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Original Article

Pseudo-duals and closeness of continuous g-frames in Hilbert spaces

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ABSTRACT: The paper deals with pseudo-duals of continuous g-frames and their characterizations in Hilbert spaces. Mainly, the pseudo-duals constructed by bounded operators inserted between the synthesis and analysis operators of the Bessel mappings are considered. Duals and approximate duals, which are two important classes of pseudo-duals, are also studied here. Moreover, the concepts of closeness and nearness of continuous g-frames are focused and some of their properties are obtained. It is shown that there are close relationships between the closeness and nearness of g-frames and their approximate duals. Also, the above-mentioned concepts are related to the notions of partial equivalence, equivalent frames, and continuous Riesz-type g-frames.

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1. Introduction

Discrete and continuous frames arise in many applications in both pure and applied mathematics and, in particular, they play important roles in digital signal processing and scientific computations. The notion of discrete frames was first introduced by Duffin and Shaeffer in [12] to study some deep problems in nonharmonic Fourier series, see also [10, 24]. A generalization of frames to a family indexed by a locally compact space endowed with a Radon measure was proposed by Kaiser in [15] and independently by Ali, Antoine and Gazeau in [2]. These frames are known as continuous frames. Gabardo and Han called these frames "frames associated with measurable spaces", see [13]. If in the definition of a continuous frame the measurable space is assumed to be a countable indexed set equipped with the counting measure, the continuous frame will be a discrete frame. In 2006, discrete g-frame as a generalization of discrete frames was introduced and investigated by Sun in [22] and then, the notion of continuous g-frames was introduced in [1]. Throughout this paper, $\mathcal{H}$ is a separable complex Hilbert space, (Ω, μ) is a measure space with positive measure μ and $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ is a family of separable Hilbert spaces. We denote the space of all

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bounded linear operators from $\mathcal{H}$ into $\mathcal{H}_\omega$ by $B(\mathcal{H},\mathcal{H}_\omega)$ and we denote $B(\mathcal{H},\mathcal{H})$ by $B(\mathcal{H})$. Also, the kernel of a bounded operator T is denoted by $\ker T$ and we show the range of T by $Rng(T)$.

Definition 1.1. *We say that $\Lambda = \{\Lambda_\omega \in B(\mathcal{H},\mathcal{H}_\omega), \quad \omega \in \Omega\}$ is a continuous g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ if*

(1) *For each $f \in \mathcal{H}$, $\{\Lambda_\omega f\}_{\omega\in\Omega}$ is measurable, i.e., the mapping $\omega \mapsto \langle \Lambda_\omega f, g_\omega\rangle$ is measurable, for each f in $\mathcal{H}$ and $\{g_\omega\}_{\omega\in\Omega}$ with $g_\omega \in \mathcal{H}_\omega$.*
(2) *There are two numbers $0 < A_\Lambda \le B_\Lambda < \infty$ such that*

$$A_\Lambda\|f\|^2 \le \int_\Omega \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega) \le B_\Lambda\|f\|^2, \qquad f \in \mathcal{H}. \tag{1}$$

We call A_Λ and B_Λ the lower and upper continuous g-frame bounds, respectively. Λ is called an A_Λ-tight continuous g-frame if $A_\Lambda = B_\Lambda$ and a Parseval continuous g-frame if $A_\Lambda = B_\Lambda = 1$. If the right hand inequality in (1) holds for all $f \in \mathcal{H}$, we say that Λ is a continuous g-Bessel family for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ with the bound B_Λ.

Proposition 1.2. *Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H},\mathcal{H}_\omega), \quad \omega \in \Omega\}$ be a continuous g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$. Then, there exists a unique positive and invertible operator $S_\Lambda : \mathcal{H} \longrightarrow \mathcal{H}$ such that for each $f, g \in \mathcal{H}$,*

$$\langle S_\Lambda f, g\rangle = \int_\Omega \langle \Lambda_\omega^*\Lambda_\omega f, g\rangle \mathrm{d}\mu(\omega)$$

and $A_\Lambda \cdot Id_{\mathcal{H}} \le S_\Lambda \le B_\Lambda \cdot Id_{\mathcal{H}}$, where $Id_{\mathcal{H}}$ is the identity operator on $\mathcal{H}$. The operator S_Λ is called the continuous g-frame operator of Λ. Also, we have

$$\langle f, g\rangle = \int_\Omega \langle S_\Lambda^{-1} f, \Lambda_\omega^*\Lambda_\omega g\rangle \mathrm{d}\mu(\omega) = \int_\Omega \langle f, \Lambda_\omega^*\Lambda_\omega S_\Lambda^{-1} g\rangle \mathrm{d}\mu(\omega), \qquad f, g \in \mathcal{H}. \tag{2}$$

Let the space

$$\hat{K} = \Big\{F \in \prod_{\omega\in\Omega} \mathcal{H}_\omega \quad : F \text{ is measurable}, \int_\Omega \|F(\omega)\|^2 \mathrm{d}\mu(\omega) < \infty\Big\}.$$

Obviously, $\hat{K}$ is a Hilbert space with pointwise operations and the inner product given by

$$\langle F, G\rangle = \int_\Omega \langle F(\omega), G(\omega)\rangle \mathrm{d}\mu(\omega), \qquad F, G \in \hat{K}.$$

Proposition 1.3. *Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H},\mathcal{H}_\omega), \quad \omega \in \Omega\}$ be a continuous g-Bessel family for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$. Then, the mapping $T_\Lambda : \hat{K} \longrightarrow \mathcal{H}$ defined by*

$$\langle T_\Lambda F, g\rangle = \int_\Omega \langle \Lambda_\omega^* F(\omega), g\rangle \mathrm{d}\mu(\omega), \qquad F \in \hat{K}, \quad g \in \mathcal{H},$$

is a linear and bounded operator with $\|T_\Lambda\| \le \sqrt{B_\Lambda}$. Moreover, for any $g \in \mathcal{H}$ and $\omega \in \Omega$, we get

$$T_\Lambda^*(g)(\omega) = \Lambda_\omega g.$$

The operators T_Λ and T_Λ^* in Proposition 1.3 are called the synthesis and analysis operators of Λ, respectively.

Definition 1.4. *If $\Lambda = \{\Lambda_\omega \in B(\mathcal{H},\mathcal{H}_\omega), \quad \omega \in \Omega\}$ and $\Theta = \{\Theta_\omega \in B(\mathcal{H},\mathcal{H}_\omega), \quad \omega \in \Omega\}$ are two continuous g-Bessel families for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$, such that*

$$\langle f, g\rangle = \int_\Omega \langle \Theta_\omega f, \Lambda_\omega g\rangle \mathrm{d}\mu(\omega), \qquad f, g \in \mathcal{H},$$

then Θ is called a dual of Λ. Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H},\mathcal{H}_\omega), \quad \omega \in \Omega\}$ be a continuous g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$. Then $\widetilde{\Lambda} = \{\Lambda_\omega S_\Lambda^{-1} \in B(\mathcal{H},\mathcal{H}_\omega), \quad \omega \in \Omega\}$ is a continuous g-frame for $\mathcal{H}$ with respect to $\{\mathcal{H}_\omega\}_{\omega\in\Omega}$ and by (2), $\widetilde{\Lambda}$ is a dual of Λ. We call $\widetilde{\Lambda}$ the canonical dual of Λ.

According to Definition 1.4, one can expect a dual of a g-frame to provide a representation for every element in the underlying Hilbert space in terms of an integral (for continuous g-frames) or a series (for discrete g-frames). This role can also be played by pseudo-duals and approximate duals. So far, different kinds of duals, pseudo-duals and approximate duals for frames and their generalizations have been introduced. For more information, we refer to [4, 8, 9, 14, 16, 18, 19, 20, 21, 23, 25].

In this paper, we focus on Q-duals, Q-approximate duals and Q-pseudo-duals of continuous g-frames, where Q is a bounded operator inserted between the synthesis and analysis operators.

