AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 5(4) (2024) 337-359
https://doi.org/10.22060/AJMC.2024.23042.1222

Original Article

A matrix approach to multi-term fractional differential equations using two new diffusive representations for the Caputo fractional derivative

Hassan Khosravian-Arab*[a], Mehdi Dehghan[b]

[a] *Department of Applied Mathematics and Computer Science, Faculty of Mathematics and Statistics, University of Isfahan, Isfahan, Iran*
[b] *Department of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), Iran*

ABSTRACT: In the last decade, there has been a surge of interest in application of fractional calculus in various areas such as, mathematics, physics, engineering, mechanics and etc. So, numerical methods have rapidly been developed to handle problems containing fractional derivatives (or integrals). Due to the fact that all the operators which appear in fractional calculus are non-local, so, the classical linear multi-step methods have some difficulties from the (time/space) computational complexity point of view. Recently, two new non-classical methods or diffusive based methods have been proposed by the authors to approximate the Caputo fractional derivatives. Here, the main aim of this paper is to use these methods to solve linear multi-term fractional differential equations numerically. To reach our aim, an efficient matrix approach has been provided to solve some well-known multi-term fractional differential equations.

Review History:

Received:07 March 2024
Revised:31 August 2024
Accepted:12 September 2024
Available Online:01 October 2024

Keywords:

Non-locality property
Caputo fractional derivative
Yuan-Agrawal method
Diffusive representation
Time-consuming methods

MSC (2020):

26A33; 65D30; 65D25; 65D32.

1. Introduction

In the last decades, there has seen a rapid surge of literature on the importance of fractional calculus and their broad applications (See for instance, [20, 21]). In fact, many real world problems can been modeled successfully by the following multi-term fractional differential equation (MTFDE) with constant coefficients and homogeneous initial conditions:

$$\sum_{r=1}^{M} c_r \, {}^{C}D_{a^+}^{\alpha_r} y(t) = f(t), \;\; 0 < \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_M, \;\; y(a) = y^{(1)}(a) = \cdots = y^{(L-1)}(a) = 0, \;\; L = \lceil \alpha_M \rceil, \tag{1}$$

*Corresponding author.
E-mail addresses: h.khosravian@sci.ui.ac.ir, mdehghan@aut.ac.ir, mdehghan.aut@gmail.com

where $c_r \in \mathbb{R}$ and $f(t)$ is given continuous function and moreover, ${}^{C}D_{a+}^{\alpha_r}$ denotes Caputo fractional derivative of order α_r with starting point a [10, 20]:

$$ {}^{C}D_{a+}^{\alpha_r}y(t) = \frac{1}{\Gamma(\lceil \alpha_r \rceil - \alpha_r)} \int_a^t (t-\tau)^{\lceil \alpha_r \rceil - \alpha_r - 1} y^{(\lceil \alpha_r \rceil)}(\tau)\, d\tau, $$

where $\Gamma(.)$ is the Euler's Gamma function.

As we are aware, the first step to solve MTFDE (1) numerically, is to present a numerical method that approximates the Caputo fractional derivative(s) ${}^{C}D_{a+}^{\alpha}y(t)$. Thanks to the well-known property of the Caputo fractional derivative (which is called non-locality), the numerical methods to approximate the Caputo operator have faced with some difficulties from the (space/time) computational complexities viewpoint [11, 13].

Several numerical methods based on the local and global methods have been introduced to solve MTFDE (1) (See [22] for various numerical methods to solve fractional differential equations). The proposed methods can be categorized in two classes [22]:

- **Direct Methods**: In these methods we first approximate the Caputo fractional derivatives directly and thus the numerical schemes can be obtained easily.
- **Indirect Methods**: In these methods, first, the obtained problem (1) is transformed into the fractional integral equation and then by using a suitable method to discretize the obtained fractional integral, we can get the numerical schemes.

The same feature of these methods is that to handle the non-locality of the fractional differential (or integral) operators, they need a relatively large amount of time and/or computer memory [13, 11].

To overcome this drawback, a new representation of the Caputo fractional derivative (which is so-called as diffusive representation (DR), infinite state representation (ISR) or memory free formulation (MFF)) was introduced by Yuan and Agrawal in [1, 29]. They have shown that the Caputo fractional derivative can be represented as:

$$ {}^{C}D_{0+}^{\alpha}y(t) = \int_0^{+\infty} \phi(\omega,t)\, d\omega, \tag{2} $$

where $\phi(\omega,t)$ for $\omega \in (0,+\infty)$ called the observed system's infinite states at time t. They also showed that the function $\omega \in (0,+\infty)$ satisfies the inhomogeneous first order differential equation in the following form:

$$ \frac{\partial}{\partial t}\phi(\omega,t) = h_1(\omega)\phi(\omega,t) + h_2(\omega)y'(t),\ \ \phi(\omega,0) = 0, \tag{3} $$

with certain functions $h_1; h_2 : (0,+\infty) \to \mathbb{R}$. K. Diethelm in his papers [9, 11, 13] has shown that the new representation, in fact, has some good features such that:

- The non-locality property of the Caputo fractional derivative is removed.
- Due to the local property of (2), the (time/space) computational complexity of the new representation to approximate the Caputo fractional derivative can be decreased substantially (See [7, 11, 14, 15]).

Some improvements and modifications of (2) have been proposed in [1, 3, 4, 5, 6, 13, 16, 23, 24, 26, 27, 28, 29].

The outline of this paper is as follows. In Section 2 we briefly review some important properties of the sine and cosine diffusive representations (SDR and CDR) to approximate the Caputo fractional derivative. In Section 3, an efficient matrix approach for SDR and CDR together with some extensions of the matrix approach in order to use for MTFDEs (1) have been made successfully. Numerical results are given in Section 4. Some concluding remarks and future works are provided at the end.

2. The sine and cosine diffusive representations

In this section, we briefly review two newly introduced diffusive representations (DRs) for Caputo fractional derivative which we will name the sine and cosine diffusive representations (SDR and CDR). So we will provide here the definitions and some related theorems. For more details see [17]. First, it is worthy to address the improved version of Yuan and Agrawal (YA) diffusive representation [29] for approximating the Caputo fractional derivatives which proposed by K. Diethelm [8].

Theorem 2.1 (Improved Yuan and Agrawal approach (YA)). *Let* $\alpha > 0$. *Then we have:*

$$ {}^{C}D_{0+}^{\alpha} y(t) = \int_0^\infty z^{2\alpha-2\lceil\alpha\rceil+1}\omega^{YA}(z,t)\,dz, $$

where

$$ \omega^{YA}(z,t) = (-1)^{\lfloor\alpha\rfloor}\frac{2\sin(\pi\alpha)}{\pi}\left(\int_0^t e^{-(t-\tau)z^2} y^{(\lceil\alpha\rceil)}(\tau)\,d\tau\right). $$

It is easy to verify that $\omega^{YA}(z,t)$ *satisfies the following differential equation:*

$$ \frac{\partial\omega^{YA}}{\partial t} + z^2\omega^{YA} = (-1)^{\lfloor\alpha\rfloor}\frac{2\sin(\pi\alpha)}{\pi} y^{(\lceil\alpha\rceil)}(t),\ \omega^{YA}(z,0) = 0. \tag{4} $$

Proof. See [8, 29] for the proof of this theorem. □

Now, we are going to define two new DRs to approximate the Caputo fractional derivative of order $0 < \alpha < 1$.

Theorem 2.2 (The cosine diffusive representation (CDR)). *For* $0 < \alpha < 1$, *one can see*

$$ {}^{C}D_{0+}^{\alpha} y(t) = \frac{2\sin(\frac{\pi\alpha}{2})}{\pi}\int_0^\infty z^{\alpha-1}\left(\int_0^t \cos\left((t-\tau)z\right) y'(\tau)\,d\tau\right) dz = \int_0^\infty z^{\alpha-1}\omega^{C}(z,t)\,dz, \tag{5} $$

where

$$ \omega^{C}(z,t) = \frac{2\sin(\frac{\pi\alpha}{2})}{\pi}\left(\int_0^t \cos\left((t-\tau)z\right) y'(\tau)\,d\tau\right). $$

We also point out that for a given function y *for which its second derivative exists on* $[0,T]$, $\omega^C(z,t)$ *(for fixed* $z > 0$*) satisfies the following second-order differential equation:*

$$ \begin{cases} \dfrac{\partial^2\omega^C}{\partial t^2} + z^2\omega^C = \dfrac{2\sin(\frac{\pi\alpha}{2})}{\pi} y''(t), \\ \omega^C(z,0) = 0,\ \dfrac{\partial}{\partial t}\omega^C(z,0) = \dfrac{2\sin(\frac{\pi\alpha}{2})}{\pi} y'(0). \end{cases} \tag{6} $$

Proof. See [17] for the proof of theorem. □

Theorem 2.3 (The sine diffusive representation (SDR)). *For* $0 < \alpha < 1$, *we have*

$$ {}^{C}D_{0+}^{\alpha} y(t) = \frac{2\cos(\frac{\pi\alpha}{2})}{\pi}\int_0^\infty z^{\alpha-1}\left(\int_0^t \sin\left((t-\tau)z\right) y'(\tau)\,d\tau\right) dz = \int_0^\infty z^{\alpha}\omega^{S}(z,t)\,dz, \tag{7} $$

where

$$ \omega^{S}(z,t) = \frac{2\cos(\frac{\pi\alpha}{2})}{z\pi}\left(\int_0^t \sin\left((t-\tau)z\right) y'(\tau)\,d\tau\right). $$

It should be noted that for a given function y *for which its first derivative exists on* $[0,T]$, $\omega^S(z,t)$ *(for fixed* $z > 0$*) satisfies the following second-order differential equation:*

$$ \begin{cases} \dfrac{\partial^2\omega^S}{\partial t^2} + z^2\omega^S = \dfrac{2\cos(\frac{\pi\alpha}{2})}{\pi}\ y'(t), \\ \omega^S(z,0) = \dfrac{\partial}{\partial t}\omega^S(z,0) = 0. \end{cases} \tag{8} $$

Proof. For the proof of this theorem see [17]. □

We see that the classical DRs of Caputo fractional derivative are usually coupled with a first-order differential equation (See the first order differential equation (3)), so from the numerical point of view, it is worthwhile to convert the second-order differential equations (6) and (8) to the system of first-order differential equations as follows.

Theorem 2.4. *Let* $0 < \alpha < 1$.

- *For a given function y for which its second derivative exists on $[0,T]$, and $\omega^C(z,t) = x_1(z,t)$ (for fixed $z > 0$), then $x_1(z,t)$ satisfies in the following system of first-order differential equations:*

$$\begin{cases} \dfrac{\partial x_1}{\partial t} = x_2(z,t), \\ \dfrac{\partial x_2}{\partial t} = -z^2 x_1(z,t) + \dfrac{2\sin(\frac{\pi\alpha}{2})}{\pi} y''(t), \\ x_1(z,0) = 0,\ x_2(z,0) = \dfrac{2\sin(\frac{\pi\alpha}{2})}{\pi} y'(0). \end{cases} \tag{9}$$

- *For a given function y for which its first derivative exists on $[0,T]$, and assume $\omega^S(z,t) = x_1(z,t)$ (for fixed $z > 0$), where $x_1(z,t)$ satisfies in the following system of first-order differential equations:*

$$\begin{cases} \dfrac{\partial x_1}{\partial t} = x_2(z,t), \\ \dfrac{\partial x_2}{\partial t} = -z^2 x_1(z,t) + \dfrac{2\cos(\frac{\pi\alpha}{2})}{\pi}\ y'(t), \\ x_1(z,0) = x_2(z,0) = 0. \end{cases} \tag{10}$$

Proof. The proofs are straightforward. □

In the following remark, we summarize some issues related to the mentioned theorems.

