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Original Article

Fixed k-watchman routes under the Min-Max criterion in staircase polygons

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ABSTRACT: In this paper, the problem of multiple watchman routes in staircase polygons is studied. The watchman route problem (WRP) is a variation of the art gallery problem (AGP) in computational geometry, where each point in the given polygon must be visible from at least one point along the route taken by one of the watchmen. A greedy algorithm is presented for the min-max criterion, where we minimize the maximum route length. We assume some starting points of the watchmen may dominate the others. This algorithm finds an optimal solution in $O(n^2 \cdot k^2 \cdot \log n)$ time, where n represents the number of vertices of the give polygon, and k represents the number of watchmen.

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1. Introduction

Visibility is an important problem in computational geometry. Two points y and z inside polygon P, are visible from (or guarded by) each other if the line segment yz lies entirely inside P [11]. The art gallery problem (AGP) was posed in 1973 by Victor Klee in a conversation with Vasek Chvátal [4] who asked how many guards are needed to guard a gallery (which is modeled by a polygon).

The watchman route problem (WRP) is a variation of the AGP, introduced by Chin and Ntafos in 1988 [5]. Given a connected polygonal domain P, in the WRP the goal is computing a shortest path or route for a mobile guard (the "watchman") which every point of P is visible from at least one point of the route [2, 9, 19]. They showed that this problem is NP-hard for polygons with holes. Dumitrescu revised NP-hardness proof of this problem [9].

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If the starting point of the watchman is given, then the problem is called a fixed or anchored watchman route, otherwise it is called a floating watchman route [8, 21].

The k-watchman route problem is computing k closed routes in an environment so that the routes together guard every point of P, and a length measure on the routes is minimized. Two standard measures are considered, the min-max measure, where we want to minimize the length of the longest route, and the min-sum measure, where we want to minimize the sum of the length of these routes [21]. Applications of the WRP include security and monitoring, time and energy optimization and efficient simulation [26]. In this paper, we study the fixed k-watchman route problem in staircase polygons, with respect to the min-max optimization criterion.

This paper is organized as follows: In Section 2, the literature review of the WRP is presented. Section 3 gives the preliminaries and the problem statement. An optimal min-max algorithm is then proposed for the problem in Section 4. The conclusion and the future works are presented in Section 5.

2. Related Works

Dumitrescu proved NP-hardness of the watchman route problem for polygons with holes [9]. The WRP is NP-hard even for convex polygons with convex holes. The WRP remains NP-hard for rectilinear polygons with rectilinear holes. Chin and Ntafos presented an $O(n \log \log n)$-time algorithm that computes optimum routes in simple rectilinear polygons [27]. In another paper [5] in 1991, they presented an $O(n^4 \cdot \log \log n)$-time algorithm to find the shortest fixed watchman route in a simple polygon through a point s. They conjectured that this problem can be solved in $O(n^2 \cdot \log \log n)$ time [5].

Tan *et al.* [32] fixed the errors that appeared in the previously published papers [5, 27], concerning the WRP. They gave an $O(n^4)$ time solution to the WRP. Hammar and Nilsson [13] found out that the algorithms in [5] and [32] were flawed and fixed them. They concluded that the time-complexity of the shortest floating watchman route algorithm is $O(n^4)$ [13]. Tan *et al.* fixed their previous error in [33].

Nilsson and Wood considered the k-WRP in spiral polygons with the min-sum criterion, and provided a $\theta(n^2)$-time algorithm for this problem [24]. Nilsson and Schuierer gave an $O(n^2 \log n)$-time algorithm to compute the optimum float k-watchman routes in a histogram polygon, under min-max criterion [23]. In [26], Packer presented heuristics to compute k-watchman routes in polygons possibly with holes. Nilsson and Packer gave a polynomial-time 7.1416-approximation algorithm to compute a pair of routes that together guard a simple polygon, under min-max criterion [21]. Bagheri *et al.* [2] investigated the k-WRP in staircase polygons and proposed an $O(n^2 \cdot min\{m, n\})$-time algorithm under the min-sum criterion, where m and n are the number of watchmen and vertices, respectively.

The works in the field of k-WRP are summarized in Table 1.

