^i, r_i x_2^i, \ldots, r_i x_{l_i}^i\} \right) \cup \left(\bigcup_{i=1}^{n-1} \{w_i r_i, w_i r_{i+1}\} \right).$$

Let

$$W_0 = \{r_1, r_2, \ldots, r_n\} \cup \left(\bigcup_{i=1}^{n} \{x_1^i, x_2^i, \ldots, x_{l_i - 1}^i\} \right).$$

Note that $N[w_i] \cap B = \{r_i, r_{i+1}\}$ for each $i \in \{1, 2, \ldots, n-1\}$ and $N[x_{l_j}^j] \cap B = \{r_j\}$ for each $j \in \{1, 2, \ldots, n\}$. Thus W_0 is an adjacency resolving set for G and hence,

$$\mathrm{adim}(G) \leq |W_0| = n + \Sigma_{i=1}^{n}(l_i - 1).$$

Now suppose that B is an adjacency basis for G. Since vertices $x_1^i, x_2^i, \ldots, x_{l_i}^i$ are twins for each $i \in \{1, 2, \ldots, n\}$, we must have $|B \cap \{x_1^i, x_2^i, \ldots, x_{l_i}^i\}| \geq l_i - 1$. Without loss of generality, we can suppose that $\{x_1^i, x_2^i, \ldots, x_{l_i - 1}^i\} \subseteq B$ for each $i \in \{1, 2, \ldots, n\}$, and hence $|B| \geq \sum_{i=1}^{n}(l_i - 1)$. Since B is an adjacency basis for G and $N[x_{l_i}^i] \cap B \neq N[x_{l_j}^j] \cap B$ when $i \neq j$, we obtain $|B \cap \{r_i, x_{l_i}^i, r_j, x_{l_j}^j\}| \geq 1$. This implies that

$$|B \cap \{r_1, x_{l_1}^1, r_2, x_{l_2}^2, \ldots, r_n, x_{l_n}^n\}| \geq n - 1.$$

If $|B \cap \{w_1, w_2, \ldots, w_n\}| \geq 1$, then we obtain

$$|B| \geq \sum_{i=1}^{n}(l_i - 1) + (n-1) + 1 = n + \sum_{i=1}^{n}(l_i - 1),$$

which leads to the identity $\mathrm{adim}(G) = n + \sum_{i=1}^{n}(l_i - 1)$ and completes the proof. Similarly, if $|B \cap \{r^i, x_{l_i}^i\}| \geq 1$ for each $i \in \{1, 2, \ldots, n\}$, then we see that $|B| \geq \sum_{i=1}^{n}(l_i - 1) + n$. Also, if there exists $i \in \{1, 2, \ldots, n\}$ such that $\{r^i, x_{l_i}^i\} \subseteq B$, then we obtain $|B| \geq \sum_{i=1}^{n}(l_i - 1) + n$ which completes the proof.
Hence, hereafter we must assume that $B \cap \{w_1, w_2, \ldots, w_n\} = \emptyset$, $|B \cap \{r^i, x_{l_i}^i\}| \neq 2$ for each $i \in \{1, 2, \ldots, n\}$, and there exists $j \in \{1, 2, \ldots, n\}$ such that $B \cap \{r_j, x_{l_j}^j\} = \emptyset$. In the following we show that this is impossible. Consider the following two cases:

Case 1. $j = 1$.
If $r_2 \notin B$, then $N[w_1] \cap B = \emptyset = N[x_{l_1}^1] \cap B$, a contradiction. Thus, $r_2 \in B$ and hence $x_{l_2}^2 \notin B$. This implies that $N[w_1] \cap B) = \{r_2\} = N[x_{l_2}^2] \cap B$, a contradiction.

Case 2. $j \neq 1$.
If $r_{j-1} \notin B$, then $N[w_{j-1}] \cap B = \emptyset = N[x_{l_j}^j] \cap B$, a contradiction. Thus, $r_{j-1} \in B$ and hence $x_{l_{j-1}}^{j-1} \notin B$. This implies that

$$N[w_{j-1}] \cap B = \{r_{j-1}\} = N[x_{l_{j-1}}^{j-1}] \cap B,$$

a contradiction. □

For the adjacency dimension of graphs defined in Definition 4 we have the following theorem.