Remark 2.5. *Some important remarks should be addressed here:*

- *As it can be seen from Theorems 2.2 and 2.3, the SDR and CDR are both coupled with a second-order differential equation while the other representations are usually coupled with a first-order differential equation. But as we stated in Theorem 2.4, the second-order differential equations (6) and (8) can be easily converted to a system of first-order differential equations (See Theorem 2.4). Thus, the SDR and CDR methods can be also considered as the classical DRs.*
- *The second derivative of the given function $y(t)$, which appears in Eq. (6) can be considered as a drawback of CDR, but, in fact, in application we don't need to evaluate $y''(t)$.*

The error bound of the YA, CDR and SDR methods is given in the following theorem.

Theorem 2.6. *[17] Let $0 < \alpha < 1$. If a A-stable one-step implicit method of order p with the step size $h < N^{-2}$, (where N is the number of integration points in the generalized Gauss-Laguerre formula) is used for Eqs. (6) and (8), then the overall error analysis of CDR and SDR approximation formulae satisfies:*

- *If $y(t) \in C^2[0,T]$ and $y(0) = y'(0) = 0$, then for $t \in [0,T]$, we have (for the CDR method):*

$$\left| R^{\alpha}_{C,N,h} y(t) \right| = \mathcal{O}(N^{\alpha-2}) + \mathcal{O}(h^p) \int_0^{4N} e^{3Tz^2}\, dz.$$

- *If $y(t) \in C^1[0,T]$ and $y(0) = 0$ then for $t \in [0,T]$, we have (for the SDR method):*

$$\left| R^{\alpha}_{S,N,h} y(t) \right| = \mathcal{O}(N^{\alpha-1}) + \mathcal{O}(h^p) \int_0^{4N} e^{3Tz^2}\, dz.$$

Proof. See [17] for the details. □

Theorem 2.7. *[8] Let $0 < \alpha < 1$ and $y(t) \in C^1[0,T]$. If a A-stable one-step implicit method of order p with the step size $h < N^{-2}$ is used for Eq. (4), then the overall error analysis of improved YA approximation formula satisfies:*

$$\left| R^{\alpha}_{N,h} y(t) \right| = \mathcal{O}(N^{2\alpha-2}) + \mathcal{O}(h^p) \int_0^{4N} e^{3Tz^2}\, dz,$$

for $t \in [0,T]$.

Proof. See Theorem 6 of [8]. □

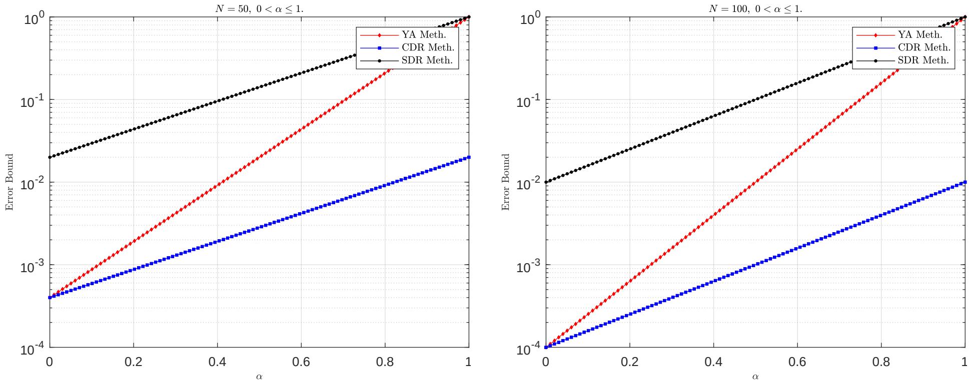

Figure 1: Error bounds of the three methods YA, CDR and SDR for $\alpha \in [0, 1]$ with $N = 50$ and $N = 100$.

The error bounds of the improved YA, CDR and SDR methods for $\alpha \in [0, 1]$ and $N = 50$ and $N = 100$ are plotted in Fig. 1. This figure obviously states that the convergence rate of the SDR method is very slow, so the following improved version suggested by the authors of [17].

Theorem 2.8 (The improved sine diffusive representation (ISDR)). *For $0 < \alpha < 1$, we have*

$$ {}^{C}D^{\alpha}_{0+}y(t) = \frac{4\cos(\frac{\pi\alpha}{2})}{\pi} \int_0^{\infty} \theta^{2\alpha-1} \left(\int_0^t \sin\left((t-\tau)\theta^2\right) y'(\tau)\, d\tau \right) d\tau = \int_0^{\infty} \theta^{2\alpha-1} \omega^{IS}(\theta, t)\, d\theta, $$

where

$$ \omega^{IS}(\theta, t) = \frac{4\cos(\frac{\pi\alpha}{2})}{\pi} \left(\int_0^t \sin\left((t-\tau)\theta^2\right) y'(\tau)\, d\tau \right). $$

Also, for a given function y for which its first derivative exists on $[0, T]$, $\omega^{IS}(\theta, t)$ (for fixed $\theta > 0$) satisfies the following second-order differential equation:

$$ \begin{cases} \dfrac{\partial^2 \omega^{IS}}{\partial t^2} + \theta^4 \omega^{IS} = \dfrac{4\cos(\frac{\pi\alpha}{2})}{\pi}\, \theta^2 y'(t), \\ \omega^{IS}(\theta, 0) = \dfrac{\partial}{\partial t}\omega^{IS}(\theta, 0) = 0. \end{cases} \tag{11} $$

Proof. The proof is straightforward. □

The next theorem, gives the error bound of the new improved SDR method.

Theorem 2.9. *Let $0 < \alpha < 1$. If a A-stable one-step implicit method of order p with the step size $h < N^{-4}$, (where N is the number of integration points in the generalized Gauss-Laguerre formula) is used for Eq. (11), then for $y(t) \in C^1[0, T]$, $y(0) = 0$ and $t \in [0, T]$, we have the overall error analysis of ISDR approximation formula:*

$$ \left| R^{\alpha}_{IS,N,h} y(t) \right| = \mathcal{O}(N^{2\alpha-2}) + \mathcal{O}(h^p) \int_0^{4N} e^{3Tz^4}\, dz. $$

Proof. The proof is obtained by the similar fashion which used for Theorem 2.6. □

Remark 2.10. *For stiff differential equations (9), (10), and (11), numerical methods with large regions of absolute stability are recommended to avoid the need for small step sizes h. In such cases, the backward Euler method is preferred over the trapezoidal method for solving stiff differential equations [2].*

The numerical parts of the newly presented methods CDR and SDR are given in the next section.

3. The numerical part: Matrix approach to solve multi-term fractional differential equation

The main objective of this section is to solve multi-term fractional differential equation (1) by the methods CDR and SDR. From the numerical analysis point of view, the well-known matrix approach gives an efficient and fast tool to apply numerical methods to solve various problems.

3.1. A matrix approach for CDR and SDR methods:

As we aware, one of the most interesting tools, from the numerical analysis viewpoint, to apply the proposed methods for various problems is to generate their matrix forms. So, we set the target of this section to obtain the matrix form of the new approximation formulae CDR and SDR to approximate the Caputo fractional derivative of order $\alpha \in (0,1)$. To do so, we start with the CDR formula. As stated in [17] the CDR formula can be approximated as:

$$ {}^{C}D_{0+}^{\alpha} y(t)\Big|_{t=t_k} = \int_0^{\infty} z^{\alpha-1} e^{-z} \left[e^{z} \omega^C(z, t_k) \right] dz \approx \sum_{i=1}^{N} w_i^{(\alpha-1)} \; e^{z_i^{(\alpha-1)}} \omega^C(r_i^{(\alpha-1)}, t_k), \tag{12} $$

where $r_i^{(\alpha-1)}$ and $w_i^{(\alpha-1)}$ are the Generalized Gauss-Laguerre (G-G-L) nodes and weights associated with the weight function $w(z) = z^{\alpha-1}e^{-z}$, respectively.

So, the matrix form of the Caputo fractional derivatives of order $\alpha \in (0,1)$ based on the CDR approximation formula (12) can be written as follows:

$$ {}^{C}\mathbf{y}^{(\alpha)} \approx \left(\mathbf{M}^C\right)^T \mathbf{w}^C, $$

where

$$ {}^{C}\mathbf{y}^{(\alpha)} = \begin{bmatrix} y^{(\alpha)}(t_1) \\ y^{(\alpha)}(t_2) \\ \vdots \\ y^{(\alpha)}(t_n) \end{bmatrix}, \quad \mathbf{w}^C = \begin{bmatrix} w_1^{(\alpha-1)} \; e^{r_1^{(\alpha-1)}} \\ w_2^{(\alpha-1)} \; e^{r_2^{(\alpha-1)}} \\ \vdots \\ w_N^{(\alpha-1)} \; e^{r_N^{(\alpha-1)}} \end{bmatrix}^T, \quad \mathbf{M}^C = \begin{bmatrix} \omega^C(r_1^{(\alpha-1)}, t_1) & \omega^C(r_1^{(\alpha-1)}, t_2) & \cdots & \omega^C(r_1^{(\alpha-1)}, t_n) \\ \omega^C(r_2^{(\alpha-1)}, t_1) & \omega^C(r_2^{(\alpha-1)}, t_2) & \cdots & \omega^C(r_2^{(\alpha-1)}, t_n) \\ \vdots & \vdots & \cdots & \\ \omega^C(r_N^{(\alpha-1)}, t_1) & \omega^C(r_N^{(\alpha-1)}, t_2) & \cdots & \omega^C(r_N^{(\alpha-1)}, t_n) \end{bmatrix}. $$

Now, it remains to compute the unknown matrix $\mathbf{M}^C$. So, we need to evaluate the values $\omega^C(r_i^{(\alpha-1)}, t_k)$, $i = 1, 2, \dots, N$, $k = 1, 2, \dots, n$ numerically. So, we are going to use Eq. (9). The mentioned system of differential equations can get the matrix form as:

$$ {}^{C}\mathbf{A}\,{}^{C}\mathbf{Z}'(\mathbf{t}) = {}^{C}\mathbf{B}\,{}^{C}\mathbf{Z}(\mathbf{t}) + {}^{C}\mathbf{F}(\mathbf{t}), \quad {}^{C}\mathbf{Z}(\mathbf{0}) = \mathbf{Z_0}, \tag{13} $$

where matrices ${}^{C}\mathbf{A}$ and ${}^{C}\mathbf{B}$, and vectors ${}^{C}\mathbf{Z}(\mathbf{t})$ and ${}^{C}\mathbf{F}(\mathbf{t})$ are given as:

$$ {}^{C}\mathbf{A} = \mathbf{I}_{2N\times 2N}, \quad {}^{C}\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}_{2N\times 2N}, \quad B_{11} = B_{22} = [\mathbf{0}]_{N\times N}, \quad B_{12} = \mathbf{I}_{N\times N}, $$

and

$$ B_{21} = \begin{bmatrix} -\left[r_1^{(\alpha-1)}\right]^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & -\left[r_N^{(\alpha-1)}\right]^2 \end{bmatrix}_{N\times N}, \quad {}^{C}\mathbf{Z}(\mathbf{t}) = \begin{bmatrix} x_1(r_1^{(\alpha-1)}, t) \\ \vdots \\ x_1(r_N^{(\alpha-1)}, t) \\ x_2(r_1^{(\alpha-1)}, t) \\ \vdots \\ x_2(r_N^{(\alpha-1)}, t) \end{bmatrix}_{2N\times 1}, \quad {}^{C}\mathbf{F}(\mathbf{t}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{2\sin(\frac{\pi\alpha}{2})}{\pi} y''(t) \\ \vdots \\ \frac{2\sin(\frac{\pi\alpha}{2})}{\pi} y''(t) \end{bmatrix}_{2N\times 1}. $$

We also point out that $\mathbf{I}_{N\times N}$ stands for the identity matrix.