Table 1: The various time bounds of the watchman route problem

Reference	Time-complexity	Problem	Problem space	(min-sum/min-max) criterion	fixed/float
[27]	$O(n \cdot \log \log n)$	1-WRP	Simple rectilinear polygons	-	float
[5]	$O(n^4 \cdot \log \log n)$	1-WRP	Simple polygons	-	fixed
[13]	$O(n^4)$	1-WRP	Simple polygon	-	fixed
[24]	$\theta(n^2)$	k-WRP	Spiral polygons	min-sum	float
[23]	$O(n^2 \cdot \log n)$	k-WRP	Histogram polygons	min-max	float
[21]	7.1416-approximation	2-WRP	Simple polygons	min-max	float
[2]	$O(n^2 \cdot \min\{k, n\})$	k-WRP	Staircase polygons	min-sum	fixed

As listed in Table 1, there are few works in the literature on k-watchman routes problem and there are still some challenges in this problem. The problem is NP-hard when dealing with simple polygons. However, for certain types of polygons, the problem can be solved in polynomial time. Therefore, a solution for this problem under the min-max criterion for staircase polygons is presented in this paper.

In [29, 17, 30] the heuristic search was used for solving the watchman route problem. Nilsson [22] introduced the k-transmitter watchman route problem, where a watchman route can intersect polygon boundary at most k times. Similar problems such as multi-robot routing problems [18, 6, 28, 1, 7, 3, 15, 20], watchman path problem [12, 35, 10], robot localization [25], and other variants of the WRP [31, 14, 34] were studied in the literature.

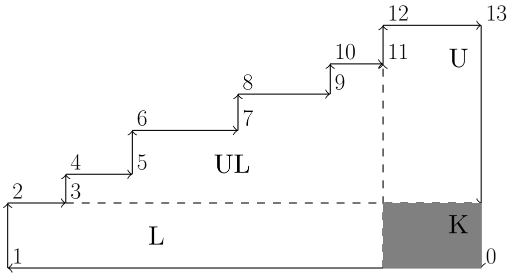


Figure 1: A staircase polygon with 14 vertices. The two essential cuts are shown by dashed lines and the gray area shows the kernel.

3. Preliminaries

A *polygon* is a closed two-dimensional figure composed of straight line-segments that meet at their endpoints. The line-segments of polygons are called *sides*, and each endpoint is called a *vertex*. A *simple polygon*, is a polygon that does not intersect itself and has no hole. A *rectilinear polygon* is a polygon that all of whose sides meet at right angles. A *staircase polygon*, as shown in Figure 1, is a rectilinear polygon consisting of three parts, a long horizontal line-segment (*horizontal base*), a long vertical line-segment (*vertical base*) and a chain consisting of alternating horizontal and vertical line-segments. The chain is monotone with respect to both x- and y-axis.

A simple polygon P is *star-shaped* if there is a point in P that can see all the points of it. The set of points with this property is called *kernel* (see Figure 1). Any staircase polygon is *star-shaped*. Extending the edges of the kernel of a staircase polygon divides the polygon into four sub-polygons, a rectangle above the kernel, is denoted by U, a rectangle left to the kernel, is denoted by L, a staircase sub-polygon on the upper-left corner of the kernel, is denoted by UL, and the kernel itself, is denoted by K (see Figure 1). The vertex v of polygon P is called *reflex* if the internal angle at v is greater than 180^o, otherwise it is called *convex*. Let u be a vertex of P adjacent to a reflex vertex v of polygon P. Let v' denotes the boundary point of P that is hit by the ray shot at v in the direction from u to v [31]. In this case the line-segment $C = vv'$ partitions P into two parts. C is called a cut of P and v is called a defining vertex of C [16, 31]. The part of P that not containing u is called the *essential part of* C, and is denoted by $P(C)$ [16, 31]. A cut C dominates a cut C' if $P(C)$ contains $P(C')$. A cut is called *essential cut* if it is not dominated by another cut [16, 31]. As shown in Figure 1, there are two essential cuts for any staircase polygon with $n > 4$ vertices.

Consider a staircase polygon P with $n = 2i, i > 2$ vertices and $k > 1$ watchmen inside the polygon. We denote j-th vertex of P by P_j. The start location of each watchman is given. Suppose that the horizontal base of the staircase polygon is located on the bottom, and the vertical base is on the right. The intersection point of the vertical and horizontal base is called *origin vertex* which is indexed with number 0 in the polygon. We also suppose that the direction of the polygon edges is in the clockwise order. We denote the x-coordinate of a point p by $x(p)$, and the y-coordinate by $y(p)$. The goal is to find a route for each watchman, that satisfies the optimization criterion (min-max or min-sum), and each point inside the polygon is visible at least by one watchman moving along its route. There are two optimization criteria, minimizing the length of the longest route (min-max) and minimizing the sum of route lengths (min-sum).