2. Pseudo-duals, duals and approximate duals of continuous g-frames

In the present section, we get some properties of Q-duals, Q-approximate duals and Q-pseudo-duals of continuous g-frames. Mainly, their stability and their characterizations are considered. Some of the obtained results are analogous to the ones stated in [4, 19].

Definition 2.1. *Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{H}_\omega), \quad \omega \in \Omega\}$ and $\Gamma = \{\Gamma_\omega \in B(\mathcal{H}, \mathcal{H}_\omega), \quad \omega \in \Omega\}$ be two continuous g-Bessel families and $Q \in B(\hat{K})$. Then*

(i) *Λ is said to be a Q-pseudo-dual for Γ if the operator $S_{\Lambda Q \Gamma} := T_\Lambda Q T_\Gamma^*$ is invertible.*
(ii) *Λ is said to be a Q-approximate dual for Γ if $\|T_\Lambda Q T_\Gamma^* - Id_{\mathcal{H}}\| < 1$.*
(iii) *Λ is said to be a Q-dual for Γ if $T_\Lambda Q T_\Gamma^* = Id_{\mathcal{H}}$.*

Theorem 2.2. *Let Λ and Γ be two continuous g-Bessel families and let $T \in B(\mathcal{H})$. Then $\Lambda T := \{\Lambda_\omega T \in B(\mathcal{H}, \mathcal{H}_\omega), \quad \omega \in \Omega\}$ and $\Gamma T := \{\Gamma_\omega T \in B(\mathcal{H}, \mathcal{H}_\omega), \quad \omega \in \Omega\}$ are two continuous g-Bessel families. Moreover, if ΛT is a Q-pseudo-dual for ΓT, then T is left-invertible.*

Proof. Since the mapping $\omega \mapsto \|\Lambda_\omega f\|$ is measurable for each $f \in \mathcal{H}$, the mapping $\omega \longmapsto \|\Lambda_\omega T f\|$ is also measurable. Then, we have

$$\int_\Omega \|\Lambda_\omega(Tf)\|^2 \mathrm{d}\mu(\omega) \leq B_\Lambda \|Tf\|^2 \leq B_\Lambda \|T\|^2 \|f\|^2.$$

Thus ΛT is a continuous g-Bessel family, so $T_{\Lambda T}$ is bounded and it is easy to see that $T_{\Lambda T} = T^* T_\Lambda$. Similarly, ΓT is a continuous g-Bessel family with $T_{\Gamma T} = T^* T_\Gamma$. Therefore, for each $f \in \mathcal{H}$, we have

$$T_{\Lambda T} Q T_{\Gamma T}^*(f) = (T^* T_\Lambda) Q (T_\Gamma^* T) f = T^*(T_\Lambda Q T_\Gamma^*) T f.$$

Since ΛT is a Q-pseudo-dual of ΓT, $T_{\Lambda T} Q T_{\Gamma T}^*$ is invertible which implies that T is left-invertible. □

Proposition 2.3. *Let the continuous g-Bessel family Λ be a Q-pseudo-dual for the continuous g-Bessel family Γ and let $T \in B(\mathcal{H})$. If T is invertible, then ΛT is a Q-pseudo-dual for ΓT.*

Proof. As we see in Theorem 2.2, ΛT and ΓT are two continuous g-Bessel families, so $T_{\Gamma T}$ and $T_{\Lambda T}$ are bounded, $T_{\Lambda T} = T^* T_\Lambda, T_{\Gamma T} = T^* T_\Gamma$ and $T_{\Lambda T} Q T_{\Gamma T}^* = T^* T_\Lambda Q T_\Gamma^* T$. Now, since T and $T_\Lambda Q T_\Gamma^*$ are invertible, we conclude that $T_{\Lambda T} Q T_{\Gamma T}^*$ is invertible which is equivalent to say that ΛT is a Q-pseudo-dual of ΓT. □

Theorem 2.4. *Let $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{H}_\omega), \quad \omega \in \Omega\}$, $\Gamma = \{\Gamma_\omega \in B(\mathcal{H}, \mathcal{H}_\omega), \quad \omega \in \Omega\}$ and $\Theta = \{\Theta_\omega \in B(\mathcal{H}, \mathcal{H}_\omega), \quad \omega \in \Omega\}$ be three continuous g-Bessel families. Then*

(i) *$\Lambda - \Gamma := \{\Lambda_\omega - \Gamma_\omega \in B(\mathcal{H}, \mathcal{H}_\omega), \quad \omega \in \Omega\}$ and $\Theta - \Gamma := \{\Theta_\omega - \Gamma_\omega \in B(\mathcal{H}, \mathcal{H}_\omega), \quad \omega \in \Omega\}$, that for every $f \in \mathcal{H}$*

$$(\Theta_\omega - \Gamma_\omega) f = \Theta_\omega f - \Gamma_\omega f,$$

and

$$(\Lambda_\omega - \Gamma_\omega) f = \Lambda_\omega f - \Gamma_\omega f,$$

are two continuous g-Bessel families.
(ii) *If $\|T_\Theta Q T_\Gamma^*\| < 1$ and Θ is a Q-dual for Λ, then Θ is a Q-approximate dual for $\Lambda - \Gamma$.*
(iii) *If $\|T_\Gamma Q T_\Lambda^*\| < 1$ and Θ is a Q-dual for Λ, then $\Theta - \Gamma$ is a Q-approximate dual for Λ.*

Proof. (i) It is easy to verify that for each $f \in \mathcal{H}$, the mappings $\begin{matrix}\Omega \to \mathbb{R}^+ \\ \omega \mapsto \|(\Lambda_\omega - \Gamma_\omega) f\|\end{matrix}$ and $\begin{matrix}\Omega \to \mathbb{R}^+ \\ \omega \mapsto \|(\Theta_\omega - \Gamma_\omega) f\|\end{matrix}$ are measurable, also we have

$$\begin{aligned}\int_\Omega \|(\Lambda_\omega - \Gamma_\omega) f\|^2 \mathrm{d}\mu(\omega) &\leq \int_\Omega (\|(\Lambda_\omega f\| + \|\Gamma_\omega f\|)^2 \mathrm{d}\mu(\omega) \\ &\leq B_\Lambda \|f\|^2 + B_\Gamma \|f\|^2 + 2\Big(\int_\Omega \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega)\Big)^{\frac{1}{2}} \Big(\int_\Omega \|\Gamma_\omega f\|^2 \mathrm{d}\mu(\omega)\Big)^{\frac{1}{2}} \\ &\leq B_\Lambda \|f\|^2 + B_\Gamma \|f\|^2 + 2\sqrt{B_\Lambda B_\Gamma} \|f\|^2,\end{aligned}$$

so $\Lambda - \Gamma$ is a continuous g-Bessel family. Similarly $\Theta - \Gamma$ is a continuous g-Bessel family.

(ii) It is clear that $T^*_{\Lambda-\Gamma} = T^*_\Lambda - T^*_\Gamma$, so

$$T_\Theta Q T^*_{\Lambda-\Gamma} = T_\Theta Q(T^*_\Lambda - T^*_\Gamma) = T_\Theta Q T^*_\Lambda - T_\Theta Q T^*_\Gamma.$$

Since Θ is a Q-dual for Λ, we have $T_\Theta Q T^*_{\Lambda-\Gamma} = Id_{\mathcal{H}} - T_\Theta Q T^*_\Gamma$. Consequently

$$\|T_\Theta Q T^*_{\Lambda-\Gamma} - Id_{\mathcal{H}}\| = \|T_\Theta Q T^*_\Gamma\| < 1,$$

which means that Θ is a Q-approximate dual of $\Lambda - \Gamma$.