Now, we face with a first order differential equation (13) in the matrix form. To solve it numerically, various numerical time integrations such as: rectangular or trapezoidal rules can be performed. So, let $t \in [0, T]$ and h stands for the step size with:

$$ h = \frac{T}{n-1}, \quad n \in \mathbb{N}. $$

Now, we define $t_k = (k-1)h$, so we have $t_1 = 0$ and $t_n = T$.

Using the rectangular or trapezoidal integration rules we get the well-known backward Euler method:

$$ {}^{C}\mathbf{Z}(\mathbf{t_{i+1}}) = \left({}^{C}\mathbf{A} - h\,{}^{C}\mathbf{B}\right)^{-1} \left[{}^{C}\mathbf{A}\,{}^{C}\mathbf{Z}(\mathbf{t_i}) + h\,{}^{C}\mathbf{F}(\mathbf{t_{i+1}})\right], \; i = 1, 2, \dots, n-1, \tag{14} $$

and trapezoidal method:

$$ {}^{C}\mathbf{Z}(\mathbf{t_{i+1}}) = \left({}^{C}\mathbf{A} - \frac{h}{2}{}^{C}\mathbf{B}\right)^{-1} \left[\left({}^{C}\mathbf{A} + \frac{h}{2}{}^{C}\mathbf{B}\right){}^{C}\mathbf{Z}(\mathbf{t_i}) + \frac{h}{2}\left({}^{C}\mathbf{F}(\mathbf{t_{i+1}}) + {}^{C}\mathbf{F}(\mathbf{t_i})\right)\right], \; i = 1, 2, \dots, n-1, \tag{15} $$

respectively, and subject to the following initial condition:

$$^{C}\mathbf{Z}(\mathbf{0}) = \begin{bmatrix} x_1(r_1^{(\alpha-1)},0) \\ \vdots \\ x_1(r_N^{(\alpha-1)},0) \\ x_2(r_1^{(\alpha-1)},0) \\ \vdots \\ x_2(r_N^{(\alpha-1)},0) \end{bmatrix}_{2N\times 1},$$

Finally, if we set:

$$^{C}\mathbf{Z} = [\mathbf{Z}(\mathbf{t_1})\ \mathbf{Z}(\mathbf{t_2})\ \dots\ \mathbf{Z}(\mathbf{t_n})]_{2N\times n}, \tag{16}$$

then we have

$$\mathbf{M}^{C} = {}^{C}\mathbf{Z}(1:N,:).$$

We also point out that $^{C}\mathbf{Z}(1:N,:)$ stands for the rows 1 to N and all columns of the matrix $^{C}\mathbf{Z}$.

For the SDR method, we use the following formula:

$$^{C}D_{0^+}^{\alpha}y(t)\Big|_{t=t_k} = \int_0^{\infty} z^{\alpha}e^{-z}\left[e^{z}\omega^{S}(z,t_k)\right]\,dz \approx \sum_{i=1}^{N} w_i^{(\alpha)}\ e^{z_i^{(\alpha)}}\omega^{S}(r_i^{(\alpha)},t_k),$$

where $r_i^{(\alpha)}$ and $w_i^{(\alpha)}$ are the Generalized Gauss-Laguerre (G-G-L) nodes and weights associated with the weight function $w(z) = z^{\alpha}e^{-z}$.

Therefore, the matrix form of the Caputo fractional derivatives of order $\alpha \in (0,1)$ based on the SDR approximation formula is obtained as follows:

$$^{S}\mathbf{y}^{(\alpha)} \approx \left(\mathbf{M}^{S}\right)^{T}\mathbf{w}^{S},$$

where

$$^{S}\mathbf{y}^{(\alpha)} = \begin{bmatrix} y^{(\alpha)}(t_1) \\ y^{(\alpha)}(t_2) \\ \vdots \\ y^{(\alpha)}(t_n) \end{bmatrix},\ \mathbf{w}^{S} = \begin{bmatrix} w_1^{(\alpha)}\ e^{r_1^{(\alpha)}} \\ w_2^{(\alpha)}\ e^{r_2^{(\alpha)}} \\ \vdots \\ w_N^{(\alpha)}\ e^{r_N^{(\alpha)}} \end{bmatrix}^{T},\ \mathbf{M}^{S} = \begin{bmatrix} \omega^{S}(r_1^{(\alpha)},t_1) & \omega^{S}(r_1^{(\alpha)},t_2) & \cdots & \omega^{S}(r_1^{(\alpha)},t_n) \\ \omega^{S}(r_2^{(\alpha)},t_1) & \omega^{S}(r_2^{(\alpha)},t_2) & \cdots & \omega^{S}(r_2^{(\alpha)},t_n) \\ \vdots & \vdots & \cdots & \\ \omega^{S}(r_N^{(\alpha)},t_1) & \omega^{S}(r_N^{(\alpha)},t_2) & \cdots & \omega^{S}(r_N^{(\alpha)},t_n) \end{bmatrix}.$$

Now, it remains to evaluate the values $\omega^{S}(r_i^{(\alpha)},t_k),\ i=1,2,\dots,N,\ k=1,2,\dots,n$. As it is observed before, the function $\omega^{S}(z,t)$ satisfies a system of first order differential equations (10).

Similarly, system (10) can be seen as the following matrix form:

$$^{S}\mathbf{A}{}^{S}\mathbf{Z}'(\mathbf{t}) = {}^{S}\mathbf{B}{}^{S}\mathbf{Z}(\mathbf{t}) + {}^{S}\mathbf{F}(\mathbf{t}),\ {}^{S}\mathbf{Z}(\mathbf{0}) = \mathbf{0},$$

where matrices $^{S}\mathbf{A}$ and $^{S}\mathbf{B}$, and vectors $^{S}\mathbf{Z}(\mathbf{t})$ and $^{S}\mathbf{F}(\mathbf{t})$ are given as:

$$^{S}\mathbf{A} = \mathbf{I}_{2N\times 2N},\ {}^{S}\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}_{2N\times 2N},\ B_{11} = B_{22} = [\mathbf{0}]_{N\times N},\ B_{12} = \mathbf{I}_{N\times N},$$

and

$$B_{21} = \begin{bmatrix} -\left[r_1^{(\alpha)}\right]^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & -\left[r_N^{(\alpha)}\right]^2 \end{bmatrix}_{N\times N},\ {}^{S}\mathbf{Z}(\mathbf{t}) = \begin{bmatrix} x_1(r_1^{(\alpha)},t) \\ \vdots \\ x_1(r_N^{(\alpha)},t) \\ x_2(r_1^{(\alpha)},t) \\ \vdots \\ x_2(r_N^{(\alpha)},t) \end{bmatrix}_{2N\times 1},\ {}^{S}\mathbf{F}(\mathbf{t}) = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \frac{2\cos(\frac{\pi\alpha}{2})}{\pi}y'(t) \\ \vdots \\ \frac{2\cos(\frac{\pi\alpha}{2})}{\pi}y'(t) \end{bmatrix}_{2N\times 1}.$$

Now, using the backward Euler and trapezoidal methods (14) and (15), the matrix $^{S}\mathbf{Z}$ which is defined as:

$$^{S}\mathbf{Z} = [\mathbf{Z}(\mathbf{t_1})\ \mathbf{Z}(\mathbf{t_2})\ \dots\ \mathbf{Z}(\mathbf{t_n})]_{2N\times n}, \tag{17}$$

is obtained. Thus

$$\mathbf{M}^S = {}^S\mathbf{Z}(1:N,:).$$

Due to the fact that many real problems which were modeled by Caputo fractional derivative, contain the Caputo fractional derivatives of order $\alpha \in (1,2)$. So, our next goal is to extend the approximation formulae CDR and SDR for the case $\alpha \in (1,2)$.

3.2. Approximation of Caputo fractional derivatives of order $\alpha \in (1,2)$ based on CDR and SDR methods

In various areas of mechanics and engineering we encounter with fractional models of order $\alpha \in (1,2)$. So, it is essential to extend the CDR and SDR approximation formulae for the mentioned case. For this reason, we present the following lemma which plays a key role to extend these formulae.

Lemma 3.1. *Let $n-1 < \alpha < n \leq m,\ m,n \in \mathbb{N}$. If we have*

$$y^{(s)}(0) = 0,\ \ s = n, n+1, \ldots, m,$$

then

$$ {}^C D^{\alpha}_{0^+} y^{(m)}(t) = \frac{d^m}{dt^m}\left({}^C D^{\alpha}_{0^+} y(t)\right) = {}^C D^{\alpha+m}_{0^+} y(t).$$

For special case $0 < \alpha < 1$ and $m = 1$, if $y'(0) = 0$, then we arrive at:

$$ {}^C D^{1+\alpha}_{0^+} y(t) = \frac{d}{dt}\left({}^C D^{\alpha}_{0^+} y(t)\right).$$

Proof. See [25] for the details. □

Using the above lemma, the following important theorem can be concluded.

Theorem 3.2. *Let $0 < \alpha < 1$ and $y(0) = y'(0) = 0$. Then we have:*

$$ {}^C D^{1+\alpha}_{0^+} y(t) = \int_0^{\infty} z^{\alpha-1} e^{-z}\left[e^z \frac{d}{dt}\left[\omega^C(z,t)\right]\right] dz,$$

and

$$ {}^C D^{1+\alpha}_{0^+} y(t) = \int_0^{\infty} z^{\alpha} e^{-z}\left[e^z \frac{d}{dt}\left[\omega^S(z,t)\right]\right] dz.$$

Proof. The proof of this theorem can be easily obtained if we differentiate both sides of Eqs. (5) and (7) with respect to t. □

Theorem 3.3. *Let $0 < \alpha < 1$ and $y(0) = y'(0) = 0$. Then the Caputo fractional derivatives of order $1+\alpha$ for which $\alpha \in (0,1)$ based on the extended CDR and SDR methods (which denoted by ECDR and ESDR) are as follows:*

$$ {}^{EC}\mathbf{y}^{(\alpha+1)} \approx \left(\mathbf{M}^{EC}\right)^T \mathbf{w}^C, \quad {}^{ES}\mathbf{y}^{(\alpha+1)} \approx \left(\mathbf{M}^{ES}\right)^T \mathbf{w}^S,$$

where

$$\mathbf{M}^{EC} = {}^C\mathbf{Z}(N+1:2N,:),\ \mathbf{M}^{ES} = {}^S\mathbf{Z}(N+1:2N,:),$$

where ${}^C\mathbf{Z}$ and ${}^S\mathbf{Z}$ are defined in (16) and (17), respectively.

Remark 3.4. *The above theorem gives a distinct advantage of the CDR and SDR methods. In fact, when we solve the systems of first order differential equations (13) and (3.1), numerically, the solutions not only give us the approximations of Caputo fractional derivative of order $\alpha \in (0,1)$ but also provide the approximations of Caputo fractional derivative of order $1+\alpha,\ \alpha \in (0,1)$ as well.*

3.3. Matrix approach for multi-term fractional differential equations

The main purpose of this section is to apply the CDR (ECDR), SDR (ESDR) and ISDR methods to solve multi-term fractional differential equation (MTFDE) (1). For the reason, consider the following MTFDE with constant coefficients:

$$my''(t)+c_1{}^C D_{0^+}^{1+\beta}y(t)+c_2{}^C D_{0^+}^{\beta}y(t)+c_3{}^C D_{0^+}^{\alpha}y(t)+ky(t)=f(t),\ y(0)=y'(0)=0,\ 0<\alpha,\beta<1,\ t\in[0,T], \tag{18}$$

where $m,\ c_1,\ c_2,\ c_3$ and k are some real constants. To solve the above MTFDE, we assume that:

$$t_1=0,\ t_n=T,\quad t_i=(i-1)h,\ i=2,3,\dots,n-1,\ h=\frac{T}{n-1}.$$

We are going to solve the mentioned system of differential equations by the CDR method. For other methods (SDR, ISDR), the same fashion can be easily followed.