Let w be a set of k points inside P which we consider as starting points for the watchmen. We denote the i-th watchman starting point by w_i. For two points a and b, if $x(a) < x(b)$ and $y(a) > y(b)$ then point b *dominates* point a. In this paper, we assume that some of the watchmen's starting points may dominate the others.

4. The Proposed Algorithm

In this section, we present an algorithm for optimizing the min-max criterion. The idea of our algorithm is to partition the polygon P into smaller sub-polygons, and each watchman guards a sub-polygon. As indicated in Figure 2, we will find the left and the right bounds for each sub-polygon. Our goal is to find the best partitioning that optimizes the min-max criterion.

4.1. Arrangement of Starting Points

Some of the given starting points may dominate the others. We preprocess them and make a new arrangement of the given starting points. We sort the starting points by their y-coordinates in ascending order. If two points have the same y-coordinate, we then sort them by their x-coordinates in descending order.

A starting point that is not dominated by the others is called non-dominated. A non-dominated starting point is called pivot point, if it is immediately on the right side of an unseen convex vertex. Non-dominated starting

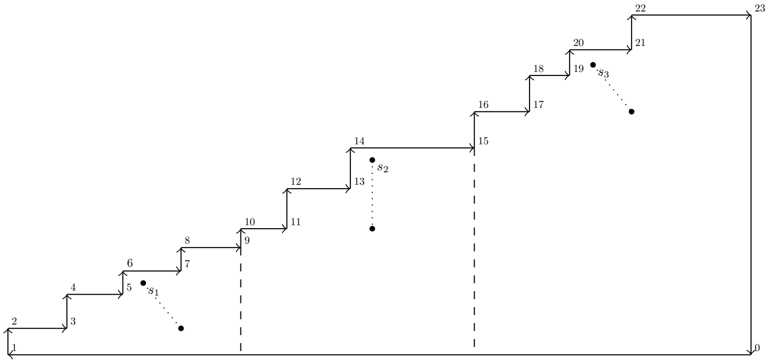


Figure 2: partitioning a staircase polygon with 3 watchmen and calculating each fixed watchman route separately, $L_1 = 2, R_1 = 8, L_2 = 10, R_2 = 14, L_3 = 16, R_3 = 22$

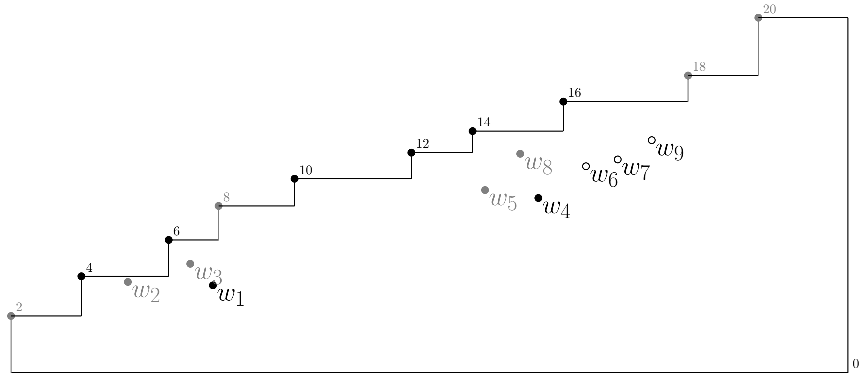


Figure 3: Pivot points are marked by filled circles and follower points are marked by hollow circles. Each unseen convex vertices are colored gray.

points between two consecutive pivot points p_1 and p_2 are called the *follower points* of p_1, where p_1 is before p_2 in the sorted list. If a pivot point has some follower points, then we do not consider the starting points that are dominated by it. Otherwise we should also consider the dominated starting points. As illustrated in Figure 3, w_1 and w_4 are pivot points. There is no follower for w_1. Starting points w_6, w_7 and w_9 are the followers of w_4. Starting points w_2, w_3 are dominated by w_1 and cannot be discarded because w_1 has no follower. The points w_5 and w_8 are dominated by w_4 and can be discarded because w_4 has some followers. The watchman at a pivot point is called *pivot watchman* and the watchmen at a follower point is called *follower watchman*.

We define the following attributes for each pivot point w_i:

- $next(i)$: The index of the next pivot point in the sorted list of starting points. If w_i is the last pivot point, then $next(i)$ is zero.
- $prev(i)$: The index of the previous pivot point in the sorted list of starting points. If w_i is the first pivot point, then $prev(i)$ is zero.
- $last(i)$: The index of the last follower of w_i.
- d_i^+: A starting point that is dominated by w_i and is vertically closest to w_i. If this point does not exist then the value of d_i^+ is null.
- d_i^-: A starting point that is dominated by w_i and is horizontally closest to w_i. If this point does not exist then the value of d_i^- is null. We may have $d_i^- = d_i^+$.