Theorem 2.10. $\mathrm{adim}(T(K_{1,l_1} \circ K_{1,l_2} \circ \cdots \circ K_{1,l_n})) = (\Sigma_{i=1}^{n} l_i) - n.$

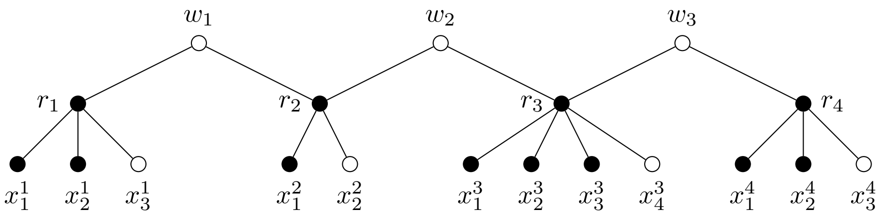

Figure 3: $T(K_{1,3} : K_{1,2} : K_{1,4} : K_{1,3})$

Proof. Let $G = T(K_{1,l_1} \circ K_{1,l_2} \circ \cdots \circ K_{1,l_n})$ and assume that (see Figure 4)

$$V(G) = \{w_1, w_2, \ldots, w_{n-1}\} \cup \{x_0^1, x_0^2, \ldots, x_0^n\} \cup \left(\bigcup_{i=1}^{n} \{x_1^i, x_2^i, \ldots, x_{l_i}^i\}\right),$$

and

$$E(G) = \left(\bigcup_{i=1}^{n} \{x_0^i x_1^i, x_0^i x_2^i, \ldots, x_0^i x_{l_i}^i\}\right) \cup \left(\bigcup_{i=1}^{n-1} \{w_i x_{l_i}^i, w_i x_1^{i+1}\}\right).$$

At first, let (see Figure 4)

$$\begin{aligned} W_0 \quad = \quad & \{w_1, w_2, \ldots, w_{n-1}\} \cup \{x_0^1, x_0^2, \ldots, x_0^{n-1}\} \cup \{x_1^1, x_2^1, \ldots, x_{l_1-2}^1\} \\ & \cup \left(\bigcup_{i=2}^{n-1} \{x_2^i, x_3^i, \ldots, x_{l_i-2}^i\}\right) \cup \{x_3^n, x_4^n, \ldots, x_{l_n}^n\}. \end{aligned}$$

Note that

$$|W_0| = 2(n-1) + (l_1 - 2) + \Sigma_{i=2}^{n-1}(l_i - 3) + (l_n - 2) = (\Sigma_{i=1}^{n} l_i) - n.$$

It is straightforward to see that W_0 is an adjacency resolving set for G and hence,

$$\operatorname{adim}(G) \leq |W_0| = (\Sigma_{i=1}^{n} l_i) - n.$$

Suppose that B is an adjacency basis for G. In the first star $l_1 - 1$ leaves are twins, in the i-th star $l_i - 2$ leaves are twins for each $i \in \{2, 3, \ldots, n-1\}$, and in the last star $l_n - 1$ leaves are twins. Thus

$$|B \cap \{x_1^1, x_2^1, \ldots, x_{l_1-1}^1\}| \geq l_1 - 2, \quad |B \cap \{x_2^n, x_3^n, \ldots, x_{l_n}^n\})| \geq l_n - 2,$$

and

$$|B \cap (\cup_{i=2}^{n-1} \{x_2^i, x_3^i, \ldots, x_{l_i-1}^i)\}| \geq \Sigma_{i=2}^{n-1}(l_i - 3).$$

Hence, we obtain

$$|B| \geq (l_1 - 2) + (l_n - 2) + (\Sigma_{i=2}^{n-1}(l_i - 3)) = (\Sigma_{i=1}^{n} l_i) - 3n + 2.$$

Moreover, without loss of generality we can assume that

$$\{x_1^1, x_2^1, \ldots, x_{l_1-2}^1\} \subseteq B, \quad \{x_3^n, x_4^n, \ldots, x_{l_n}^n\} \subseteq B,$$

and

$$\{x_2^i, x_3^i, \ldots, x_{l_i-2}^i\} \subseteq B$$

for each $2 \leq i \leq n-1$. Let

$$y_1 = x_{l_1-1}^1, \ y_2 = x_{l_2-1}^2, \ldots, \ y_{n-1} = x_{l_{n-1}-1}^{n-1}, \ y_n = x_2^n.$$