In order to handle problem (18) numerically, we first convert it into a system of (ordinary and fractional) differential equations. So, let $y'(t)=v(t)$, then we get:

$$\begin{cases} y'(t)=v(t), \\ mv'(t)=f(t)-c_1{}^C D_{0^+}^{1+\beta}y(t)-c_2{}^C D_{0^+}^{\beta}y(t)-c_3{}^C D_{0^+}^{\alpha}y(t)-ky(t), \end{cases}\qquad y(0)=v(0)=0. \tag{19}$$

As it is stated in (12), we have:

$${}^C D_{0^+}^{\beta}y(t)\approx\sum_{i=1}^{N}w_i^{(\beta-1)}\ e^{z_i^{(\beta-1)}}\,x_1^{(\beta)}(r_i^{(\beta-1)},t), \tag{20}$$

$${}^C D_{0^+}^{\beta+1}y(t)\approx\sum_{i=1}^{N}w_i^{(\beta-1)}\ e^{z_i^{(\beta-1)}}\,x_2^{(\beta)}(r_i^{(\beta-1)},t), \tag{21}$$

where $r_i^{(\beta-1)}$ and $w_i^{(\beta-1)}$ are used for the Gauss-Laguerre nodes and weights associated with the weight function $w(z)=z^{\beta-1}e^{-z}$ and also $x_1^{(\beta)}(z,t)$ and $x_2^{(\beta)}(z,t)$ satisfy the following first order differential equations:

$$\begin{cases} \dfrac{\partial x_1^{(\beta)}}{\partial t}=x_2^{(\beta)}(z,t), \\[2ex] \dfrac{\partial x_2^{(\beta)}}{\partial t}=-z^2x_1^{(\beta)}(z,t)+\dfrac{2\sin(\frac{\beta\pi}{2})}{\pi}y''(t), \\[2ex] x_1^{(\beta)}(z,0)=x_2^{(\beta)}(z,0)=0. \end{cases}$$

Moreover, we also have:

$${}^C D_{0^+}^{\alpha}y(t)\approx\sum_{i=1}^{N}w_i^{(\alpha-1)}\ e^{z_i^{(\alpha-1)}}\,x_1^{(\alpha)}(r_i^{(\alpha-1)},t), \tag{22}$$

where $r_i^{(\alpha-1)}$ and $w_i^{(\alpha-1)}$ are used for the Gauss-Laguerre nodes and weights associated with the weight function $w(z)=z^{\alpha-1}e^{-z}$ and also $x_1^{(\alpha)}(z,t)$ satisfies the following system of first order differential equations:

$$\begin{cases} \dfrac{\partial x_1^{(\alpha)}}{\partial t}=x_2^{(\alpha)}(z,t), \\[2ex] \dfrac{\partial x_2^{(\alpha)}}{\partial t}=-z^2x_1^{(\alpha)}(z,t)+\dfrac{2\sin(\frac{\alpha\pi}{2})}{\pi}y''(t), \\[2ex] x_1^{(\alpha)}(z,0)=x_2^{(\alpha)}(z,0)=0. \end{cases}$$

Substituting approximations (20), (21) and (22) into (19), the following first order differential equation in the matrix form can be followed:

$$\mathbf{A}\mathbf{Z}'(\mathbf{t})=\mathbf{B}\mathbf{Z}(\mathbf{t})+\mathbf{F}(\mathbf{t}),\ \mathbf{Z}(\mathbf{0})=\mathbf{Z_0}, \tag{23}$$

where matrices $\mathbf{A}$ and $\mathbf{B}$, and vectors $\mathbf{Z(t)}$ and $\mathbf{F(t)}$ are given as:

$$\mathbf{A} = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} \\ A_{21} & A_{22} & A_{23} & A_{24} & A_{25} \\ A_{31} & A_{32} & A_{33} & A_{34} & A_{35} \\ A_{41} & A_{42} & A_{43} & A_{44} & A_{45} \\ A_{51} & A_{52} & A_{53} & A_{54} & A_{55} \end{bmatrix}_{(4N+2)\times(4N+2)},$$

where

$$A_{11} = \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix},\ A_{12} = A_{13} = A_{14} = A_{15} = [\mathbf{0}]_{2\times N},$$
$$A_{21} = [\mathbf{0}]_{N\times 2},\ A_{22} = \mathbf{I}_{N\times N},\ A_{23} = A_{24} = A_{25} = [\mathbf{0}]_{N\times N},$$
$$A_{31} = \begin{bmatrix} 0 & -\frac{2\sin(\frac{\beta\pi}{2})}{\pi} \\ \vdots & \vdots \\ 0 & -\frac{2\sin(\frac{\beta\pi}{2})}{\pi} \end{bmatrix}_{N\times 2},\ A_{32} = [\mathbf{0}]_{N\times N},\ A_{33} = \mathbf{I}_{N\times N},\ A_{34} = A_{35} = [\mathbf{0}]_{N\times N},$$

$$A_{41} = [\mathbf{0}]_{N\times 2},\ A_{42} = A_{43} = [\mathbf{0}]_{N\times N},\ A_{44} = \mathbf{I}_{N\times N},\ A_{45} = [\mathbf{0}]_{N\times N},$$
$$A_{51} = \begin{bmatrix} 0 & -\frac{2\sin(\frac{\pi\alpha}{2})}{\pi} \\ \vdots & \vdots \\ 0 & -\frac{2\sin(\frac{\pi\alpha}{2})}{\pi} \end{bmatrix}_{N\times 2},\ A_{52} = A_{53} = A_{54} = [\mathbf{0}]_{N\times N},\ A_{55} = \mathbf{I}_{N\times N},$$

and for matrix $\mathbf{B}$, we arrive at;

$$\mathbf{B} = \begin{bmatrix} B_{11} & B_{12} & B_{13} & B_{14} & B_{15} \\ B_{21} & B_{22} & B_{23} & B_{24} & B_{25} \\ B_{31} & B_{32} & B_{33} & B_{34} & B_{35} \\ B_{41} & B_{42} & B_{43} & B_{44} & B_{45} \\ B_{51} & B_{52} & B_{53} & B_{54} & B_{55} \end{bmatrix}_{(4N+2)\times(4N+2)},$$

where

$$B_{11} = \begin{bmatrix} 0 & 1 \\ -k & 0 \end{bmatrix},\quad B_{12} = \begin{bmatrix} 0 & \cdots & 0 \\ -c_2 w_1^{(\beta-1)}\ e^{r_1^{(\beta-1)}} & \cdots & -c_2 w_N^{(\beta-1)}\ e^{r_N^{(\beta-1)}} \end{bmatrix}_{2\times N},$$
$$B_{13} = \begin{bmatrix} 0 & \cdots & 0 \\ -c_1 w_1^{(\beta-1)}\ e^{r_1^{(\beta-1)}} & \cdots & -c_1 w_N^{(\beta-1)}\ e^{r_N^{(\beta-1)}} \end{bmatrix}_{2\times N},$$
$$B_{14} = \begin{bmatrix} 0 & \cdots & 0 \\ -c_3 w_1^{(\alpha-1)}\ e^{r_1^{(\alpha-1)}} & \cdots & -c_3 w_N^{(\alpha-1)}\ e^{r_N^{(\alpha-1)}} \end{bmatrix}_{2\times N},\quad B_{15} = [\mathbf{0}]_{2\times N},$$
$$B_{21} = [\mathbf{0}]_{N\times 2},\quad B_{22} = [\mathbf{0}]_{N\times N},\quad B_{23} = \mathbf{I}_{N\times N},\quad B_{24} = B_{25} = [\mathbf{0}]_{N\times N},$$

$$B_{31} = [\mathbf{0}]_{N\times 2},\ B_{32} = \begin{bmatrix} -\left[r_1^{(\beta-1)}\right]^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & -\left[r_N^{(\beta-1)}\right]^2 \end{bmatrix}_{N\times N},\ B_{33} = B_{34} = B_{35} = [\mathbf{0}]_{N\times N},$$
$$B_{41} = [\mathbf{0}]_{N\times 2},\ B_{42} = B_{43} = B_{44} = [\mathbf{0}]_{N\times N},\ B_{45} = \mathbf{I}_{N\times N},$$
$$B_{51} = [\mathbf{0}]_{N\times 2},\ B_{52} = B_{53} = [\mathbf{0}]_{N\times N},\ B_{54} = \begin{bmatrix} -\left[r_1^{(\alpha-1)}\right]^2 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & -\left[r_N^{(\alpha-1)}\right]^2 \end{bmatrix}_{N\times N},\ B_{55} = [\mathbf{0}]_{N\times N},$$

and

$$\mathbf{Z}(\mathbf{t})=\begin{bmatrix} y(t)\\ v(t)\\ -----\\ x_1^{(\beta)}(r_1^{(\beta-1)},t)\\ \vdots\\ x_1^{(\beta)}(r_N^{(\beta-1)},t)\\ -----\\ x_2^{(\beta)}(r_1^{(\beta-1)},t)\\ \vdots\\ x_2^{(\beta)}(r_N^{(\beta-1)},t)\\ -----\\ x_1^{(\alpha)}(r_1^{(\alpha-1)},t)\\ \vdots\\ x_1^{(\alpha)}(r_N^{(\alpha-1)},t)\\ -----\\ x_2^{(\alpha)}(r_1^{(\alpha-1)},t)\\ \vdots\\ x_2^{(\alpha)}(r_N^{(\alpha-1)},t)\end{bmatrix}_{(4N+2)\times 1},\quad \mathbf{F}(\mathbf{t})=\begin{bmatrix}0\\ f(t)\\ 0\\ \vdots\\ 0\\ \vdots\\ 0\end{bmatrix}_{(4N+2)\times 1},\quad \mathbf{Z_0}=\begin{bmatrix}0\\ 0\\ 0\\ \vdots\\ 0\\ \vdots\\ 0\end{bmatrix}_{(4N+2)\times 1}.$$

We also point out that $\mathbf{I}_{N\times N}$ stands for the identity matrix and where $\left\{r_i^{(\beta-1)},w_i^{(\beta-1)}\right\}$ and $\left\{r_i^{(\alpha-1)},w_i^{(\alpha-1)}\right\}$ for $i=1,\dots,N$ are the Gauss-Laguerre nodes and weights associated with the weight functions $w(z)=z^{\beta-1}e^{-z}$ and $w(z)=z^{\alpha-1}e^{-z}$, respectively. By the use of the well-known backward Euler and trapezoidal methods as presented in (14) and (15), the approximate solution of MTFDE (18) is obtained.

4. Numerical results

This section is devoted to verify the obtained theoretical results numerically. So, to have a good comparison, we split this section into three subsections. In the first subsection, we report the time running of the proposed methods. In the second subsection, the matrix approach is used to approximate the Caputo fractional derivatives of some given functions and in the third one, the proposed approach is applied to solve some fractional differential equations, numerically.