The following attribute is defined for any point w_i:

- $parent(i)$: The index of the corresponding pivot point of w_i.

For each pivot starting point i, we define two indices L_i and R_i to define the left and the right bounds. L_i and R_i refer to indices of the convex vertices on the chain of the given polygon (Figure 2). These indices are defined

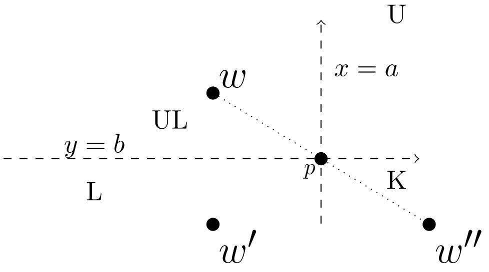


Figure 4: The intersection point p of the two essential cuts, is the closest point to visit the cuts by the watchman standing on w.

such that in a solution of the watchman route (possibly optimal) we have the following relations:

$$L_i = \begin{cases} 2, & \text{if } prev(i) = 0 \\ R_{prev(i)} + 2, & \text{otherwise} \end{cases}$$

$$R_i = \begin{cases} n-2, & \text{if } next(i) = 0 \\ L_{next(i)} - 2, & \text{otherwise} \end{cases}$$

where k is the number of watchmen, n is the number of vertices of the polygon and i is the index of a pivot starting point.

4.2. Fixed 1-Watchman Route

First, we give an algorithm to solve the fixed 1-watchman route problem. The FW algorithm which is given by Equation 1, solves this problem in a staircase polygon. In Lemma 4.1, we prove that the watchman should go to the nearest point on the kernel and return back to its starting point.

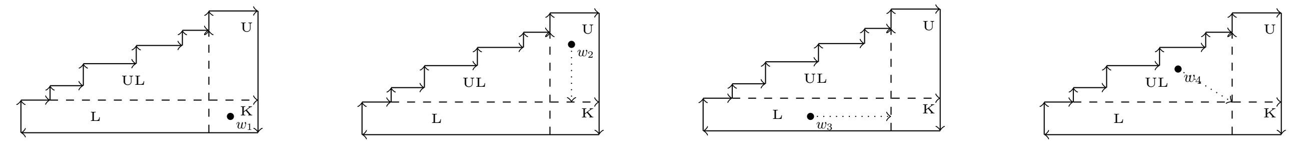
Lemma 4.1. *In the fixed 1-watchman route problem, the nearest point of the kernel to the watchman is the destination point of the watchman.*

Proof. As shown in Figure 4, if the watchman starting point is in the L, U or K area of the staircase polygon P then the lemma is trivial. Now suppose that the watchman starting point is in the UL area of P. We must show that the optimal route must touch the upper-left corner of the kernel and returns back to the starting point. Suppose that $w = (x_1, y_1)$ is the starting point of the watchman in the UL area. To guard P, the watchman should touch two essential cuts of P and then returns back to w. Let $y = b$ and $x = a$ be the horizontal and the vertical essential cuts of P, respectively. The point $p = (a, b)$ which is the upper-left corner of the kernel, is the intersection point of the two essential cuts. Thus, the mirror image of w against $y = b$ is $w' = (x_1, 2 \cdot b - y_1)$ and the mirror image of w' against $x = a$ is $w'' = (2 \cdot a - x_1, 2 \cdot b - y_1)$. The straight line ww'' crosses p, because the line $[w, p]$ has the same slope as the line $[p, w'']$ has and the points w, p and w'' are collinear. So the lemma is proved. ☐

$$FW(i, a, b) = |(w_i, q)|, \text{where } q = \begin{cases} w_i, & i \in K \\ (x(w_i), b), & i \in U \\ (a, y(w_i)), & i \in L \\ (a, b), & i \in UL \end{cases} \tag{1}$$

The coordinate of the upper-left corner of the kernel is denoted by (a, b). There are four possible cases that are illustrated in Figure 5. Case 1: w_i is located in the kernel (see Figure 5a), in the sub-polygon K, then no movement is required by the watchman. Case 2: The watchman is located above the kernel (see Figure 5b), in the sub-polygon U, then the watchman should go vertically to the kernel. Case 3: the watchman is located on the left of the kernel (see Figure 5c), in the sub-polygon L, then the watchman should go horizontally to the kernel. Case 4: the watchman is located at the upper-left corner of the kernel (see Figure 5d), in the sub-polygon UL, then by Lemma 4.1, the watchman should have a diagonal movement to the kernel. Consider Figure 5, the guards w_1, w_2, w_3 and w_4 are, respectively, in the cases 1, 2, 3 and 4.