(iii) Assume that $\|T_\Gamma Q T^*_\Lambda\| < 1$ and Θ is a Q-dual for Λ which is $T_\Theta Q T^*_\Lambda = Id_{\mathcal{H}}$. Hence

$$T_{\Theta-\Gamma} Q T^*_\Lambda = T_\Theta Q T^*_\Lambda - T_\Gamma Q T^*_\Lambda = Id_{\mathcal{H}} - T_\Gamma Q T^*_\Lambda,$$

so

$$\|T_{\Theta-\Gamma} Q T^*_\Lambda - Id_{\mathcal{H}}\| = \|T_\Gamma Q T^*_\Lambda\| < 1,$$

and the result follows.

□

Theorem 2.5. *Let $\Gamma = \{\Gamma_\omega \in B(\mathcal{H}, \mathcal{H}_\omega) : \omega \in \Omega\}$ be a continuous g-Bessel family. Then, the following statements are equivalent:*

(i) Γ *possesses a Q-dual, for some $Q \in B(\hat{K})$.*
(ii) Γ *possesses a Q-approximate dual, for some $Q \in B(\hat{K})$.*
(iii) Γ *possesses a Q-pseudo-dual, for some $Q \in B(\hat{K})$.*
(iv) Γ *is a continuous g-frame.*
(v) Γ *is a Q-dual of itself, for some positive operator $Q \in B(\hat{K})$.*
(vi) Γ *is a Q-dual of itself, for some self-adjoint operator $Q \in B(\hat{K})$.*
(vii) Γ *is a Q-approximate dual of itself, for some self-adjoint operator $Q \in B(\hat{K})$.*
(viii) *There exist some $0 < \varepsilon < 1$ and some self-adjoint operator $Q \in B(\hat{K})$ such that $(1-\varepsilon)\cdot Id_{\mathcal{H}} \le S_{\Gamma Q \Gamma} \le (1+\varepsilon)\cdot Id_{\mathcal{H}}$.*
(ix) Γ *is a Q-approximate dual of itself, for some positive operator $Q \in B(\hat{K})$.*
(x) *There exist some $0 < \varepsilon < 1$ and some positive operator $Q \in B(\hat{K})$ such that $(1-\varepsilon)\cdot Id_{\mathcal{H}} \le S_{\Gamma Q \Gamma} \le (1+\varepsilon)\cdot Id_{\mathcal{H}}$.*
(xi) Γ *is a Q-pseudo-dual of itself, for some positive operator $Q \in B(\hat{K})$.*
(xii) Γ *is a Q-pseudo-dual of itself, for some self-adjoint operator $Q \in B(\hat{K})$.*

Proof. The implications (i) ⇒ (ii) and (ii) ⇒ (iii) are obvious.
(iii) ⇒ (iv). Suppose that there exists some $Q \in B(\hat{K})$ such that Λ is a Q-pseudo-dual of Γ, which is, $S_{\Lambda Q \Gamma} := T_\Lambda Q T^*_\Gamma$ is invertible. Then, for each $f \in \mathcal{H}$, we have

$$\|f\| = \|S^{-1}_{\Lambda Q \Gamma} S_{\Lambda Q \Gamma} f\| \le \|S^{-1}_{\Lambda Q \Gamma}\| \|T_\Lambda\| \|Q\| \|T^*_\Gamma f\|,$$

so

$$\frac{\|f\|^2}{\|S^{-1}_{\Lambda Q \Gamma}\|^2 \|T_\Lambda\|^2 \|Q\|^2} \le \int_\Omega \|\Gamma_\omega f\|^2 \mathrm{d}\mu(\omega) \le B_\Gamma \|f\|^2.$$

(iv) ⇒ (v). Assume that $\Gamma = \{\Gamma_\omega \in B(\mathcal{H}, \mathcal{H}_\omega) : \omega \in \Omega\}$ is a continuous g-frame for $\mathcal{H}$, so $\widetilde{\Gamma} := \{\Gamma_\omega S^{-1}_\Gamma \in B(\mathcal{H}, \mathcal{H}_\omega) : \omega \in \Omega\}$ is a continuous g-frame for $\mathcal{H}$, and $\widetilde{\Gamma}$ is a dual for Γ, so $T_{\widetilde{\Gamma}}$ and $T^*_{\widetilde{\Gamma}}$ are bounded. Then $\begin{array}{c} Q : \hat{K} \to \hat{K} \\ Q := T^*_{\widetilde{\Gamma}} T_{\widetilde{\Gamma}} \end{array}$ is a positive operator and we have

$$T_\Gamma Q T^*_\Gamma = T_\Gamma (T^*_{\widetilde{\Gamma}} T_{\widetilde{\Gamma}}) T^*_\Gamma = Id_{\mathcal{H}}.$$

The implications (v) ⇒ (vi), (vi) ⇒ (vii), (v) ⇒ (ix), (ix) ⇒ (xi), (xi) ⇒ (xii) and (vii) ⇒ (xii) are trivial.
(xii) ⇒ (i). Assume that $Q \in B(\hat{K})$ is a self-adjoint operator and Γ is a Q-pseudo-dual of itself and $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{H}_\omega) : \omega \in \Omega\}$ where $\Lambda_\omega := \Gamma_\omega (T_\Gamma Q T^*_\Gamma)^{-1}$. For every $f \in \mathcal{H}$, we have

$$\int_\Omega \|\Lambda_\omega f\|^2 \mathrm{d}\mu(\omega) = \int_\Omega \|\Gamma_\omega (T_\Gamma Q T^*_\Gamma)^{-1} f\|^2 \mathrm{d}\mu(\omega) \le B_\Gamma \|(T_\Gamma Q T^*_\Gamma)^{-1} f\|^2 \le B_\Gamma \|(T_\Gamma Q T^*_\Gamma)^{-1}\|^2 \|f\|^2.$$

Thus Λ is a continuous g-Bessel family. On the other hand, for each $F \in \hat{K}$ and $g \in \mathcal{H}$, we get

$$\begin{aligned}\langle T_\Lambda F, g\rangle &= \int_\Omega \langle \Lambda_\omega^* F(\omega), g\rangle \mathrm{d}\mu(\omega)\\ &= \int_\Omega \langle F(\omega), \Lambda_\omega g\rangle \mathrm{d}\mu(\omega)\\ &= \int_\Omega \langle F(\omega), \Gamma_\omega (T_\Gamma Q T_\Gamma^*)^{-1} g\rangle \mathrm{d}\mu(\omega)\\ &= \int_\Omega \langle \Gamma_\omega^* F(\omega), (T_\Gamma Q T_\Gamma^*)^{-1} g\rangle \mathrm{d}\mu(\omega)\\ &= \langle T_\Gamma F, (T_\Gamma Q T_\Gamma^*)^{-1} g\rangle\\ &= \langle ((T_\Gamma Q T_\Gamma^*)^{-1})^* T_\Gamma F, g\rangle\\ &= \langle (T_\Gamma Q^* T_\Gamma^*)^{-1} T_\Gamma F, g\rangle\\ &= \langle (T_\Gamma Q T_\Gamma^*)^{-1} T_\Gamma F, g\rangle.\end{aligned}$$