If $B \cap \{x_0^i, y_i, x_0^j, y_j\} = \emptyset$ for some $i \neq j$, then $N[y_i] \cap B = \emptyset = N[y_j] \cap B$, which is a contradiction. Thus, for each $i \neq j$ we must have $|B \cap \{x_0^i, y_i, x_0^j, y_j\}| \geq 1$ and hence,

$$\left| B \cap \left(\bigcup_{i=1}^{n} \{x_0^i, y_i, x_0^j, y_j\}\right) \right| \geq n - 1.$$

If $B \cap \{x^i_{l_i}, w_i, x^{i+1}_1\} = \emptyset$ for some i, then $N[w_i] \cap B = \emptyset$. Since B is an adjacency basis for G, for each $i \neq j$ we must have $|B \cap \{x^i_{l_i}, w_i, x^{i+1}_1\}| \geq 1$ or $|B \cap \{x^j_{l_j}, w_j, x^{j+1}_1\}| \geq 1$. Hence,

$$\left| B \cap \left(\bigcup_{i=1}^{n-1} \{x^i_{l_i}, w_i, x^{i+1}_1\} \right) \right| \geq n-2.$$

These facts imply that

$$|B| \geq \big((\Sigma_{i=1}^n l_i) - 3n + 2 \big) + (n-1) + (n-2) = (\Sigma_{i=1}^n l_i) - n - 1.$$

But, since $B \cap \{x^i_{l_i}, w_i, x^{i+1}_1\} = \emptyset$ and $B \cap \{x^j_0, y_j\} = \emptyset$, where $i \in \{1, 2, \ldots, n-1\}$ and $j \in \{1, 2, \ldots, n\}$, lead to the contradction $N[w_i] \cap B = \emptyset = N[y_j] \cap B$, actually one more vertex must be in B and hence,

$$|B| \geq (\Sigma_{i=1}^n l_i) - n,$$

which completes the proof. □

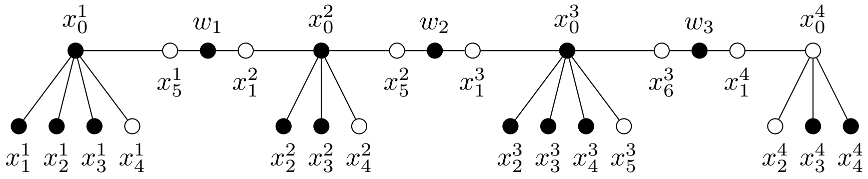

Figure 4: $T(K_{1,5} \circ K_{1,5} \circ K_{1,6} \circ K_{1,4})$

Finally, in what follows we determine $\operatorname{adim}(G)$ when G is a shrub graph.

Theorem 2.11. *The adjacency dimension of the shrub graph is given by*

$$\operatorname{adim}\big(St(l_1, l_2, \ldots, l_n)\big) = \begin{cases} (\sum_{i=1}^n l_i) - 1 & n \geq 3 \\ l_1 + l_2 & n = 2. \end{cases}$$

Proof. Let $G = St(l_1, l_2, \ldots, l_n)$. Assume that $V(G) = \{v_0\} \cup \big(\cup_{i=1}^n \{x^i_0, x^i_1, \ldots, x^i_{l_i}\} \big)$ and (see Figure 5)

$$E(G) = \big(\cup_{i=1}^n \{v_0 x^i_0\} \big) \cup \big(\cup_{i=1}^n \{x^i_0 x^i_1, x^i_0 x^i_2, \ldots, x^i_0 x^i_{l_i}\} \big).$$

If $n \geq 3$, then let

$$W_0 = \{x^1_0, x^2_0, \ldots, x^{n-1}_0\} \cup \big(\cup_{i=1}^n \{x^i_1, x^i_2, \ldots, x^i_{l_i - 1}\} \big).$$