4.1. The CPU *times of the new methods*

In this section we are going to compare the (CPU) time running of the proposed methods with the classical ones which are based on the convolution quadrature rules. Thus, especially for $0<\alpha<1$, two simple and well-known methods Grünwald-Letnikov (GL):

$$ {}^{C}D_{0^+}^{\alpha}y(t)\Big|_{t=t_{n+1}}\approx\frac{1}{h^{\alpha}}\sum_{k=1}^{n}(-1)^{k-1}\binom{\alpha}{k-1}y((n-k)h),\ y(0)=0,\ h=\frac{T}{n},\ t_k=(k-1)h,$$

and $L1$

$$ {}^{C}D_{0^+}^{\alpha}y(t)\Big|_{t=t_{n+1}}\approx\frac{1}{\Gamma(2-\alpha)h^{\alpha}}\sum_{k=1}^{n}\left[(n-k+1)^{1-\alpha}-(n-k)^{1-\alpha}\right]\left[y(kh)-y((k-1)h)\right],$$

have been selected (See [22] for more details).

Example 4.1. *Consider the following functions and their Caputo fractional derivatives [17]:*

$$\begin{aligned}
&y(t) = t^{1.6},\ {}^{C}D^{\alpha}_{0+}y(t) = \frac{\Gamma(2.6)}{\Gamma(2.6-\alpha)}t^{1.6-\alpha},\ \alpha = 0.4,\ t \in [0,1],\\
&y(t) = t^{3},\ {}^{C}D^{\alpha}_{0+}y(t) = \frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha},\ \alpha = 0.6,\ t \in [0,1],\\
&y(t) = \sin t,\ {}^{C}D^{\alpha}_{0+}y(t) = t^{1-\alpha}\sum_{k=0}^{+\infty}\frac{(-t)^{2k}}{\Gamma(2k+2-\alpha)} = t^{1-\alpha}E_{2,2-\alpha}(-t^2),\ \alpha = 0.5,\ t \in [0,1],\\
&y(t) = t^{\frac{\nu}{2}}J_{\nu}(2\sqrt{t}),\ {}^{C}D^{\alpha}_{0+}y(t) = t^{\frac{\nu-\alpha}{2}}J_{\nu-\alpha}(2\sqrt{t}),\ \nu = 3,\ \alpha = 0.5,\ t \in [0,1],
\end{aligned}$$

where $E_{\alpha,\beta}(z)$ and $J_{\nu}(z)$ are the two-parameter Mittag-Leffler function and the Bessel function of the first kind, respectively.

Tables 1-4 report the time running of the YA, CDR, SDR, GL, and L1 methods for varying values of n. Large n values lead to faster growth in CPU *time for the classical GL and L1 methods. Furthermore, Fig. 2 depicts the time running of all the methods.*

Finally, we compare the maximum errors of the approximations obtained by the YA, CDR, and SDR methods together with GL and L1 methods. These results are depicted in Figure 3.

Table 1: The CPU times of the approximations of Caputo fractional derivative of function $y(t) = t^{1.6},\ t \in [0,1]$ with $\alpha = 0.4$ by the methods GL, $L1$ and the three methods YA, CDR and SDR (using backward Euler method) versus various values of n with $N = 50,\ b = 1$.

	CPU time (Second)				
Methods	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$	$n = 5000$
GL [22]	0.0115	0.0238	0.1475	1.9025	41.0767
$L1$ [22]	0.0097	0.0533	0.2458	2.3638	41.9368
YA (G-G-L) [8, 9]	0.0113	0.0170	0.0189	0.0298	0.0547
CDR (G-G-L)	0.0395	0.0756	0.1364	0.2776	0.6447
SDR (G-G-L)	0.0340	0.0681	0.1315	0.2531	0.6342

Table 2: The CPU times of the approximations of Caputo fractional derivative of function $y(t) = t^{3},\ t \in [0,1]$ with $\alpha = 0.6$ by the methods GL, $L1$ and the three methods YA, CDR and SDR (using backward Euler method) versus various values of n with $N = 50,\ b = 1$.

	CPU time (Second)				
Methods	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$	$n = 5000$
GL [22]	0.0138	0.0237	0.1519	1.9302	37.6966
$L1$ [22]	0.0094	0.0575	0.2473	2.6898	41.7627
YA (G-G-L) [8, 9]	0.0128	0.0153	0.0189	0.0295	0.0545
CDR (G-G-L)	0.0427	0.0711	0.1329	0.2644	0.6345
SDR (G-G-L)	0.0324	0.0702	0.1295	0.2617	0.6351

Table 3: The CPU times of the approximations of Caputo fractional derivative of function $y(t) = \sin t, \ t \in [0,1]$ with $\alpha = 0.5$ by the methods GL, $L1$ and the three methods YA, CDR and SDR (using backward Euler method) versus various values of n with $N = 50, \ b = 1$.

	CPU time (Second)				
Methods	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$	$n = 5000$
GL [22]	0.0083	0.0239	0.1546	1.8655	41.9035
$L1$ [22]	0.0037	0.0202	0.1142	1.8213	37.7475
YA (G-G-L) [8, 9]	0.0128	0.0171	0.0178	0.0315	0.0580
CDR (G-G-L)	0.0346	0.0830	0.1344	0.2685	0.6498
SDR (G-G-L)	0.0323	0.0869	0.1352	0.2580	0.6409

Table 4: The CPU times of the approximations of Caputo fractional derivative of function $y(t) = t^{\frac{\nu}{2}} J_\nu(2\sqrt{t}), \ t \in [0,1]$ with $\nu = 3, \ \alpha = 0.5$ by the methods GL, $L1$ and the three methods YA, CDR and SDR (using backward Euler method) versus various values of n with $N = 50, \ b = 1$.

	CPU time (Second)				
Methods	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$	$n = 5000$
GL [22]	0.0131	0.0265	0.1493	1.8656	38.6438
$L1$ [22]	0.0037	0.0265	0.1177	1.8298	38.3127
YA (G-G-L) [8, 9]	0.0135	0.0244	0.0206	0.0329	0.0649
CDR (G-G-L)	0.0390	0.0814	0.1411	0.2766	0.6730
SDR (G-G-L)	0.0329	0.0681	0.1380	0.2625	0.6521

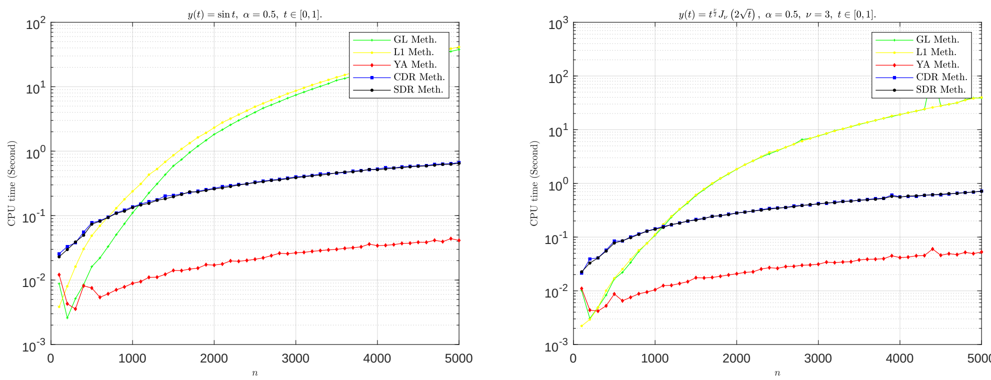

Figure 2: The CPU times of the approximations of Caputo fractional derivative of two functions obtained by the methods GL, $L1$ and the three methods YA, CDR and SDR (using backward Euler method) versus various values of n with $N = 50, \ b = 1$.

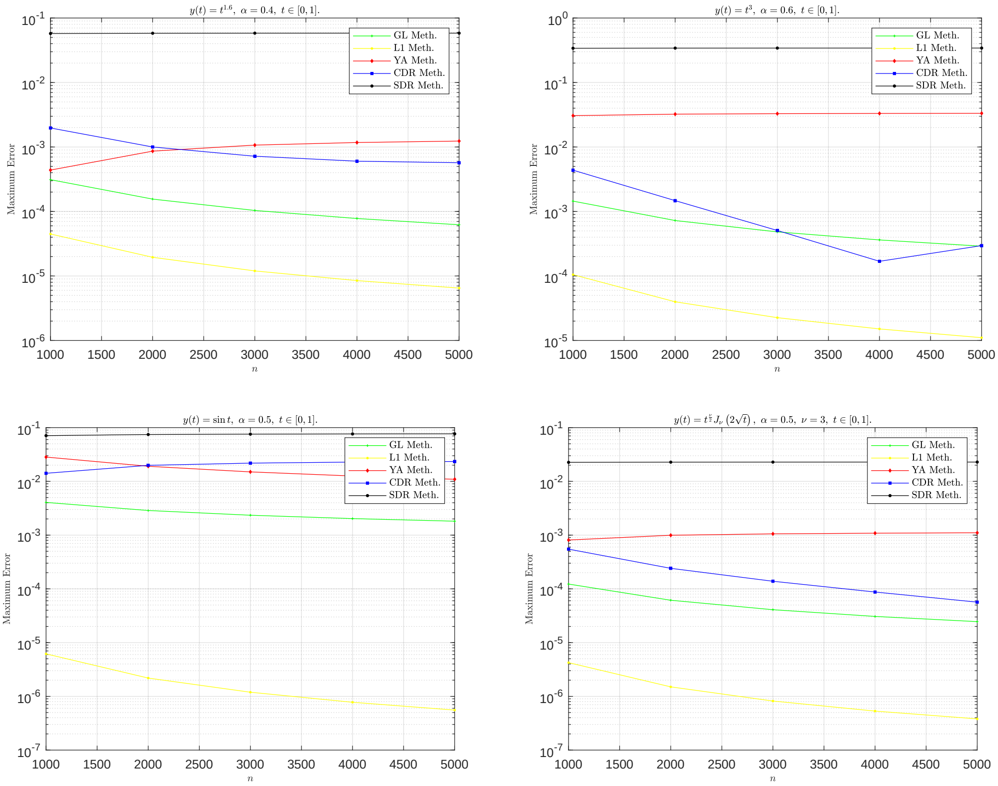

Figure 3: The maximum errors of the approximations of Caputo fractional derivative of two functions obtained by the methods GL, $L1$ and the three methods YA, CDR and SDR (using backward Euler method) versus various values of n with $N = 50,\ b = 1$.

Remark 4.1. *It is worthy to point out that the error term of the methods YA, CDR, SDR and ISDR consists of two parts: The first part depends on the quadrature nodes N, and the second part depends on the ODE solver step size h (See Theorems 2.5, 2.6 and 2.8). Therefore, comparing the DR methods with the classical methods that are usually dependent only on h is not a valid approach (See Figure 3).*

4.2. Approximation of the Caputo fractional derivatives

This subsection is focused on the use of the proposed approach to approximate the Caputo fractional derivatives based on the CDR, SDR and ISDR methods. To due so, we proceed with the following example.

Example 4.2. *Consider the following functions [17]:*

$$
\begin{aligned}
&y(t) = t^{1.6},\ {}^{C}D^{\alpha}_{0+}y(t) = \frac{\Gamma(2.6)}{\Gamma(2.6-\alpha)}t^{1.6-\alpha},\ \alpha = 0.4,\ t \in [0,1],\\
&y(t) = t^{3},\ {}^{C}D^{\alpha}_{0+}y(t) = \frac{\Gamma(4)}{\Gamma(4-\alpha)}t^{3-\alpha},\ \alpha = 0.6,\ t \in [0,1],\\
&y(t) = \sin t,\ {}^{C}D^{\alpha}_{0+}y(t) = t^{1-\alpha}\sum_{k=0}^{+\infty}\frac{(-t)^{2k}}{\Gamma(2k+2-\alpha)} = t^{1-\alpha}E_{2,2-\alpha}(-t^2),\ \alpha = 0.5,\ t \in [0,1],\\
&y(t) = t^{\frac{\nu}{2}}J_\nu(2\sqrt{t}),\ {}^{C}D^{\alpha}_{0+}y(t) = t^{\frac{\nu-\alpha}{2}}J_{\nu-\alpha}(2\sqrt{t}),\ \nu = 3,\ \alpha = 0.5,\ t \in [0,1],
\end{aligned}
$$

where $E_{\alpha,\beta}(z)$ and $J_\nu(z)$ are the two-parameter Mittag-Leffler function and the Bessel function of the first kind, respectively.