Hence $(T_\Gamma Q T_\Gamma^*)^{-1} T_\Gamma = T_\Lambda$, so we have

$$T_\Lambda Q T_\Gamma^* = ((T_\Gamma Q T_\Gamma^*)^{-1} T_\Gamma) Q T_\Gamma^* = (T_\Gamma Q T_\Gamma^*)^{-1}(T_\Gamma Q T_\Gamma^*) = Id_{\mathcal{H}},$$

which means that Λ is a Q-dual of Γ.
(vii) $\Rightarrow$ (viii). Let $Q \in B(\hat{K})$ be a self-adjoint operator and let Γ be a Q-approximate dual of itself, so $\|T_\Gamma Q T_\Gamma^* - Id_{\mathcal{H}}\| < 1$. If $T_\Gamma Q T_\Gamma^* = Id_{\mathcal{H}}$, then for each $0 < \varepsilon < 1$ we have $(1-\varepsilon)\cdot Id_{\mathcal{H}} \le T_\Gamma Q T_\Gamma^* \le (1+\varepsilon)\cdot Id_{\mathcal{H}}$. Let $0 < \varepsilon := \|T_\Gamma Q T_\Gamma^* - Id_{\mathcal{H}}\| < 1$. Since $T_\Gamma Q T_\Gamma^* - Id_{\mathcal{H}}$ is self-adjoint, we get $(T_\Gamma Q T_\Gamma^* - Id_{\mathcal{H}}) \le \| T_\Gamma Q T_\Gamma^* - Id_{\mathcal{H}} \| \cdot Id_{\mathcal{H}}$, also we have $(Id_{\mathcal{H}} - T_\Gamma Q T_\Gamma^*) \le \| Id_{\mathcal{H}} - T_\Gamma Q T_\Gamma^* \| \cdot Id_{\mathcal{H}}$, so we conclude that $-\varepsilon \cdot Id_{\mathcal{H}} \le T_\Gamma Q T_\Gamma^* - Id_{\mathcal{H}} \le \varepsilon \cdot Id_{\mathcal{H}}$.
(viii) $\Rightarrow$ (vii). If there exist $0 < \varepsilon < 1$ and some self-adjoint operator $Q \in B(\hat{K})$ such that

$$(1-\varepsilon)\cdot Id_{\mathcal{H}} \le T_\Gamma Q T_\Gamma^* \le (1+\varepsilon)\cdot Id_{\mathcal{H}},$$

then

$$-\varepsilon \cdot Id_{\mathcal{H}} \le T_\Gamma Q T_\Gamma^* - Id_{\mathcal{H}} \le \varepsilon \cdot Id_{\mathcal{H}}.$$

Let $T := T_\Gamma Q T_\Gamma^* - Id_{\mathcal{H}}$, then for each $f \in \mathcal{H}$, we have

$$-\varepsilon\langle f, f\rangle \le \langle Tf, f\rangle \le \varepsilon\langle f, f\rangle,$$

so $\|T\| \le \varepsilon$ and we conclude that $\|T_\Gamma Q T_\Gamma^* - Id_{\mathcal{H}}\| \le \varepsilon < 1$, therefore Γ is a Q-approximate dual of itself. The equivalence of (ix) and (x) can be obtained similar to the equivalence of (vii) and (viii). ☐

Proposition 2.6. *Let Λ and Γ be two continuous g-Bessel families and $Q \in B(\hat{K})$. Then, the following statements are equivalent:*

(i) *Λ is a Q-pseudo-dual (Q-approximate dual) of Γ.*
(ii) *There exists some invertible operator $T \in B(\mathcal{H})$ (there exists $T \in B(\mathcal{H})$ with $\|T^{-1} - Id_{\mathcal{H}}\| < 1$) such that $\Lambda \circ T := \{\Lambda_\omega \circ T\}_{\omega\in\Omega}$ is a Q-dual of Γ.*
(iii) *Γ is a continuous g-frame and there exist an invertible operator $S \in B(\mathcal{H})$ ($S \in B(\mathcal{H})$ with $\|S - Id_{\mathcal{H}}\| < 1$) and some $R \in B(\hat{K}, \mathcal{H})$ such that*

$$T_\Lambda Q = S(S_\Gamma^{-1} T_\Gamma + R(Id_{\hat{K}} - T_\Gamma^* S_\Gamma^{-1} T_\Gamma)).$$

Proof. (i) $\Rightarrow$ (ii). Since Λ is a Q-pseudo-dual (Q-approximate dual) of Γ, the operator $T_\Gamma Q^* T_\Lambda^*$ is invertible ($\|T_\Gamma Q^* T_\Lambda^* - Id_{\mathcal{H}}\| < 1$). Let $T := (T_\Gamma Q^* T_\Lambda^*)^{-1}$. Then, it is easy to see that $\Lambda \circ T$ is a continuous g-Bessel family and $T_{\Lambda\circ T} = T^* T_\Lambda$. Thus

$$T_{\Lambda\circ T} Q T_\Gamma^* = T^* T_\Lambda Q T_\Gamma^* = (T_\Lambda Q T_\Gamma^*)^{-1}(T_\Lambda Q T_\Gamma^*) = Id_{\mathcal{H}}.$$

This means that $\Lambda \circ T$ is a Q-dual of Γ.
(ii) $\Rightarrow$ (iii). Since $\Lambda \circ T$ is a Q-dual of Γ, by Theorem 2.5, Γ is a continuous g-frame and $T^* T_\Lambda Q T_\Gamma^* = Id_{\mathcal{H}}$, so $T_\Lambda Q T_\Gamma^* = (T^*)^{-1}$. Let $S := {T^*}^{-1}$, $R := T^* T_\Lambda Q \in B(\hat{K}, \mathcal{H})$, then

$$\begin{aligned}S(S_\Gamma^{-1} T_\Gamma + R(Id_{\hat{K}} - T_\Gamma^* S_\Gamma^{-1} T_\Gamma)) &= {T^*}^{-1}\Big((T_\Gamma T_\Gamma^*)^{-1} T_\Gamma + T^* T_\Lambda Q(Id_{\hat{K}} - T_\Gamma^*(T_\Gamma T_\Gamma^*)^{-1} T_\Gamma)\Big)\\ &= {T^*}^{-1}(T_\Gamma T_\Gamma^*)^{-1} T_\Gamma + T_\Lambda Q - T_\Lambda Q T_\Gamma^*(T_\Gamma T_\Gamma^*)^{-1} T_\Gamma\\ &= {T^*}^{-1}(T_\Gamma T_\Gamma^*)^{-1} T_\Gamma + T_\Lambda Q - {T^*}^{-1}(T_\Gamma T_\Gamma^*)^{-1} T_\Gamma\\ &= T_\Lambda Q.\end{aligned}$$

(iii) $\Rightarrow$ (i). If Γ is a continuous g-frame and there are operators $S \in B(\mathcal{H})$ and $R \in B(\hat{K}, \mathcal{H})$ such that S is invertible ($\|S - Id_{\mathcal{H}}\| < 1$) and

$$T_\Lambda Q = S(S_\Gamma^{-1} T_\Gamma + R(Id_{\hat{K}} - T_\Gamma^* S_\Gamma^{-1} T_\Gamma)),$$

then we get

$$\begin{aligned} T_\Lambda Q T_\Gamma^* &= S(T_\Gamma T_\Gamma^*)^{-1} T_\Gamma T_\Gamma^* + SRT_\Gamma^* - SRT_\Gamma^* (T_\Gamma T_\Gamma^*)^{-1} T_\Gamma T_\Gamma^* \\ &= S + SRT_\Gamma^* - SRT_\Gamma^* = S. \end{aligned}$$

Since S is invertible ($\|S - Id_{\mathcal{H}}\| < 1$), Λ is a Q-pseudo-dual (Q-approximate dual) of Γ. □

Proposition 2.7. *Let Γ and Λ be two continuous g-Bessel families, and let $Q \in B(\hat{K})$. Then Λ is a Q-dual of Γ if and only if Γ is a continuous g-frame and there exists some $R \in B(\hat{K}, \mathcal{H})$ such that*

$$T_\Lambda Q = (T_\Gamma T_\Gamma^*)^{-1} T_\Gamma + R(Id_{\hat{K}} - T_\Gamma^* (S_\Gamma^{-1}) T_\Gamma). \tag{3}$$

Proof. Suppose that Λ is a Q-dual of Γ, so $T_\Lambda Q T_\Gamma^* = Id_{\mathcal{H}}$. Then, by Theorem 2.5, Γ is a continuous g-frame and for $R := T_\Lambda Q$ we have

$$S_\Gamma^{-1} T_\Gamma + R - RT_\Gamma^* S_\Gamma^{-1} T_\Gamma = R = T_\Lambda Q.$$

For the converse, if there exists some $R \in B(\hat{K}, \mathcal{H})$ such that (3) holds, then

$$T_\Lambda Q T_\Gamma^* = (T_\Gamma T_\Gamma^*)^{-1} T_\Gamma T_\Gamma^* + RT_\Gamma^* - RT_\Gamma^* (T_\Gamma T_\Gamma^*)^{-1} T_\Gamma T_\Gamma^* = Id_{\mathcal{H}}.$$

Hence, we conclude that Λ is a Q-dual of Γ. □

3. closeness and nearness of continuous g-frames

As we know, the well-known Paley-Wiener theorem concerns the stability of Riesz bases which are sufficiently close to an orthonormal basis of the underlying Hilbert space. Some general versions of Paley-Wiener theorem for Hilbert space frames were presented in [6, 7]. Then, the closeness and the nearness of two discrete frames and two continuous frames were introduced and studied in [3, 5], see also [4, 11, 17, 19].