The horizontal movement of a watchman with starting point w_i to a vertical cut $x = a$ is denoted by $H(i, a)$ in Equation (2). If $x(w_i) > a$, then $H(i, a)$ is zero, otherwise this is the horizontal distance of w_i to the line $x = a$.



(a) The watchman is located in K.

(b) The watchman is located in U.

(c) The watchman is located in L.

(d) The watchman is located in UL.

Figure 5: The four possible cases of the route in the fixed 1-watchman route problem.

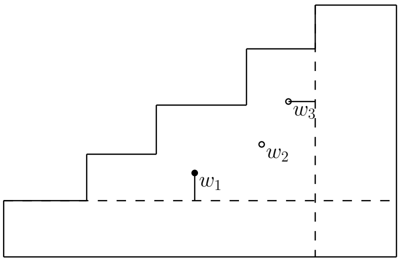
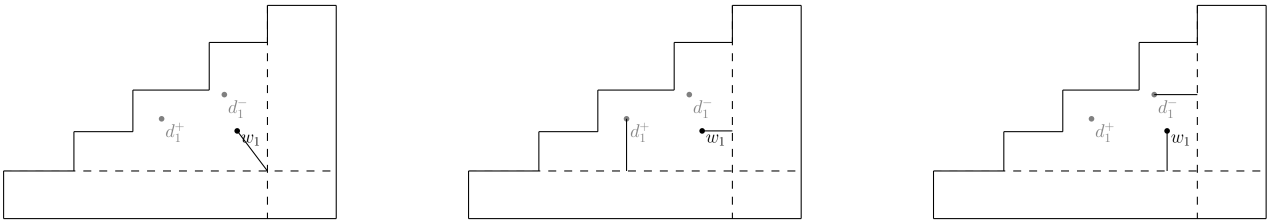


Figure 6: Guard a sub-polygon by a watchman at pivot point w_1 and its last followers.

The vertical movement of a watchman with starting point w_i to a horizontal cut $y = b$ is denoted by $V(i, b)$ in Equation 3. If $y(w_i) < b$, then $V(i, b)$ is zero, otherwise this is the vertical distance of w_i to the line $y = b$.

$$H(i,a) = \begin{cases} 0, & x(w_i) > a \\ |[w_i, (a, y(w_i))]|, & \text{otherwise,} \end{cases} \tag{2}$$

$$V(i,b) = \begin{cases} 0, & y(i) < b \\ |[w_i, (x(i), b)]|, & \text{otherwise.} \end{cases} \tag{3}$$



(a) The watchman at pivot point w_1 moves diagonaly to the kernel.

(b) The watchman at pivot point w_1 guards the vertical cut and the watchman d_i^+ guards the horizontal cut.

(c) The watchman at pivot point w_1 guards the horizontal cut and the watchman d_i^- guards the vertical cut.

Figure 7: There are 3 cases to guard a sub-polygon by a pivot point with no follower point

We denote the route of the pivot watchman i inside its sub-polygon by $r(i, L_i, R_i)$, and let $i(r) = i$, $R(r) = R_i$, and $L(r) = L_i$. The length of a route r is denoted by $|r|$ and can be calculated by Equation (4). There are four possible cases to guard a sub-polygon by a pivot point.

Case 1. If there are some follower point for a pivot point, then the sub-polygon can be guarded by vertical movement of pivot point to the horizontal essential cut and horizontal movement of the last follower to the vertical essential cut. This case is included in the first case of Equation (4) and is shown in Figure 6.

Case 2. If there is no follower point for a pivot point w_i (e.g, $last(i) = i$), then an option is to move the watchman from the pivot point to the closest point to the kernel. This case is shown in Figure 7a and is included in the second case of Equation (4).

Case 3. If there is no follower point for a pivot point w_i and there exists d_i^+, then another option is to move the watchman i, horizontally from the pivot point to the vertical essential cut and moving the watchman d_i^+ vertically to the horizontal essensional cut. This case is shown in Figure 7b and is included in the third case of Equation (4).