Relative errors obtained by the methods YA, CDR, SDR and ISDR for $N = 50,\ n = 10^4$ have been reported in Fig. 4.

In real word problems, we usually face with the problems with long domains ($t >> 1$). So, our next aim is to examine the four methods improved YA, CDR, SDR and ISDR for the mentioned problems (with long domains $t > 1$). Numerical results obtained by the four methods on domain $t \in [0, 50]$ are plotted in Fig. 5. It can be easily observed from this figure that the methods CDR, SDR and ISDR for $t > 5$ may lead to inaccurate results.

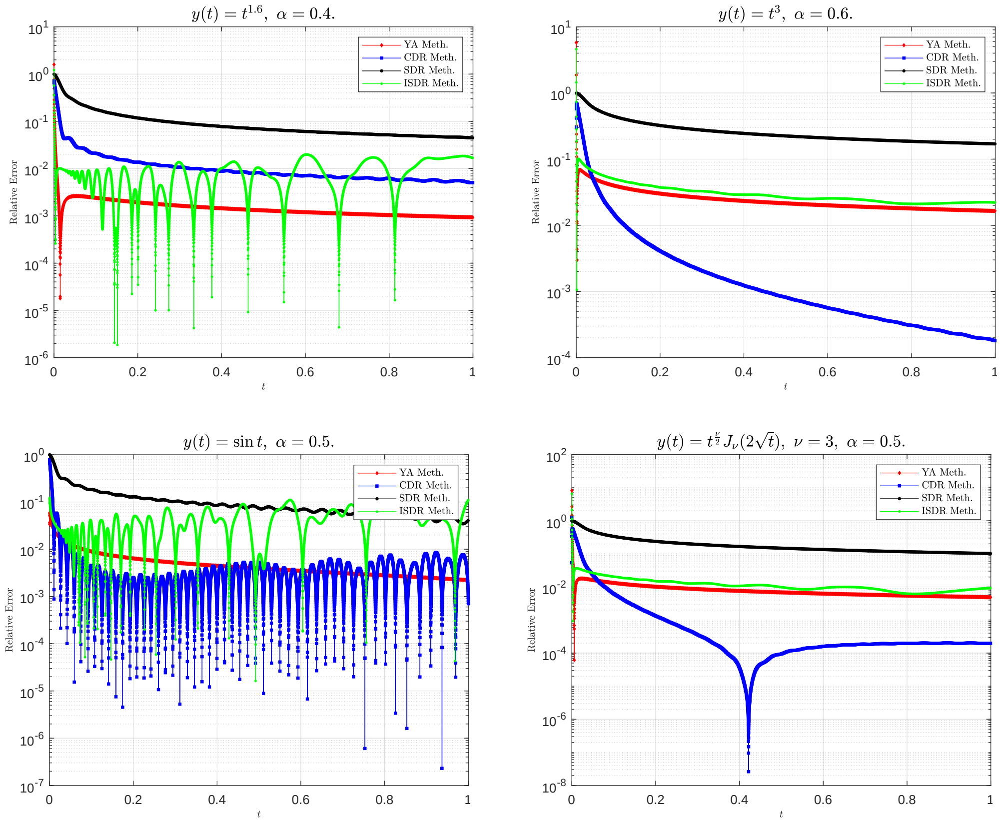

Figure 4: Relative errors obtained by four methods improved YA, CDR, SDR and ISDR with $n = 10^4$ and $N = 50$.

In what follows, a simple and useful idea to improve the accuracy of the obtained results for large values of t is suggested. The idea which can be used here is based on the change of variable:

$$z = \frac{\theta}{b}, \ \ \theta \in [0, b].$$

Numerical results obtained by the methods CDR, SDR and ISDR on domain $t \in [0, 50]$ after using the mentioned change of variable are plotted in Fig. 6. It can be observed from this figure that the accuracy of the obtained results by three methods CDR, SDR and ISDR using the mentioned change of variable is in good agreement with the analytical solution.

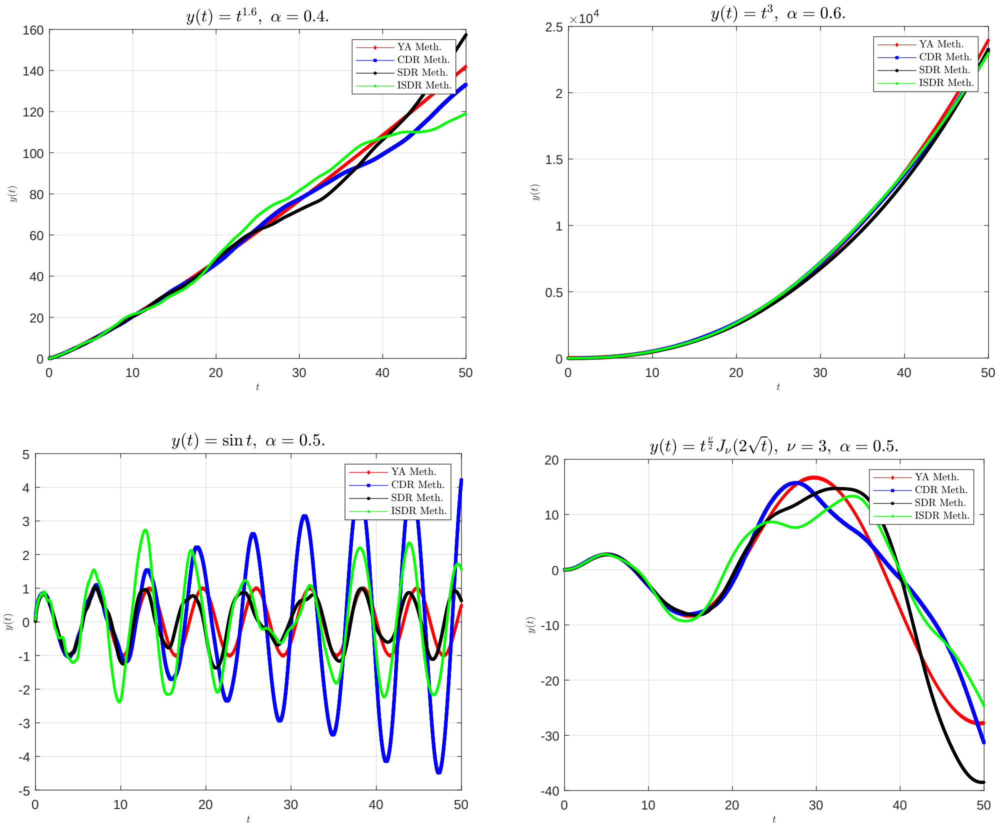

Figure 5: Numerical solutions obtained by four methods YA, CDR, SDR and ISDR (without the change of variable) with $n = 10^4$ and $N = 50$ on $[0, 50]$.

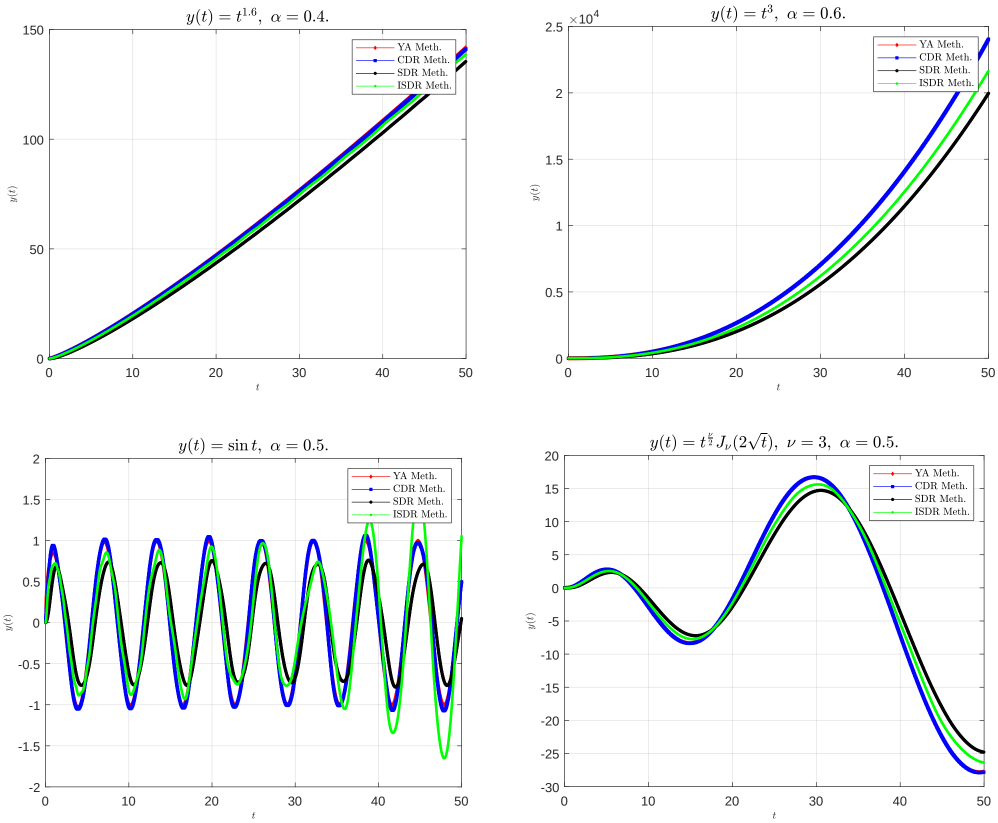

Figure 6: Numerical solutions obtained by four methods YA, CDR, SDR and ISDR (using the change of variable) with $n = 10^4$ and $N = 50$ on $[0, 50]$.

4.3. Multi-term fractional differential equations

In this subsection, the proposed numerical methods have testified with some examples. So, to have a good comparison, the methods CDR, SDR and ISDR are compared with improved Yuan and Agrawal method (YA). It is worthy to point out that the trapezoidal method is carried out to solve system of first order differential equations (23).

Example 4.3. *For the first example we consider the following MTFDE:*

$$my''(t) + c_1 {}^C D_{0^+}^{1+\beta} y(t) + c_2 {}^C D_{0^+}^{\beta} y(t) + c_3 {}^C D_{0^+}^{\alpha} y(t) + ky(t) = f(t),\ y(0) = y'(0) = 0,\ 0 < \alpha, \beta < 1,\ t \in [0, 3],$$

with various values of the parameters m, c_1, c_2, c_3 and k with the exact solution $y(t) = t^3$. For the reader's convenience, we consider the following cases for the unknown parameters:

1. *For the first case, consider $m = k = 1$ and $c_1 = c_2 = c_3 = 0$, then we obtain the following ordinary differential equation:*
$$y''(t) + y(t) = f(t),\ y(0) = y'(0) = 0,\ t \in [0, 3].$$
2. *For the second case, the use of $m = k = 1$, $c_1 = c_2 = 0$, $c_3 = 1$ and $\alpha = 0.5$, yields the following fractional differential equation:*
$$y''(t) + {}^C D_{0^+}^{0.5} y(t) + y(t) = f(t),\ y(0) = y'(0) = 0,\ t \in [0, 3].$$
3. *For the third case, selecting $m = k = 1$, $c_1 = 1$, $c_2 = c_3 = 0$ and $\beta = 0.5$, concludes the following fractional differential equation:*
$$y''(t) + {}^C D_{0^+}^{1.5} y(t) + y(t) = f(t),\ y(0) = y'(0) = 0,\ t \in [0, 3].$$
4. *For the fourth case, having $m = k = 1$, $c_1 = c_2 = 1$, $c_3 = 0$ and $\beta = 0.5$, gives the following fractional differential equation:*
$$y''(t) + {}^C D_{0^+}^{1.5} y(t) + {}^C D_{0^+}^{0.5} y(t) + y(t) = f(t),\ y(0) = y'(0) = 0,\ t \in [0, 3].$$
5. *For the fifth case, consider $m = k = 1$, $c_1 = c_2 = c_3 = 1$ and $\beta = 0.75$, $\alpha = 0.5$, then we have the following fractional differential equation:*
$$y''(t) + {}^C D_{0^+}^{1.75} y(t) + {}^C D_{0^+}^{0.75} y(t) + {}^C D_{0^+}^{0.5} y(t) + y(t) = f(t),\ y(0) = y'(0) = 0,\ t \in [0, 3].$$
6. *For the sixth case, consider $m = 0$, $k = 1$, $c_1 = 1$, $c_2 = c_3 = 0$ and $\beta = 0.75$, then we have the following fractional differential equation:*
$${}^C D_{0^+}^{1.75} y(t) + y(t) = f(t),\ y(0) = y'(0) = 0,\ t \in [0, 3].$$

Relative errors of the approximate solutions of the mentioned cases obtained by YA, CDR, SDR and ISDR methods (without the change of variable) by the trapezoidal method for $n = 10^4$ together with the $N = 50$-points generalized Gauss-Laguerre quadrature rule are plotted in Fig. 7.