Here, the concepts of closeness, nearness and partial equivalence for continuous g-frames are considered and some of their properties are obtained. In particular, we extend the obtained results in [3] to continuous g-frames and we use them in the next section.

Definition 3.1. *Let $\Gamma = \{\Gamma_\omega \in B(\mathcal{H}, \mathcal{H}_\omega) : \omega \in \Omega\}$ and $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{H}_\omega) : \omega \in \Omega\}$ be two continuous g-Bessel families. Then, we say that Γ is close to Λ if there exists some $\lambda \geq 0$ with*

$$\|T_\Gamma \varphi - T_\Lambda \varphi\| \leq \lambda \|T_\Lambda \varphi\|,$$

for each $\varphi \in \hat{K}$. The infimum of such λ's is called the closeness bound and it is denoted by $C(\Gamma, \Lambda)$.

Definition 3.2. *Let Γ and Λ be two continuous g-Bessel families. Then, we say that Λ and Γ are near if Λ is close to Γ and Γ is close to Λ.*

Definition 3.3. *Let Γ and Λ be two continuous g-Bessel families. Then, Λ is called to be partial equivalent with Γ if there exists some bounded operator T on $\mathcal{H}$ with $\Gamma_\omega = \Lambda_\omega T$, for each $\omega \in \Omega$. Moreover, if T is invertible, then we say that Λ and Γ are equivalent via T.*

If $\Gamma_\omega = \Lambda_\omega T$, for each $\omega \in \Omega$, then

$$\langle T_\Gamma F, g \rangle = \int_\Omega \langle \Lambda_\omega^* F(\omega), Tg \rangle \mathrm{d}\mu(\omega) = \langle T^* T_\Lambda F, g \rangle, \qquad F \in \hat{K}, \quad g \in \mathcal{H}.$$

Hence Λ is partial equivalent with Γ via T if and only if $T_\Gamma = T^* T_\Lambda$. The same identity holds for the equivalent continuous g-frames via an invertible operator T.

Lemma 3.4. *Let Λ and Γ be two continuous g-frames. Then,*

(i) *The continuous g-frame Λ is partial equivalent with Γ if and only if $Rng(T_\Gamma^*) \subseteq Rng(T_\Lambda^*)$.*

(ii) *The continuous g-frames Λ and Γ are equivalent if and only if $Rng(T_\Gamma^*) = Rng(T_\Lambda^*)$.*

Proof. (i) Assume that Λ is partial equivalent with Γ via a bounded operator T. Hence, $T_\Gamma = T^*T_\Lambda$ and consequently $T_\Gamma^* = T_\Lambda^* T$ which implies the inclusion $Rng(T_\Gamma^*) \subseteq Rng(T_\Lambda^*)$.
Conversely, let $Rng(T_\Gamma^*) \subseteq Rng(T_\Lambda^*)$. For each f in $\mathcal{H}$, we get $T_\Lambda^* S_\Lambda^{-1} T_\Lambda(T_\Lambda^* f) = T_\Lambda^* f$. This means that for every element g in $Rng(T_\Lambda^*)$, the equality $T_\Lambda^* S_\Lambda^{-1} T_\Lambda(g) = g$ holds and since $Rng(T_\Gamma^*) \subseteq Rng(T_\Lambda^*)$, we have $T_\Lambda^* S_\Lambda^{-1} T_\Lambda T_\Gamma^* = T_\Gamma^*$. Therefore $(S_\Lambda^{-1} T_\Lambda T_\Gamma^*)^* T_\Lambda = T_\Gamma$, meaning that Λ is partial equivalent with Γ via $T := S_\Lambda^{-1} T_\Lambda T_\Gamma^*$.
Part (ii) is an immediate consequence of (i). □

Theorem 3.5. *Let Λ and Γ be two continuous g-frames. Then, Γ is close to Λ if and only if Λ is partial equivalent with Γ via a bounded operator T. In this case, we have $C(\Gamma, \Lambda) = \|Id_{\mathcal{H}} - T\|$.*

Proof. Suppose that Γ is close to Λ. Thus, there is some $\lambda \geq 0$ such that

$$\|T_\Gamma \varphi - T_\Lambda \varphi\| \leq \lambda \|T_\Lambda \varphi\|,$$

for each $\varphi \in \hat{K}$. Therefore, $\ker T_\Lambda \subseteq \ker T_\Gamma$, so $Rng(T_\Gamma^*) \subseteq Rng(T_\Lambda^*)$. Now, Lemma 3.4 yields that Λ is partial equivalent with Γ. Conversely, if Λ is partial equivalent with Γ via a bounded operator T, then for each $\varphi \in \hat{K}$, we obtain that

$$\|T_\Gamma \varphi - T_\Lambda \varphi\| = \|(T^*T_\Lambda - T_\Lambda)\varphi\| \leq \|T^* - Id_{\mathcal{H}}\| \|T_\Lambda \varphi\|.$$

Consequently, Γ is close to Λ with

$$C(\Gamma, \Lambda) \leq \|T^* - Id_{\mathcal{H}}\| = \|T - Id_{\mathcal{H}}\|.$$

Also, since T_Λ is surjective, for each f in $\mathcal{H}$, we have

$$\|(T^* - Id_{\mathcal{H}})f\| \leq C(\Gamma, \Lambda)\|f\|.$$

Thus

$$\|T - Id_{\mathcal{H}}\| = \|T^* - Id_{\mathcal{H}}\| \leq C(\Gamma, \Lambda).$$

□

Corollary 3.6. *Let Λ and Γ be two continuous g-frames. Then, Γ and Λ are near if and only if they are equivalent via an invertible operator T. In this case,*

$$\max\{C(\Gamma, \Lambda), C(\Lambda, \Gamma)\} \leq \max\{\|T - Id_{\mathcal{H}}\|, \|T^{-1} - Id_{\mathcal{H}}\|\}.$$

Proof. It follows from Theorem 3.5 that the nearness of Λ and Γ is equivalent to say that Λ is partial equivalent with Γ via a bounded operator T and Γ is partial equivalent with Λ via a bounded operator S. Thus, $T_\Gamma = T^*T_\Lambda$ and $T_\Lambda = S^*T_\Gamma$, so $T_\Gamma = T^*S^*T_\Gamma$ and $T_\Lambda = S^*T^*T_\Lambda$. Because T_Γ and T_Λ are surjective, we conclude that T is invertible with $T^{-1} = S$. Also, the inequality

$$\max\{C(\Gamma, \Lambda), C(\Lambda, \Gamma)\} \leq \max\{\|T - Id_{\mathcal{H}}\|, \|T^{-1} - Id_{\mathcal{H}}\|\},$$

follows from Theorem 3.5. □

Corollary 3.7. *Let Λ be a continuous g-frame. If $\Gamma = \{\Gamma_\omega \in B(\mathcal{H}, \mathcal{H}_\omega) : \omega \in \Omega\}$ is a continuous g-Bessel family with*

$$\|(T_\Lambda - T_\Gamma)\varphi\| \leq \lambda_1 \|T_\Lambda \varphi\| + \lambda_2 \|T_\Gamma \varphi\|, \qquad \forall \varphi \in \hat{K}, \tag{4}$$

where $0 \leq \lambda_1, \lambda_2 < 1$, then Γ is a g-frame. Moreover, $Rng(T_\Lambda^*) = Rng(T_\Gamma^*)$ and Λ and Γ are near.