Case 4. If there is no follower point for a pivot point w_i and there exists d_i^-, then the last option is to move the watchman i, vertically from the pivot point to the horizontal essential cut and moving the watchman d_i^- horizontally to the vertical essensional cut. This case is shown in Figure 7c and is included in the fourth case of Equation (4).

$$|r(i, L, R)| = \min \begin{cases} \max\{V(w_i, y_L), H(w_{last(i)}, x_R)\}, & \text{if } last(i) > i \\ FW(w_i, x_R, y_L), & \text{if } last(i) = i \\ \max\{V(d_i^+, y_L), H(w_i, x_R)\}, & \text{if } last(i) = i \text{ and } d_i^+ \neq null \\ \max\{V(w_i, y_L), H(d_i^-, x_R)\}, & \text{if } last(i) = i \text{ and } d_i^- \neq null \end{cases} \tag{4}$$

In Equation (4), we denote the y- and x-coordinates of the left and the right convex vertices by y_L and x_R, respectively.

4.3. The Initialization Algorithm

As we mentioned before, we divide the input polygon into some sub-polygons, which are bounded by two vertical edges, are defined by L_i and R_i indices. To find the optimal indices for each sub-polygon, first we initialize them to some initial values. In Equation (7), we utilize these initial values to determine possible routes for a pivot watchman. In this section we describe how we can initialize these indices.

Lemma 4.2. *For each point p inside the staircase polygon P, every point x inside P that is lying on the upper-left area of p, is guarded by p, and we say, p dominates x.*

Proof. If we pass a vertical an a horizontal line through $(x(p), y(p))$, it makes a staircase polygon that includes any point x of p that lying on the upper-left area of p. Then p is the origin of it, so guards x. ☐

We denote the initial left and right bounds of each pivot point i by L_i^I and R_i^I, respectively. By Lemma 4.2, we find the values of L_i^I and R_i^I using Equations (5) and (6). To guard the whole polygon, the lowest convex vertex on the chain must be guarded by the first pivot point or its followers, thus the value of L_1^I is initialized by 2 and for other pivot points ($i > 1$), it will be the minimum index of a convex vertex that is dominated by w_i (see Equation (5)). On the other hand, the highest convex vertex on the chain must be guarded by the last pivot point or its followers ($next(i) = 0$), thus the value of R_i^I is initialized by $n - 2$ and for other pivot watchmen, it will be the maximum index of a convex vertex that is dominated by $last(i)$ (see Equation (6)).

$$L_i^I = \begin{cases} 2, & \text{if } prev(i) = 0 \\ \min\limits_{1 \le s \le \frac{n-2}{2}} \{j = 2s \mid w_i \text{ dominates } P_j\}, & \text{otherwise,} \end{cases} \tag{5}$$

$$R_i^I = \begin{cases} n-2, & \text{if } next(i) = 0 \\ \max\limits_{1 \le s \le \frac{n-2}{2}} \{j = 2s \mid w_{last(i)} \text{ dominates } P_j\}, & \text{otherwise.} \end{cases} \tag{6}$$

The initialization algorithm is given by Algorithm 1. It determines the initial values of L_i^I and R_i^I for each watchman i standing at a pivot point, based on Equations (5) and (6). The starting points are considered one after another in order of increasing y-coordinate. Keeping track of the last convex vertex c' seen by the previously considered starting point, and the first convex vertex c seen by the current starting point allows us to determine whether a starting points is a pivot/follower point. The currently considered starting point w_i is dominated by a starting point prior to it in w if w_i's x-coordinate is smaller. In particular, the first starting point w_1 is non-dominated. We store the largest x-coordinate of the starting points considered previously in variable x, compare it with $x(w_i)$ and, if necessary, update x.