In this case, it is easy to check that W_0 is an adjacency resolving set for $St(l_1, l_2, \ldots, l_n)$ and hence,

$$\operatorname{adim}(St(l_1, l_2, \ldots, l_r)) \leq |W_0| = (n-1) + \sum_{i=1}^n (l_i - 1) = \big(\sum_{i=1}^n l_i \big) - 1.$$

If $n = 2$, then let

$$W_0 = \{x^1_0, x^2_0\} \cup \{x^1_1, x^1_2, \ldots, x^1_{l_1 - 1}\} \cup \{x^2_1, x^2_2, \ldots, x^2_{l_2 - 1}\}.$$

Again, W_0 is an adjacency resolving set for $St(l_1, l_2)$ and hence,

$$\operatorname{adim}(St(l_1, l_2)) \leq |W_0| = l_1 + l_2.$$

Thus, in both cases the upper bounds for $\operatorname{adim}(G)$ are obtained.

Now let B be an adjacency basis for $St(l_1, l_2, \ldots, l_n)$. For each $1 \leq i \leq r$, the vertices $\{x^i_1, x^i_2, \ldots, x^i_{l_i}\}$ are twins. Thus, $|B \cap \{x^i_1, x^i_2, \ldots, x^i_{l_i}\}| \geq l_i - 1$ for each $1 \leq i \leq n$. Without loss of generality, we can assume that $\{x^i_1, \ldots, x^i_{l_i - 1}\} \subseteq B$. Therefore,

$$\bigcup_{i=1}^n \{x^i_1, x^i_3, \ldots, x^i_{l_i - 1}\} \subseteq B, \quad |B| \geq \sum_{i=1}^n (l_i - 1).$$

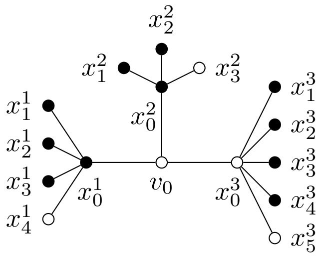

Figure 5: $st(4,3,5)$

If we have $B\cap\{x_0^i, x_{l_i}^i, x_0^j, x_{l_j}^j\}=\emptyset$ for some $i\neq j$, then we obtain $N[x_{l_i}^i]\cap B=\emptyset=N[x_{l_j}^j]\cap B$, which is a contradiction. Thus, for each $i\neq j$, $|B\cap\{x^i, x_{l_i}^i, x^j, x_{l_j}^j\}|\geq 1$. This implies that

$$\left|B\cap\{\bigcup_{i=1}^{n}\{x_0^i, x_{l_i}^i\}\}\right|\geq n-1.$$

Therefore,

$$|B|\geq\sum_{i=1}^{n}(l_i-1)+(n-1)=\big(\sum_{i=1}^{n}l_i\big)-1.$$

For the case $n\geq 3$ the lower bound is also obtained and the proof is complete.
Now assume that $n=2$. If $|B|=(l_1+l_2)-1$, then we have

$$|B\cap\{x_0^1, x_{l_1}^1, x_0^2, x_{l_2}^2\}|=1.$$

If $B\cap\{x_0^1, x_{l_1}^1, x_0^2, x_{l_2}^2\}=\{x_0^1\}$, then $N[v_0]\cap B=\{x_0^1\}=N[x_{l_1}^1]\cap B$, a contradiction.
If $B\cap\{x_0^1, x_{l_1}^1, x_0^2, x_{l_2}^2\}=\{x_{l_1}^1\}$, then $N[v_0]\cap B=\emptyset=N[x_{l_2}^2]\cap B$, a contradiction.
If $B\cap\{x_0^1, x_{l_1}^1, x_0^2, x_{l_2}^2\}=\{x_0^2\}$, then $N[v_0]\cap B=\{x_0^2\}=N[x_{l_2}^2]\cap B$, a contradiction.
If $B\cap\{x_0^1, x_{l_1}^1, x_0^2, x_{l_2}^2\}=\{x_{l_2}^2\}$, then $N[v_0]\cap B=\emptyset=N[x_{l_1}^1]\cap B$, a contradiction.
These contradictions show that $|B\cap\{x_0^1, x_{l_1}^1, x_0^2, x_{l_2}^2\}|\geq 2$ and hence, $|B|=(l_1+l_2)$ as desired. ☐

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