Proof. The fact that Γ is a continuous g-frame follows from the inequalities

$$\|T_\Gamma \varphi\| \leq \frac{1 + \lambda_1}{1 - \lambda_2} \|T_\Lambda \varphi\|,$$

$$\|T_\Lambda \varphi\| \leq \frac{1 + \lambda_2}{1 - \lambda_1} \|T_\Gamma \varphi\|$$

and the properties of the synthesis operators. Also, the inequality (4) implies that $\ker T_\Lambda = \ker T_\Gamma$, consequently, $Rng(T_\Lambda^*) = Rng(T_\Gamma^*)$ and the result follows from Lemma 3.4 and Corollary 3.6. □

4. Pseudo-duals and closeness of continuous g-frames

In this section, it is shown that there are close relationships between the pseudo-duals (mainly, duals and approximate duals) of continuous g-frames and the concepts of closeness and nearness. Indeed, the stated results in [4, 19] are generalized to continuous g-frames.

Theorem 4.1. *Let* $\Lambda = \{\Lambda_\omega \in B(\mathcal{H}, \mathcal{H}_\omega),\ \ \omega \in \Omega\}$ *and* $\Gamma = \{\Gamma_\omega \in B(\mathcal{H}, \mathcal{H}_\omega),\ \ \omega \in \Omega\}$ *be two continuous g-frames. Then, the following statements are equivalent:*

(i) Λ *is close to* Γ *with* $C(\Lambda, \Gamma) < \dfrac{1}{2}$.

(ii) Λ *and* Γ *are equivalent via an operator* T *with* $\|T - Id_{\mathcal{H}}\| < \dfrac{1}{2}$.

(iii) Λ *and* Γ *are near with* $C(\Lambda, \Gamma) < \dfrac{1}{2}$ *and* $C(\Gamma, \Lambda) < 1$.

In case the equivalent conditions are satisfied, then the following equivalent conditions hold:

(iv) Λ *(resp.* Γ*) is close to* Γ *(resp.* Λ*) with* $C(\Lambda, \Gamma) < 1$ *(resp.* $C(\Gamma, \Lambda) < 1$*).*

(v) $\ker T_\Gamma = \ker T_\Lambda$ *and if* $Q \in B(\hat{K})$*, then every* Q*-dual of* Γ *(resp.* Λ*) is a* Q*-approximate dual of* Λ *(resp.* Γ*).*

(vi) $\ker T_\Gamma = \ker T_\Lambda$ *and there exist some* $Q \in B(\hat{K})$ *and a* Q*-dual of* Γ *(resp.* Λ*) which is a* Q*-approximate dual of* Λ *(resp.* Γ*).*

(vii) $\ker T_\Gamma = \ker T_\Lambda$ *and every dual of* Γ *(resp.* Λ*) is an approximate dual of* Λ *(resp.* Γ*).*

(viii) $\ker T_\Gamma = \ker T_\Lambda$ *and there exists a dual of* Γ *(resp.* Λ*) which is an approximate dual of* Λ *(resp.* Γ*).*

(ix) Γ *(resp.* Λ*) is partial equivalent with* Λ *(resp.* Γ*) and every dual of* Γ *(resp.* Λ*) is an approximate dual of* Λ *(resp.* Γ*) .*

(x) Λ, Γ *are equivalent and every dual of* Γ *(resp.* Λ*) is an approximate dual of* Λ *(resp.* Γ*) .*

(xi) Λ, Γ *are equivalent via* T *with* $\|T - Id_{\mathcal{H}}\| < 1$ *(resp.* $\|T^{-1} - Id_{\mathcal{H}}\| < 1$*).*

Proof. (i) $\Rightarrow$ (ii). Since Λ is close to Γ with $C(\Lambda, \Gamma) < \dfrac{1}{2}$, for every $\varphi \in \hat{K}$, we get

$$\|T_\Lambda \varphi - T_\Gamma \varphi\| \leq C(\Lambda, \Gamma)\|T_\Gamma \varphi\|.$$

If $\varphi \in \ker T_\Gamma$, then $T_\Gamma \varphi = 0$ and $\|T_\Lambda \varphi\| \leq 0$, therefore $\varphi \in \ker T_\Lambda$, so $\ker T_\Gamma \subseteq \ker T_\Lambda$. If $\varphi \in \ker T_\Lambda$, then $\|T_\Gamma \varphi\|(1 - C(\Lambda, \Gamma)) \leq 0$ and since $C(\Lambda, \Gamma) < \dfrac{1}{2}$, we have $\varphi \in \ker T_\Gamma$, consequently $\ker T_\Lambda \subseteq \ker T_\Gamma$, so $\ker T_\Lambda = \ker T_\Gamma$, and so $Rng T_\Gamma^* = Rng T_\Lambda^*$. By Lemma 3.4, Γ and Λ are equivalent via $T \in B(\mathcal{H})$ with $T_\Lambda = T^* T_\Gamma$. On the other hand we know, for every $\varphi \in \hat{K}$, that

$$\|(T^* - Id_{\mathcal{H}})(T_\Gamma \varphi)\| = \|T^* T_\Gamma \varphi - T_\Gamma \varphi\| = \|T_\Lambda \varphi - T_\Gamma \varphi\| \leq C(\Lambda, \Gamma)\|T_\Gamma \varphi\|,$$

so

$$\|T - Id_{\mathcal{H}}\| = \|T^* - Id_{\mathcal{H}}\| \leq C(\Lambda, \Gamma) < \frac{1}{2}.$$

(ii) $\Rightarrow$ (i). Since Γ and Λ are equivalent via T with $\|T - Id_{\mathcal{H}}\| < \dfrac{1}{2}$ (T is invertible), we have $T^* T_\Gamma = T_\Lambda$ and for each $\varphi \in \hat{K}$ with $T_\Gamma \varphi \neq 0$ we obtain that

$$\|T_\Lambda \varphi - T_\Gamma \varphi\| = \|T^* T_\Gamma \varphi - T_\Gamma \varphi\| \leq \|T^* - Id_{\mathcal{H}}\|\|T_\Gamma \varphi\| < \frac{1}{2}\|T_\Gamma \varphi\|,$$

so Λ is close to Γ with $C(\Lambda, \Gamma) < \dfrac{1}{2}$.

(i) $\Rightarrow$ (iii). Λ is close to Γ with $C(\Lambda, \Gamma) < \dfrac{1}{2}$, so for each $\varphi \in \hat{K}$, we get

$$\begin{aligned}\|T_\Lambda \varphi - T_\Gamma \varphi\| \leq C(\Lambda, \Gamma)\|T_\Gamma \varphi\| &= C(\Lambda, \Gamma)\|T_\Gamma \varphi - T_\Lambda \varphi + T_\Lambda \varphi\| \\ &\leq C(\Lambda, \Gamma)(\|T_\Gamma \varphi - T_\Lambda \varphi\| + \|T_\Lambda \varphi\|),\end{aligned}$$

so

$$\|T_\Lambda \varphi - T_\Gamma \varphi\|(1 - C(\Lambda, \Gamma)) \leq C(\Lambda, \Gamma)\|T_\Lambda \varphi\|.$$