The running-time of Algorithm 1 is computed by aggregation of the time of sorting of the starting points and the time is spent on processing of the convex vertices, hence it takes $O(n + k \log k)$ time.

```
Algorithm 1: Initialization($P, w, k$)
  Input: $P$ (a staircase polygon with $n$ vertices), $w$ (a list of $k$ starting points of the watchmen), $k$ (the
         number of watchmen)
  Output: The initial lists $L^I$ and $R^I$, where $L^I_i$ and $R^I_i$ are indices of the vertices of $P$ that represents the
          left and the right bounds of watchman $i$. The pivot points and their followers. Initial values of
          $next$, $prev$, $parent$, $last$, $d^+$, and $d^-$.
  Sort the list w        ▷ $w_i < w_j$ if $y(w_i) < y(w_j)$ or $\big((y(w_i) = y(w_j)$ and $x(w_i) > x(w_j)\big)$
  $x \leftarrow -\infty$, $i \leftarrow 1$
  $p \leftarrow 1$                ▷ $p$ is the index of the last pivot point
  $c \leftarrow 2$                ▷ to enumerate convex vertices
  $prev(p) \leftarrow 0$
  $L^I_p \leftarrow c$            ▷ The first case of Equation 5
  for $i \leftarrow 1$ to $k$ do
      if $x(w_i) \leq x$ then
                                  ▷ $w_i$ is a dominated point
          if $d^+_p = null$ then
              $d^+_p = w_i$
          end
          if $(d^-_p = null)$ or $(x(w_p) - x(w_i) < x(w_p) - x(d^-_p))$ then
              $d^-_p = w_i$
          end
          $parent(i) \leftarrow p$
      else
          $c' \leftarrow c$
          while $c < n - 2$ and $w_i$ does not see $P_c$ do
              $c \leftarrow c + 2$        ▷ Find the first convex vertex that $w_i$ can see
          end
          if $i > 1$ and $c \neq c'$ and $c \neq c + 2$ then
              $next(p) \leftarrow i$ , $prev(i) \leftarrow p$      ▷ $w_i$ is a pivot point
              $p \leftarrow i$            ▷ save the last pivot point index in $p$
              $L^I_p \leftarrow c$        ▷ The second case of Equation 5
          end
          $parent(i) \leftarrow p$, $last(p) \leftarrow i$, $x \leftarrow x(w_i)$
          while $c < n - 2$ and $w_i$ sees $P_{c+2}$ do
              $c \leftarrow c + 2$        ▷ Find the last convex vertex that $w_i$ can see
          end
          $R^I_p \leftarrow c$            ▷ The second case of Equation 6
      end
  end
  $R^I_p \leftarrow n - 2$                ▷ The first case of Equation 6
  $next(p) \leftarrow 0$
  return $L^I$ and $R^I$
```

4.4. The Min-Max Fixed k-watchman Routes Algorithm

In this section, we present a greedy algorithm for optimizing the min-max criterion to solve the fixed k-WRP in staircase polygons. The proposed algorithm finds an optimal solution under the min-max criterion in polynomial time. The pseudo code of our algorithm, called Fixed-k-Watchman, is given in Algorithm 2.

First, we calculate the initial values of L^I and R^I using Algorithm 1. Then by Equation (8), We produce all possible routes for all watchmen in a list called *routes* based on L^I and R^I. Then iteratively, we select a route r from *routes*, such that $|r| < min$, where min is the current minimum value of min-max, initialized with ∞. We assume that r is a route for a watchman (say watchman s) in our solution with maximum length, thus, other watchmen should have same or smaller route length. By fixing the left and the right bounds of watchman s, we can find the left and the right bounds of the other watchmen using Equations (9) and (10). If the current solution in L and R guards all convex vertices on the chain (see the relations provided in the begining of the Section 4), then the optimal solution in L^o and R^o will be updated by L and R, respectively.

Theorem 4.3. *The Fixed-k-Watchman algorithm finds an optimal solution.*

Proof. The Fixed-k-Watchman generates all available routes for all watchmen in line 3 of the algorithm by Equation 8. In line 7, the algorithm minimizes the maximum route length in all available solutions. We calculate bounds of other pivot watchmen by for loops in lines 13 and 19 such that their route length is smaller than the current maximum route by Equations (9) and (10) (r has maximum length in current solution). If we reach a solution that guards vertex 2 by w_1 and vertex $n-2$ by the watchman standing at the last pivot point (e.g $w_{parent(k)}$) then we update min, L^o and R^o. Therefore, the Fixed-k-Watchman algorithm finds an optimal solution. ☐

Theorem 4.4. *The time-complexity of the Fixed-k-Watchman algorithm is $O(n^2 \cdot k^2 \cdot \log(n))$, where k is the number of guards and n is the number of vertices of the input polygon.*

Proof. The Fixed-k-Watchman algorithm given in Algorithm 2, involes $O(n + k \cdot \log k)$ time in line 2 and $O(k \cdot n^2)$ in line 4, because as stated in the third case of Equation 7, there are $O(n)$ choices for a and b, thus, $O(n^2)$ possible routes exist for each watchman. On the other hand, each loop statements in lines 13 and 19 has $O(k)$ operations. Finally, we can use a binary search in Equations (9) and (10), because the length of routes for a watchman is in increasing and/or decreasing order when the left or right bounds changes, then the operations in lines 15 and 21 require at most $O(\log n)$ time. Thus, the total time of Fixed-k-Watchman is $O(n^2 \cdot k^2 \cdot \log(n))$. ☐

