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Original Article

Homology groups and decomposition of the game complex

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ABSTRACT: In this paper, we introduce a novel simplicial complex named Game Complex for finite non-cooperative games in the strategic form. We prove that the number of Nash equilibrium in non-cooperative games with more than two players is the rank of the first homology group of the game complex. Furthermore, we give a decomposition of the game complex.

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1. Introduction

Game theory is a theoretical explanation for depicting social situations for competing players. Mathematician John von Neumann and economist Oskar Morgenstern are considered as the key pioneers of game theory, [5]. The first significant extension of the von Neumann and Morgenstern work was provided by mathematician John Nash. The key point in the game theory is a player's profit which depends on the strategy implemented by the other players. The players can not increase the payoff in the Nash equilibrium by changing unilateral decisions. Game theory has a wide range of applications, including psychology, evolutionary biology, war, politics, economics, and commerce. Game theory revolutionized economics by addressing fundamental problems in previous mathematical economic models. In business, game theory is useful for modeling competitive behaviors between agents, [3]. Ozan Candogan and his coworkers represent a new flow of preferential structure in finite strategic games, [1]. This representation decomposes an arbitrary game into a potential component, a harmonic component and a non-strategic component, each of which has a distinctive feature. The distance of an arbitrary game to a set of potential games can be defined by this canonical orthogonal decomposition. Another concept used in this article is homology. Generally, a sequence

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of algebraic objects, such as groups or modules of Abelian groups, is related to other mathematical objects, such as topological spaces by homology, [8]. Homology groups were originally defined in algebraic topology. One can distinguish two shapes by examining their holes in homology groups, [4]. Connectivity of a space can be formalized by homology in quantitative manner, [2]. However, less of the topological information can be captured by homology formalism. This paper is organized as follows. In Section 2, we present the relevant game theoretic background [1], and provide a representation of the abstract simplicial complex, [2], and the geometric simplicial complex, [6]. In the following of this section, homology and cohomology groups and their features are introduced, [4]. In Section 3, we work in the framework of non-cooperative games with pure strategies, where the game is presented as a finite dimensional simplicial complex $\mathcal{K}_E$. The utility functions (or alternately payoff functions) are included in the Game Complex $\mathcal{K}_{\mathcal{G}}$, and design a subcomplex, called the Utility complex $\mathcal{K}_{\mathcal{U}}$. Then we recognize and count the number of the Nash equilibrium in the Game Complex. Nash equilibrium is considered one of the most important concepts of game theory, which attempts to determine mathematically and logically the actions that participants of a game should take to secure the best outcomes for themselves. The existence, and the number of Nash Equilibria are analyzed by fairly simple Algebraic Topology, namely the features of the first homology group of the Utility complex. We prove that the existence of Nash equilibria is associated to the rank of the first homology group of the Utility complex for the games with the number of players greater than two. In section 4, we attempt to decompose our game into three subspaces, which we refer to as the potential, harmonic, and nonstrategic components.

2. Preliminaries

A (non-cooperative) strategic-form finite game is given by the tuple $G = \langle M, E^m, u^m\rangle$, where $M = \{1, \ldots, n\}$ is a finite set of players, E^m is a finite set of strategies (or actions) of player m, and $u^m : E \to R$ is the utility function for every $m \in M$. We use the notation $p^m \in E^m$ for a strategy of mth player. A collection of players' strategies is given by $p = \{p^m\}_{m\in M}$ which is known as a strategy profile. A collection of strategies for all players except the m-th one is indicated by $p^{-m} \in E^{-m}$. Formally, a strategy profile $p = (p^m, p^{-m})$ is a (pure) Nash equilibrium if

$$u^m(p^m, p^{-m}) \geq u^m(q^m, p^{-m}),$$

for every $q^m \in E^m$ and $m \in M$. The collection $\mathcal{A}$ is an abstract simplicial complex on S if $\mathcal{A}$ is a family of finite subsets of S, and the following conditions are satisfied,

1. If $X \in \mathcal{A}$, and $Y \subseteq X$, then $Y \in \mathcal{A}$; and
2. $\{v\} \in \mathcal{A}$ for all $v \in S$.