Since $C(\Lambda,\Gamma) < \frac{1}{2}$ and $\|T_\Lambda\varphi - T_\Gamma\varphi\| \leq \frac{C(\Lambda,\Gamma)}{1-C(\Lambda,\Gamma)}\|T_\Lambda\varphi\|$, it is obtained that $\frac{C(\Lambda,\Gamma)}{1-C(\Lambda,\Gamma)} < 1$ and as a result $C(\Gamma,\Lambda) < 1$. The implication (iii) $\Rightarrow$ (i) is obvious.
(iv) $\Rightarrow$ (v). Since Λ is close to Γ with $C(\Lambda,\Gamma) < 1$, for each $\varphi \in \hat{K}$, we have

$$\|(T_\Lambda - T_\Gamma)\varphi\| \leq C(\Lambda,\Gamma)\|T_\Gamma\varphi\|, \tag{5}$$

which yields that $\ker T_\Gamma = \ker T_\Lambda$ (see the proof of (i) $\Rightarrow$ (ii)). Now, let $Q \in B(\hat{K})$, and let Θ be a Q-dual of Γ, so $T_\Theta Q T_\Gamma^* = Id_{\mathcal{H}}$. For $\varphi := Q^* T_\Theta^* f$ we have $\varphi \in \hat{K}$ and, by (5), we get

$$\|T_\Gamma Q^* T_\Theta^* f - T_\Lambda Q^* T_\Theta^* f\| \leq C(\Lambda,\Gamma)\|T_\Gamma Q^* T_\Theta^* f\|,$$

so

$$\|T_\Lambda Q^* T_\Theta^* - Id_{\mathcal{H}}\| \leq C(\Lambda,\Gamma) < 1.$$

Therefore Λ is a Q^*-approximate dual of Θ, which is equivalent to say that Θ is a Q-approximate dual Λ.
(v) $\Rightarrow$ (vi). Since Γ is a continuous g-frame, it has a dual which is $\tilde{\Gamma} = \{\Gamma_\omega S_\Gamma^{-1} \in B(\mathcal{H},\mathcal{H}_\omega) \ : \ \omega \in \Omega\}$, so for $Q := Id_{\hat{K}}$, $\tilde{\Gamma}$ is a Q-dual of Γ, so according to the assumption $\tilde{\Gamma}$ is a Q-approximate dual of Λ.
(vi) $\Rightarrow$ (iv). According to the assumption there exist some $Q \in B(\hat{K})$ and a Q-dual of Γ like Θ which is a Q-approximate dual of Λ, so, $T_\Theta Q T_\Gamma^* = Id_{\mathcal{H}}$, $\|T_\Theta Q T_\Lambda^* - Id_{\mathcal{H}}\| < 1$. On the other hand, we have $\ker T_\Lambda = \ker T_\Gamma$ and as a result we have $RngT_\Lambda^* = RngT_\Gamma^*$. Thus, by Lemma 3.4, Γ and Λ are equivalent, so there is an invertible operator $T \in B(\mathcal{H})$ such that $T_\Lambda = T^* T_\Gamma$. Now, for $\varphi \in \hat{K}$, we get

$$\|T_\Lambda\varphi - T_\Gamma\varphi\| = \|T^*T_\Gamma\varphi - T_\Gamma\varphi\| = \|(T^* - Id_{\mathcal{H}})(T_\Gamma\varphi)\| \leq \|T^* - Id_{\mathcal{H}}\|\|T_\Gamma\varphi\|,$$

also

$$\|T^* - Id_{\mathcal{H}}\| = \|T^*T_\Gamma Q^* T_\Theta^* - Id_{\mathcal{H}}\| = \|T_\Lambda Q^* T_\Theta^* - Id_{\mathcal{H}}\| = \|T_\Theta Q T_\Lambda^* - Id_{\mathcal{H}}\| < 1.$$

Since $\|T - Id_{\mathcal{H}}\| = \|T^* - Id_{\mathcal{H}}\| < 1$, $C(\Lambda,\Gamma) < 1$ and so Λ is close to Γ with $C(\Lambda,\Gamma) < 1$.
(v) $\Rightarrow$ (vii). Since the statement is valid for every $Q \in B(\hat{K})$, it also holds for $Q = Id_{\hat{K}}$, so (vii) is obtained.
(vii) $\Rightarrow$ (viii). Since Γ has at least one dual (the canonical dual), (viii) is verified.
The implication (viii) $\Rightarrow$ (vi) is obtained using $Q := Id_{\hat{K}}$. The implication (vii) $\Rightarrow$ (ix) is an immediate consequence of Lemma 3.4
(ix) $\Rightarrow$ (x). Since Γ is partial equivalent with Λ, there exists some $T \in B(\mathcal{H})$ such that $T_\Lambda = T^*T_\Gamma$. Now, let Θ be a dual of Γ which is an approximate dual of Λ, so $T_\Theta T_\Gamma^* = Id_{\mathcal{H}}$ and $\|T_\Theta T_\Lambda^* - Id_{\mathcal{H}}\| < 1$. Thus

$$\begin{aligned}\|T - Id_{\mathcal{H}}\| &= \|(T - Id_{\mathcal{H}})^*\| = \|T^* - Id_{\mathcal{H}}\| \\ &= \|T^*T_\Gamma T_\Theta^* - Id_{\mathcal{H}}\| = \|T_\Lambda T_\Theta^* - Id_{\mathcal{H}}\| \\ &= \|(T_\Theta T_\Lambda^* - Id_{\mathcal{H}})^*\| = \|T_\Theta T_\Lambda^* - Id_{\mathcal{H}}\| < 1.\end{aligned}$$

Hence T is invertible and consequently Γ, Λ are equivalent.
(x) $\Rightarrow$ (vii). Since Γ, Λ are equivalent, by Lemma 3.4, we have $RngT_\Gamma^* = RngT_\Lambda^*$ which yields the equality $\ker T_\Gamma = \ker T_\Lambda$.
(iv) $\Rightarrow$ (xi). According to the assumption, Λ is close to Γ with $C(\Lambda,\Gamma) < 1$, so for each $\varphi \in \hat{K}$, we have

$$\|T_\Lambda\varphi - T_\Gamma\varphi\| \leq C(\Lambda,\Gamma)\|T_\Gamma\varphi\|.$$

Since $C(\Lambda,\Gamma) < 1$, by Lemma 3.4, Λ, Γ are equivalent which implies the existence of some invertible operator T on $\mathcal{H}$ with $T_\Lambda = T^*T_\Gamma$. Hence, for each $\varphi \in \hat{K}$, we get

$$\|T^*T_\Gamma\varphi - T_\Gamma\varphi\| \leq C(\Lambda,\Gamma)\|T_\Gamma\varphi\|,$$

consequently

$$\|(T^* - Id_{\mathcal{H}})T_\Gamma\varphi\| = \|T^*T_\Gamma\varphi - T_\Gamma\varphi\| \leq C(\Lambda,\Gamma)\|T_\Gamma\varphi\|,$$

meaning that $\|T - Id_{\mathcal{H}}\| < 1$.
(xi) $\Rightarrow$ (iv). Since Λ, Γ are equivalent via T with $\|T - Id_{\mathcal{H}}\| < 1$, we get $T_\Lambda = T^*T_\Gamma$, so for each $\varphi \in \hat{K}$, it is obtained that

$$\|T_\Lambda\varphi - T_\Gamma\varphi\| = \|T^*T_\Gamma\varphi - T_\Gamma\varphi\| = \|(T^* - Id_{\mathcal{H}})T_\Gamma\varphi\| \leq \|T^* - Id_{\mathcal{H}}\|\|T_\Gamma\varphi\|.$$

Since $\|T^* - Id_{\mathcal{H}}\| = \|T - Id_{\mathcal{H}}\| < 1$, Λ is close to Γ with $C(\Lambda,\Gamma) < 1$. □

Definition 4.2. *Let Γ be a continuous g-frame. Then Γ is called a Riesz-type g-frame if it has only one dual, i.e., $\tilde{\Gamma} := \{\Gamma_\omega S_\Gamma^{-1} \in B(\mathcal{H}, \mathcal{H}_\omega) \ : \ \omega \in \Omega\}$, (the canonical dual) is the only dual of Γ.*

Proposition 4.3. *Let $\Gamma = \{\Gamma_\omega \in B(\mathcal{H}, \mathcal{H}_\omega) \ : \ \omega \in \Omega\}$ be a continuous g-frame. Then, the following statements are equivalent:*