$$Routes(i) = \begin{cases} \{r(i, 2, n-2)\ \}, & \text{if } prev(i) = 0 \textit{ and } next(i) = 0 \\ \{r(i, 2, 2b) \mid b \in [\frac{R_i^I}{2}, \frac{L^I_{next(i)}-2}{2}]\}, & \text{if } prev(i) = 0 \textit{ and } next(i) \neq 0 \\ \{r(i, 2a, n-2) \mid a \in [\frac{R^I_{prev_i}+2}{2}, \frac{L_i^I}{2}]\}, & \text{if } prev(i) \neq 0 \textit{ and } next(i) = 0 \\ \{r(i, 2a, 2b) \mid a \in [\frac{R^I_{prev_i}+2}{2}, \frac{L_i^I}{2}] \wedge\ b \in [\frac{R_i^I}{2}, \frac{L^I_{next(i)}-2}{2}]\}, & \text{otherwise,} \end{cases} \tag{7}$$

$$AllRoutesOf(i) = \begin{cases} Routes(i), & \text{if } parent(i) = i \\ \emptyset, & \text{otherwise,} \end{cases} \tag{8}$$

$$L_i = \begin{cases} 0, & \text{if } i = 1 \textit{ and } |r(i, 2, R_i)| > |r| \\ 2, & \text{if } i = 1 \textit{ and } |r(i, 2, R_i)| \leq |r| \\ \min\limits_{\frac{R^I_{prev(i)}+2}{2} \leq x \leq \frac{L_i^I}{2}} \{2x \mid |r(i, 2x, R_i)| \leq |r|\}, & \text{otherwise,} \end{cases} \tag{9}$$

$$R_i = \begin{cases} 0, & \text{if } next(i) = 0 \textit{ and } |r(i, L_i, n-2)| > |r| \\ n-2, & \text{if } next(i) = 0 \textit{ and } |r(i, L_i, n-2)| \leq |r| \\ \max\limits_{\frac{R_i^I}{2} \leq x \leq \frac{L^I_{next(i)}-2}{2}} \{2x \mid |r(i, L_i, 2x)| \leq |r|\}, & \text{otherwise.} \end{cases} \tag{10}$$

Algorithm 2: Fixed-k-Watchman(P, n, w, k)

```
Input: The polygon $P$ with $n$ vertices, the watchman list $w$ of $k$ starting points of watchmen
Output: $min$ as min-max value and the lists $L^o$ and $R^o$ as optimal solution, where $L_i^o$ and $R_i^o$ are indices
        of the vertices of $P$ that represent the left and the right bounds of watchman $i$.
$min \leftarrow \infty$
$L^I, R^I \leftarrow Initialization(P, w, k)$
$routes \leftarrow \cup_{i=1}^{k} AllRoutesOf(i)$                    ▷ compute all possible routes by Equation 8
while $routes \neq \emptyset$ do
    $r \leftarrow$ select a route from $routes$;
    $routes \leftarrow routes - \{r\}$                               ▷ excluding $r$ from $routes$ set
    if $|r| < min$ then
                                                                     ▷ is feasible?
        $L_1 \leftarrow R_{parent(k)} \leftarrow 0$
        $i \leftarrow i(r)$
        $L_i \leftarrow L(r)$, $R_i \leftarrow R(r)$
        $p \leftarrow prev(i)$
        while $p > 0$ and $L_{next(p)} > 0$ do
            $R_p \leftarrow L_{next(p)} - 2$
            $L_p \leftarrow$ value of Equation 9
            $p \leftarrow prev(p)$
        end
        $p \leftarrow next(i)$
        while $p > 0$ and $R_{prev(p)} > 0$ do
            $L_p \leftarrow R_{prev(p)} + 2$
            $R_p \leftarrow$ value of Equation 10
            $p \leftarrow next(p)$
        end
        if $L_1 = 2$ and $R_{parent(k)} = n - 2$ then
                                                                     ▷ there is a solution?
            $min \leftarrow |r|$, $R^o \leftarrow R$, $L^o \leftarrow L$  ▷ replace optimal solution
        end
    end
end
return $min$, $L^o$, $R^o$                                           ▷ optimal solution
```

5. Conclusions

This paper examines the fixed k-watchman routes problem in a staircase polygon, considering the min-max criterion. In the fixed version of the problem the starting points of the watchmen are given. We have presented a greedy algorithm for optimizing the min-max criterion, where the maximum route length is minimized. Our algorithm takes $O(n^2 \cdot k^2 \cdot \log n)$ time, where n and k are the number of polygon vertices and watchmen. As future works, we suggest improving the time-complexity. Additionally, it would be beneficial to solve the problem by considering the min-sum criterion.

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