Each element of $\mathcal{A}$ is called a simplex and the set S is the vertex set of $\mathcal{A}$ and is denoted by $\mathcal{A}^{(0)}$. The dimension of the simplex $\sigma \in \mathcal{A}$ is equal to one less than the number of its vertices, that is, $|\sigma| -1$. A simplex τ is a face of σ if $\tau \subseteq \sigma$, and it is a proper face if $\tau \subset \sigma$. A **facet** σ in a complex $\mathcal{A}$ is a simplex if it is not a proper face of any other simplex in $\mathcal{A}$. The maximum dimension of any of the facets in a complex $\mathcal{A}$ is the dimension of the complex $\mathcal{A}$. If every simplex of $\mathcal{B}$ is also a simplex of $\mathcal{A}$, then the complex $\mathcal{B}$ is a subcomplex of $\mathcal{A}$. The affine combination of a finite set of points $X = \{x_0, \ldots, x_n\}$ in R^d is the weighted sum

$$y = \sum_{i=0}^{n} t_i.x_i,$$

where the coefficients t_i satisfy $t_1+t_2+\cdots+t_n = 1$, and are called the barycentric coordinates of y with respect to X. When all barycentric coordinates t_i satisfy $t_i \geq 0$, y is a convex combination of X. The set of convex combinations of X is the **convex hull** of X, and is denoted by conv$X$. The set X is affinely independent if there is no point in the set X which can be expressed as an affine combination of the others. The standard n-simplex is the convex hull of the $n + 1$ points in R^{n+1} with coordinates $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$. Generally, a geometric n-simplex, or a **geometric simplex** of dimension n, is the convex hull of any set of $n+1$ affinely independent points in R^d (in particular, we must have $d \geq n$). A **geometric simplicial complex** $\mathcal{G}$ in R^d is defined as a collection of geometric simplices, provided that any face of a $\sigma \in \mathcal{G}$ is also in $\mathcal{G}$, and for all $\sigma, \tau \in \mathcal{G}$, their intersection $\sigma \cap \tau$ is a face of each of them [6]. We can now define the underlying abstract simplicial complex $A(\mathcal{G})$ of a geometric simplicial complex $\mathcal{G}$. The vertices of $A(\mathcal{G})$ is the union of all the sets of vertices of the simplices of $\mathcal{G}$; The set $\{v_0, \ldots, v_n\}$ is a simplex of $A(\mathcal{G})$ if $\sigma = \text{conv}\{v_0, \ldots, v_n\}$ is a simplex of $\mathcal{G}$. In the opposite direction, if an **abstract simplicial complex** $\mathcal{A}$ is given with finitely many vertices, then there exist many geometric simplicial complexes $\mathcal{G}$, such that $A(\mathcal{G}) = \mathcal{A}$. The simplest construction is the standard form described in the following. Suppose that the abstract simplicial complex $\mathcal{A}$ has d vertices; correspond the standard simplices σ in R^d to the sets in the set family $\mathcal{A}$. We call $\mathcal{G}$ a **geometric realization** of $\mathcal{A}$. We will see that many of the notions defined for abstract simplicial complexes generalize in a straightforward way to geometric complexes. Geometric Realization Theorem in [2] proves that every abstract simplicial complex of dimension d has a geometric realization in R^{2d+1}. According to above descriptions, as long as there is no ambiguity, we consider the geometric and abstract simplicial complex to be the same.

Definition 2.1 ([8]). *A geometric complex $\mathcal{B}$ is called a subdivision of a geometric complex $\mathcal{A}$ if the following two conditions,* (1) *and* (2)*, are satisfied.*

1. $|\mathcal{A}| = |\mathcal{B}|$*;*
2. *Each simplex of $\mathcal{B}$ is contained in a simplex of $\mathcal{A}$.*

Suppose $\sigma = \{s_0, \ldots, s_n\}$ is an n-simplex, the barycenter of σ is defined by the point b,

$$b = \sum_{i=0}^{n} \frac{s_i}{n+1},$$

in the interior of σ. Simplest subdivision is the stellar subdivision. Given a complex G, the complex stellar G is constructed inductively over the facets of G. We insert a barycenter in each facet of G, then connect this barycenter to all the vertices of the facet.