(i) *Γ is a Riesz-type g-frame.*
(ii) *For every invertible operator T on $\mathcal{H}$, $\Gamma \circ T$ is a Riesz-type g-frame.*
(iii) *Every pseudo-dual of Γ is a Riesz-type g-frame.*
(iv) *Every approximate dual of Γ is a Riesz-type g-frame.*
(v) *Every dual of Γ is a Riesz-type g-frame.*
In case the equivalent conditions are satisfied, then the following equivalent conditions for a continuous g-frame Λ hold:
(vi) *Λ is close to Γ with $C(\Lambda, \Gamma) < 1$.*
(vii) *Λ is an approximate dual for $\tilde{\Gamma}$.*
(viii) *If $Q \in B(\hat{K})$, then every Q-dual of Γ is a Q-approximate dual for Λ.*
(ix) *Γ is partial equivalent with Λ and Λ is an approximate dual for $\tilde{\Gamma}$.*
(x) *Γ, Λ are equivalent and Λ is an approximate dual for $\tilde{\Gamma}$.*
(xi) *Γ, Λ are equivalent via T with $\|T - Id_{\mathcal{H}}\| < 1$.*

Proof. (i) $\Rightarrow$ (ii). Let Γ be a Riesz-type g-frame and let T be an invertible operator on $\mathcal{H}$. Since Γ is a continuous g-frame, the mapping $\omega \mapsto \|\Gamma_\omega f\|$ is measurable, for each $f \in \mathcal{H}$. Therefore, the mapping $\omega \mapsto \|\Gamma_\omega T f\|$ is also measurable. On the other hand, for each $f \in \mathcal{H}$, we get

$$\int_\Omega \|\Gamma_\omega T f\|^2 \mathrm{d}\mu(\omega) \leq B_\Gamma \|Tf\|^2 \leq B_\Gamma \|T\|^2 \|f\|^2. \tag{6}$$

Also, the relation

$$\|Tf\| = \|S_\Gamma^{-1} S_\Gamma T f\| \leq \|S_\Gamma^{-1}\| \|T_\Gamma\| \|T_\Gamma^* T f\|,$$

yields that

$$\|Tf\|^2 \leq \|S_\Gamma^{-1}\|^2 \|T_\Gamma\|^2 \Big(\int_\Omega \|\Gamma_\omega T f\|^2 \mathrm{d}\mu(\omega) \Big).$$

Since T is invertible, we get

$$\frac{\|f\|^2}{\|T^{-1}\|^2 \|S_\Gamma^{-1}\|^2 \|T_\Gamma\|^2} \leq \int_\Omega \|\Gamma_\omega T f\|^2 \mathrm{d}\mu(\omega) \leq B_\Gamma \|T\|^2 \|f\|^2,$$

so $\Gamma \circ T$ is a continuous g-frame. Now, let Λ be a dual of $\Gamma \circ T$. Then

$$Id_{\mathcal{H}} = T_\Lambda T_{\Gamma \circ T}^* = T_\Lambda (T^* T_\Gamma)^* = T_\Lambda T_\Gamma^* T.$$

Therefore $T^{-1} = T_\Lambda T_\Gamma^*$, so $T_{\Lambda \circ T^*} T_\Gamma^* = T T_\Lambda T_\Gamma^* = T T^{-1} = Id_{\mathcal{H}}$. Hence, $\Lambda \circ T^*$ is a dual of Γ. Since Γ is a Riesz-type g-frame, it has a dual which is $\tilde{\Gamma}$, so $\tilde{\Gamma} = \Lambda \circ T^*$, and since T is invertible, we have $\tilde{\Gamma} T^{*-1} = \Lambda$ meaning that $\Gamma \circ T$ has only one dual, consequently $\Gamma \circ T$ is a Riesz-type g-frame.
(ii) $\Rightarrow$ (i). Since for each invertible operator T on $\mathcal{H}$, $\Gamma \circ T$ is a Riesz-type g-frame, the statement is also valid for $T := Id_{\mathcal{H}}$.
(i) $\Rightarrow$ (iii). Let Λ be a pseudo-dual for Γ. Then, by Proposition 2.6, there exists an invertible operator T on $\mathcal{H}$ such that $\Lambda \circ T$ is a dual for Γ, so $\Lambda \circ T = \tilde{\Gamma}$, consequently $\Lambda = \tilde{\Gamma} \circ T^{-1}$. Now, the same argument stated for the proof of the implication (i) $\Rightarrow$ (ii) yields that Λ is Riesz-type. The implications (iii) $\Rightarrow$ (iv) $\Rightarrow$ (v) are obvious.
(v) $\Rightarrow$ (i) Since every dual of Γ is a Riesz-type g-frame, $\tilde{\Gamma} := \Gamma S_\Gamma^{-1}$ is Riesz-type, so it has only one dual. Now, it is easy to see that Γ has also only one dual. The other equivalences can be obtained similar to the proof of Theorem 4.1 and using the fact that Γ is a Riesz-type g-frame and $\tilde{\Gamma}$ is its only dual. □

Proposition 4.4. *Let Γ be a continuous g-frame. Then the following statements are equivalent:*

(i) *Γ is a continuous Riesz-type g-frame.*
(ii) *There exists some invertible operator T on $\mathcal{H}$ such that $\Gamma \circ T$ is a continuous Riesz-type g-frame.*
(iii) *There exists some pseudo-dual for Γ which is a Riesz-type g-frame.*

(iv) *There exists some approximate dual for* Γ *which is a Riesz-type g-frame.*
(v) *There exists a dual for* Γ *which is a Riesz-type g-frame.*

Proof. We can get the implication (i) $\Rightarrow$ (v) using $\tilde{\Gamma}$ (the canonical dual of Γ) as a dual of Γ which is a Riesz-type g-frame. The implications (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) are obvious.
(iii) $\Rightarrow$ (ii). Assume that there exists a pseudo-dual for Γ like Θ which is a Riesz-type g-frame. By Proposition 2.6, there exists an invertible operator $R \in B(\mathcal{H})$ such that $\Theta \circ R$ is a dual of Γ, so

$$Id_{\mathcal{H}} = T_\Gamma T^*_{\Theta \circ R} = T_\Gamma (R^* T_\Theta)^* = T_\Gamma T^*_\Theta R.$$

Since R is invertible, $T_\Gamma T^*_\Theta = R^{-1}$, so $\Gamma \circ R^*$ is a dual of Θ because

$$T_{\Gamma \circ R^*} T^*_\Theta = R T_\Gamma T^*_\Theta = R R^{-1} = Id_{\mathcal{H}}.$$

On the other hand, since Θ is a Riesz-type g-frame, we get $\Gamma \circ R^* = \tilde{\Theta} = \Theta S^{-1}_\Theta$. Now by considering $T := R^* S_\Theta$, we have $\Gamma \circ T = \Gamma \circ R^* S_\Theta = \Theta \circ S^{-1}_\Theta S_\Theta = \Theta$ and we immediately conclude that $\Gamma \circ T$ is a Riesz-type g-frame.
(ii) $\Rightarrow$ (i). Suppose that there exists an invertible operator T on $\mathcal{H}$ such that $\Lambda := \Gamma \circ T$ is a continuous Riesz-type g-frame. If Θ_1, Θ_2 are two duals for Γ, then we have

$$T_{\Theta_1} T^*_\Gamma = Id_{\mathcal{H}} = T_\Gamma T^*_{\Theta_1}.$$

Thus

$$T_{\Theta_1 \circ T^{-1*}} T^*_{\Gamma \circ T} = T^{-1} T_{\Theta_1} (T^* T_\Gamma)^* = T^{-1} T_{\Theta_1} T^*_\Gamma T = Id_{\mathcal{H}}.$$

Hence $\Theta_1 \circ T^{-1^*}$ is a dual for Λ. Similarly, we can obtain that $\Theta_2 \circ T^{-1^*}$ is also a dual for Λ. Since Λ is a continuous Riesz-type g-frame, $\Theta_1 \circ T^{-1^*} = \Theta_2 \circ T^{-1^*}$, so $\Theta_1 = \Theta_2$ which implies that Γ is a continuous Riesz-type g-frame.
□

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