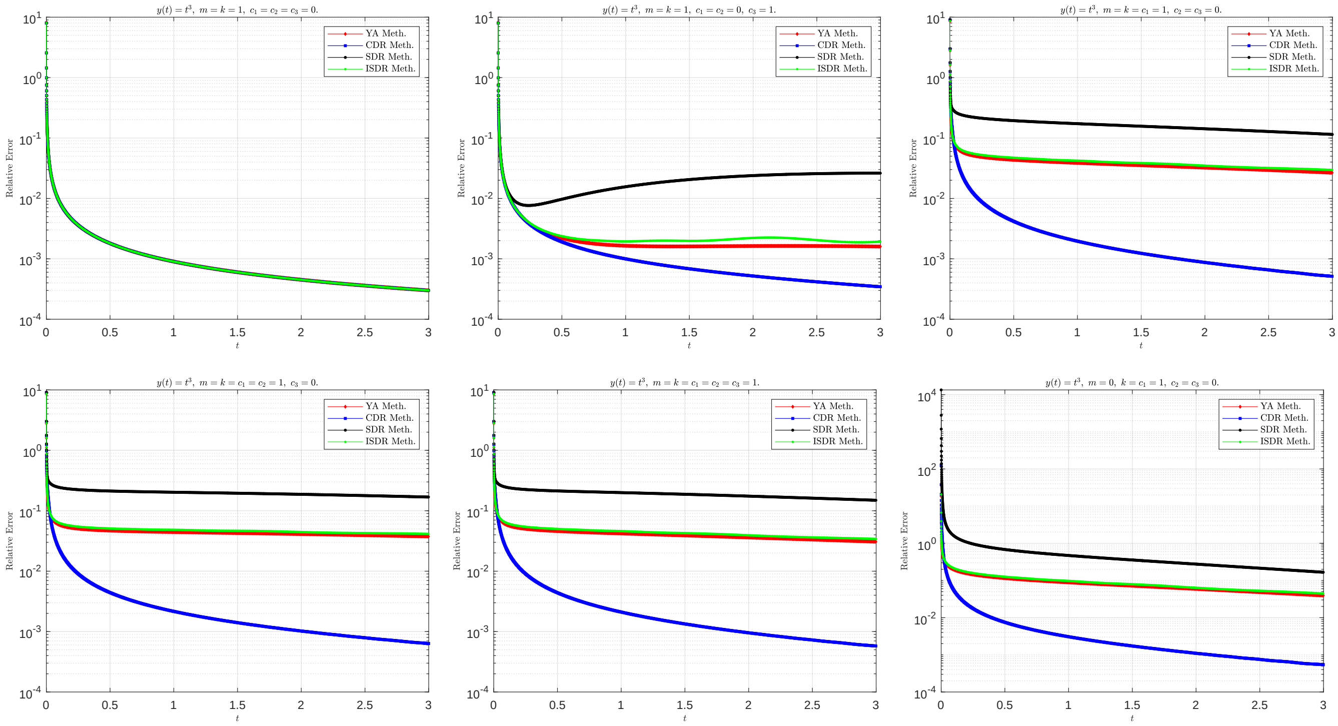

Figure 7: Relative errors obtained by four methods improved YA, CDR, SDR and ISDR (without the change of variable) with $n = 10^4$ and $N = 50$.

Example 4.4. *For the second example, consider the following fractional relaxation-oscillation equation [25, 19]:*

$$^{C}D_{0^+}^{1+\beta}y(t) + ky(t) = 1,\ \ y(0) = y'(0) = 0,\ \ 0 < \beta \leq 1.$$

The analytical solution of this problem is given as:

$$y(t) = t^{\mu}E_{\mu,\mu+1}(-kt^{\mu}),\ \ \mu = 1 + \beta,$$

where $E_{\alpha,\beta}(z)$ is the Mittag–Leffler function which is defined by the power series [25]:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(\alpha k + \beta)},\ \ \alpha, \beta > 0.$$

Numerical solutions of this problem using the methods YA, CDR, SDR and ISDR (without using the change of variable) for $\mu = 1.25,\ 1.5,\ 1.75,\ 1.95$ are depicted in Fig. 8. The improved results after using the change of variable are also plotted in Fig. 9. It can be clearly observed from these figures that the use of newly introduced change of variable can substantially improve the accuracy of numerical results.

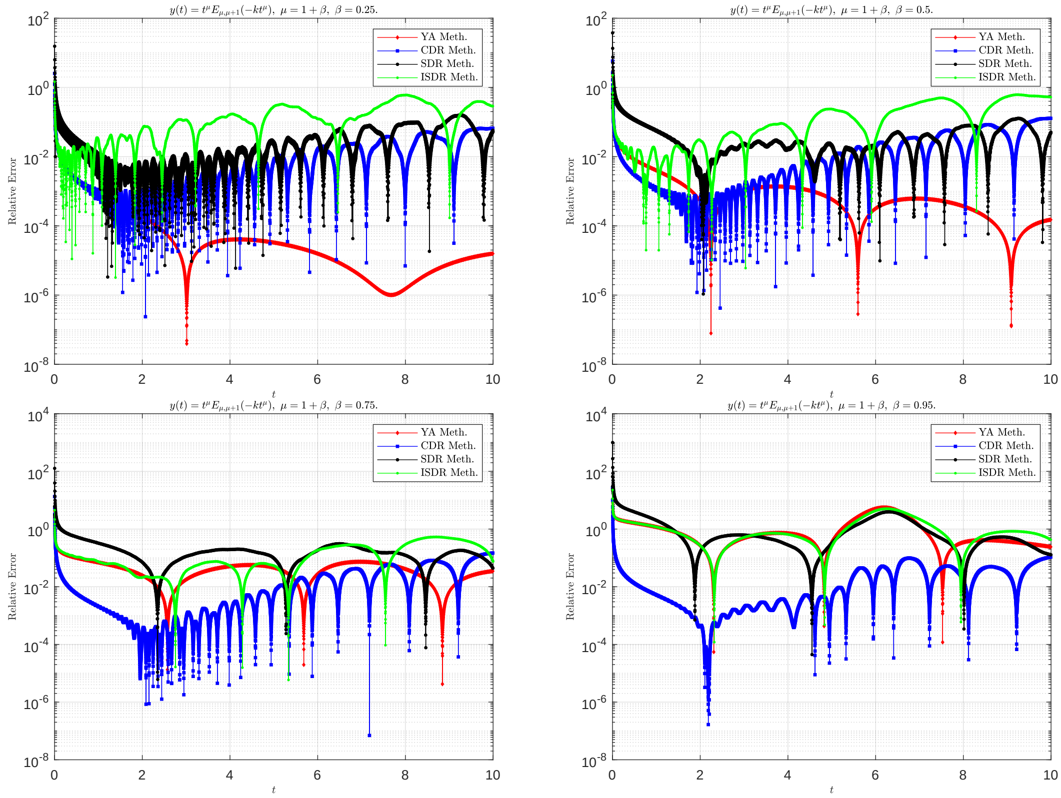

Figure 8: Relative errors obtained by four methods improved YA, CDR, SDR and ISDR (without using change of variable) with $n = 10^4$ and $N = 50$.

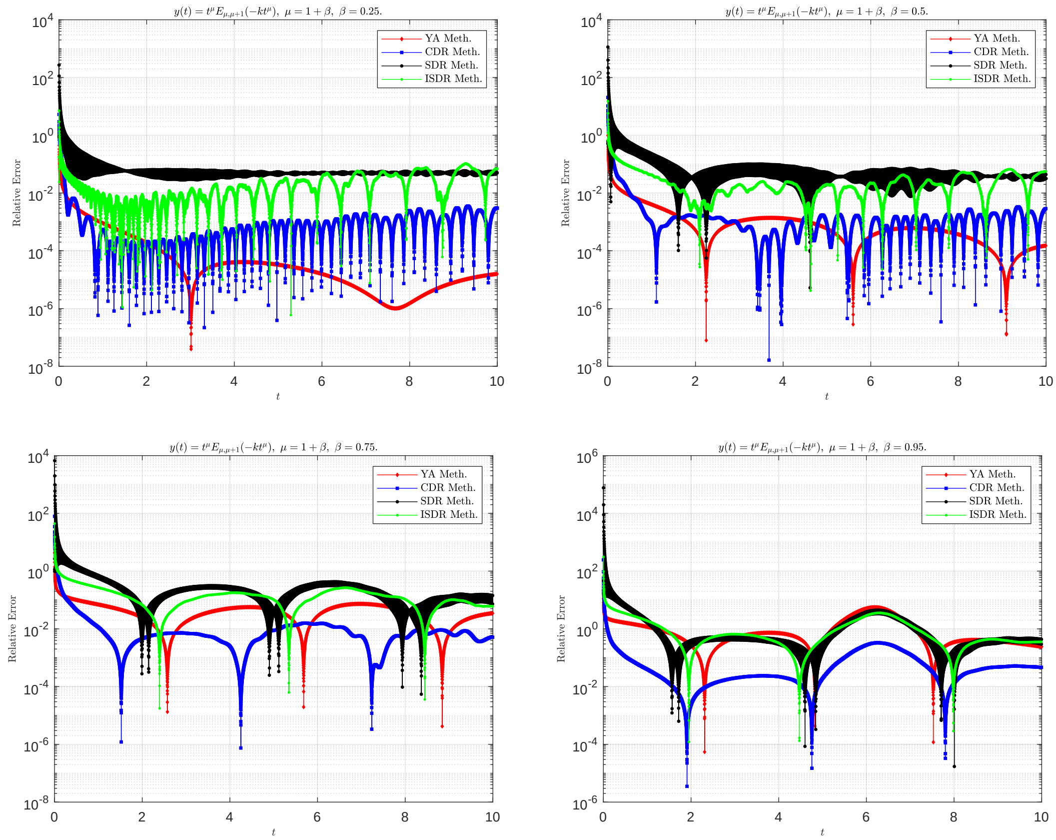

Figure 9: Relative errors obtained by four methods improved YA, CDR, SDR and ISDR (using change of variable) with $n = 10^4$ and $N = 50$.

Example 4.5 (Bagley-Torvic equation). *For the last example, we consider one of the most well-known fractional differential equations as called the Bagley-Torvic equation [10, 12, 25].*

$$Ay''(x) + B\ {}^{C}D^{\beta}_{0^+}y(x) + Cy(x) = f(x),\ 1 < \beta < 2,\ y(0) = y'(0) = 0. \tag{24}$$

In fact, the motion of an immersed plate in a non-Newtonian fluid can be modeled by (24). So, consider a rigid plate of mass M with area S, which is sufficiently large, immersed into an infinite non-Newtonian fluid as shown in Fig. 10 and connected by a massless spring of stiffness K. We also assume that a force $f(x)$ applied to the plate. It is also assumed that the spring motions do not disturb fluid.

This problem is solved numerically with $A = 1,\ B = C = 0.5$ and for

$$f(x) = \begin{cases} 8 & 0 < x < 1 \\ 0 & x \geq 1 \end{cases}$$

and different values of β by four methods improved YA, CDR, SDR and ISDR using the new change of variable. All results are shown in Fig. 11. It should be observed from this figure that, the use of change of variable may lead to get accurate results.

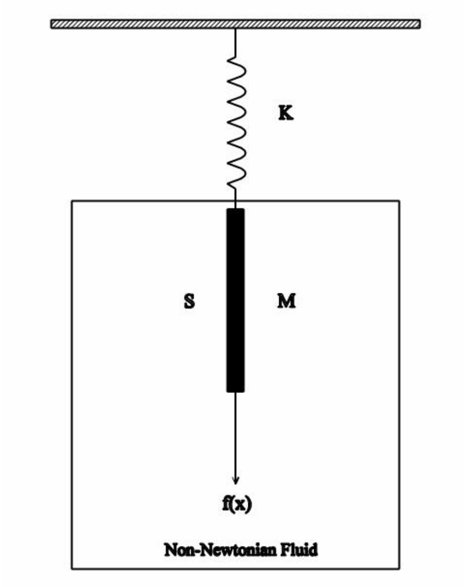

Figure 10: An immersed plate in a Non-Newtonian fluid for Example 4.5 [18].

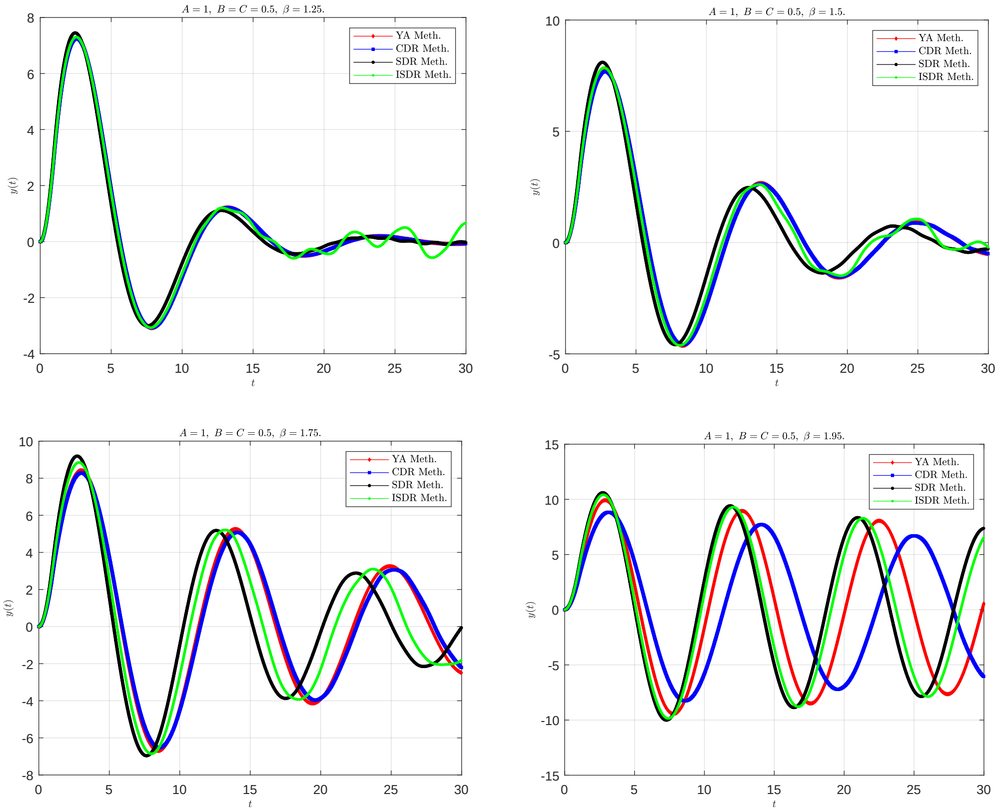

Figure 11: Numerical solutions obtained by four methods YA, CDR, SDR and ISDR (using change of variable) with $n = 10^4$ and $N = 50$.

5. Concluding remarks and future works

This paper is, in fact, a continuation of the recent work of the authors [17] to approximate the Caputo fractional derivative using two new diffusive representations (CDR and SDR). In the current paper, a matrix approach is introduced to apply the CDR and SDR (and its improvement which is denoted by ISDR) for approximation of the Caputo fractional derivatives of order $\alpha \in (0, 1)$. Then we extend the methods CDR and SDR for $\alpha \in (1, 2)$. At the end, a unified matrix approach to solve a linear multi-term fractional differential equation with constant coefficients is introduced. Some test examples are provided to verify the validity and accuracy of the proposed approach.

In the following, some future works are provided:

1. The first suggestion, is to use of CDR, SDR and ISDR for other problems such as fractional optimal control problems, fractional calculus of variations and fractional partial(-integro) differential equations.
2. As we stated in Example 4.2, the CDR and SDR (and thus ISDR) methods for large values of t can not work properly. In the current paper a simple idea (which is based on a change of variable) is carried out to overcome this difficulty. So, our second suggestion is to propose other ideas for CDR and SDR methods to get more accurate results.
3. The last suggestion is to introduce some new modifications and improvements of the CDR and SDR methods to obtain other fast and accurate numerical methods to approximate the Caputo fractional derivative based on the approaches of the papers [4, 6, 7, 8, 9, 11, 23].

6. Acknowledgments

The authors are also thankful the anonymous reviewers who helped us to improve the quality of this manuscript.

References

[1] O. P. Agrawal, *A numerical scheme for initial compliance and creep response of a system*, Mechanics Research Communications, 36 (2009), pp. 444–451.

[2] K. E. Atkinson, W. Han, and D. Stewart, *Numerical solution of ordinary differential equations*, Pure and Applied Mathematics (Hoboken), John Wiley & Sons, Inc., Hoboken, NJ, 2009.

[3] J. Audounet, D. Matignon, and G. Montseny, *Semi-linear diffusive representations for nonlinear fractional differential systems*, in Nonlinear control in the year 2000, Vol. 1 (Paris), vol. 258 of Lect. Notes Control Inf. Sci., Springer, London, 2001, pp. 73–82.

[4] D. Baffet, *A Gauss-Jacobi kernel compression scheme for fractional differential equations*, J. Sci. Comput., 79 (2019), pp. 227–248.

[5] M. Bettayeb and S. Djennoune, *Design of sliding mode controllers for nonlinear fractional-order systems via diffusive representation*, Nonlinear Dynam., 84 (2016), pp. 593–605.

[6] C. Birk and C. Song, *An improved non-classical method for the solution of fractional differential equations*, Comput. Mech., 46 (2010), pp. 721–734.

[7] K. Diethelm, *A new diffusive representation for fractional derivatives, Part I: construction, implementation and numerical examples*, in Fractional differential equations—modeling, discretization, and numerical solvers, vol. 50 of Springer INdAM Ser., Springer, Singapore, pp. 1–15.

[8] ——, *An investigation of some nonclassical methods for the numerical approximation of Caputo-type fractional derivatives*, Numer. Algorithms, 47 (2008), pp. 361–390.

[9] ——, *An improvement of a nonclassical numerical method for the computation of fractional derivatives*, J. Vib. Acoust., 131 (2009), p. 014502.

[10] ——, *The analysis of fractional differential equations*, vol. 2004 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2010. An application-oriented exposition using differential operators of Caputo type.

[11] ——, *Fast solution methods for fractional differential equations in the modeling of viscoelastic materials*, in 2021 9th International Conference on Systems and Control (ICSC), 2021, pp. 455–460.

[12] K. Diethelm and N. J. Ford, *Numerical solution of the Bagley-Torvik equation*, BIT, 42 (2002), pp. 490–507.

[13] K. Diethelm, V. Kiryakova, Y. Luchko, J. A. T. Machado, and V. E. Tarasov, *Trends, directions for further research, and some open problems of fractional calculus*, Nonlinear Dynamics, 107 (2022), pp. 3245–3270.

[14] N. J. Ford and A. C. Simpson, *The numerical solution of fractional differential equations: speed versus accuracy*, Numer. Algorithms, 26 (2001), pp. 333–346.

[15] R. Garrappa, *Numerical solution of fractional differential equations: A survey and a software tutorial*, Mathematics, 6 (2018).

[16] M. Hinze, A. Schmidt, and R. I. Leine, *Numerical solution of fractional-order ordinary differential equations using the reformulated infinite state representation*, Fract. Calc. Appl. Anal., 22 (2019), pp. 1321–1350.

[17] H. Khosravian-Arab and M. Dehghan, *The sine and cosine diffusive representations for the Caputo fractional derivative*, Appl. Numer. Math., 204 (2024), pp. 265–290.

[18] H. Khosravian-Arab, M. Dehghan, and M. R. Eslahchi, *Fractional Sturm-Liouville boundary value problems in unbounded domains: theory and applications*, J. Comput. Phys., 299 (2015), pp. 526–560.

[19] ——, *Generalized Bessel functions: theory and their applications*, Math. Methods Appl. Sci., 40 (2017), pp. 6389–6410.

[20] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and applications of fractional differential equations*, vol. 204 of North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam, 2006.

[21] C. Li and M. Cai, *Theory and numerical approximations of fractional integrals and derivatives*, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2020.

[22] C. Li and F. Zeng, *Numerical methods for fractional calculus*, Chapman & Hall/CRC Numerical Analysis and Scientific Computing, CRC Press, Boca Raton, FL, 2015.

[23] Q. X. Liu, Y. M. Chen, and J. K. Liu, *An improved Yuan–Agrawal method with rapid convergence rate for fractional differential equations*, Computational Mechanics, 63 (2019), pp. 713–723.

[24] D. Matignon, *Diffusive representations for fractional Laplacian: systems theory framework and numerical issues*, Physica Scripta, 2009 (2009), p. 014009.

[25] I. Podlubny, *Fractional differential equations*, vol. 198 of Mathematics in Science and Engineering, Academic Press, Inc., San Diego, CA, 1999. An introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications.

[26] A. Schmidt and L. Gaul, *On a critique of a numerical scheme for the calculation of fractionally damped dynamical systems*, Mech. Res. Commun., 33 (2006), pp. 99–107.

[27] A. Shen, Y. Guo, and Q. Zhang, *A novel diffusive representation of fractional calculus to stability and stabilisation of noncommensurate fractional-order nonlinear systems*, Int. J. Dyn. Control, 10 (2022), pp. 283–295.

[28] C. Trinks and P. Ruge, *Treatment of dynamic systems with fractional derivatives without evaluating memory-integrals*, Comput. Mech., 29 (2002), pp. 471–476.

[29] L. Yuan and O. P. Agrawal, *A numerical scheme for dynamic systems containing fractional derivatives*, J. Vib. Acoust., 124 (2002), pp. 321–324.

Please cite this article using:

Hassan Khosravian-Arab, Mehdi Dehghan, A matrix approach to multi-term fractional differential equations using two new diffusive representations for the Caputo fractional derivative, AUT J. Math. Comput., 5(4) (2024) 337-359
https://doi.org/10.22060/AJMC.2024.23042.1222