Assume that $\mathcal{K}$ is a simplicial complex with dimension d. A d-chain can be defined by a formal sum of d-simplices in $\mathcal{K}$. In addition, we can add two d-chains component wise, like polynomials[2]. The group of d-chains is formed by the addition operation and denoted as $C_d = C_d(\mathcal{K})$. The boundary of a d-simplex can be defined as the sum of its $(d-1)$-dimensional faces. Assume that $\sigma = \{u_0, u_1, \ldots, u_d\}$ is the simplex spanned by the listed vertices, its boundary is

$$\partial_d \sigma = \sum_{j=0}^{d} \{u_0, \ldots, \hat{u_j}, \ldots, u_d\}, \tag{1}$$

where the hat indicates that u_j is omitted.
The chain complex is the boundary homomorphism connects the sequence of chain groups.

$$\cdots \to C_{d+1} \xrightarrow{\partial_{d+1}} C_d \xrightarrow{\partial_d} C_{d-1} \to \cdots.$$

A d-chain with empty boundary, $\partial c = 0$, is called a d-cycle. The kernel of the d-th boundary homomorphism is the group of d-cycles, $Z_p = \ker \partial_p$. A d-chain which is the boundary of a $(d+1)$-chain, $c = \partial\sigma$ with $\sigma \in C_{d+1}$, is called a d-boundary. The image of the $(d+1)$th boundary homomorphism, $B_d = \mathrm{im}\partial_{d+1}$, is the group of d-boundaries. Each boundary of a $(d+1)$-chain is a d-cycle. that is, $B_d \subset Z_d$.

Definition 2.2 ([2]). *The d-th* **homology group** $H_d = H_d(\mathcal{K})$ *is defined by*

$$H_d = Z_d/B_d,$$

which is the d-th cycle group modulo the d-th boundary group. The **rank** *of H_d is $\beta_d = rank H_d = rank Z_d - rank B_d$.*

Theorem 2.3 ([8]). *Suppose that $\{C_d, \partial_d\}$ is a chain complex. Then for each d there are subgroups U_d, V_d, W_d of C_d such that*

$$C_d = U_d \oplus V_d \oplus W_d,$$

where W_d consists of all elements $c_d \in C_d$ such that some nonzero multiple of c_p belongs to B_d, and $\partial_d(U_d) \subset W_{d-1}$ and $\partial_d(V_d) = 0$ and $\partial_d(W_d) = 0$.

Assume that $\mathcal{K}$ be a simplicial complex. A d-cochain is a a homomorphism $\phi : C_d \to G$, where $G = Z_2$, and the group of d-cochains, $C^d = Hom(C_d, G)$, is consists of the d-dimensional cochains. Now, we can define the coboundary map

$$\delta^{d-1} : Hom(C_{d-1}, G) \to Hom(C_d, G),$$

or $\delta : C^{d-1} \to C^d$. In addition, $\delta o \delta : C^{d-1} \to C^{d+1}$ is the zero homomorphism. The coboundary map is a dual homomorphism of the boundary map, and its kernel is the group of cocycles and its image is the group of coboundaries, that is,

$$Z^d = \ker \delta^d, \qquad B^d = \mathrm{im}\delta^{d-1}.$$

The quotient of the d-th cocycle group modulo divided by the d-th coboundary group, $H^d = Z^d/B^d$, for all d, is called d-th **cohomology group**, [2].

3. The Nash Equilibrium in a Game Complex

In this section, we are going to introduce a simplicial complex called a game complex. After introducing our vertices and simplices, we pay a special attention to a subcomplex of the game complex we have created. At the end, we focus on the first homology group of the game complex and its subcomplex introduced; and the 1-cycles of this subcomlpex are our solution to find the Nash Equilibrium. Our goal is to construct game complex $\mathcal{K}_G$, so let $G = \langle M, E^m, u^m \rangle$ be a game and $E = \Pi_{m \in M} E^m$ be the set of strategy profiles. The set $\sigma_{p^{-m}} = \{(q^m, p^{-m}) \mid q^m \in E^m\}$ is called a **comparable strategy profile** for every $m \in M$ and $p^{-m} \in E^{-m}$.

Example 3.1. *Let G be a game with three players for which every player has two options, Ann has the moves A_1 and A_2, Beth has B_1 and B_2, and Cindy has C_1 and C_2. Assume that $p = (A_1, B_1, C_1)$, then*

$$\sigma_{p^{-1}} = \{(A_1, B_1, C_1), (A_2, B_1, C_1)\},$$
$$\sigma_{p^{-2}} = \{(A_1, B_1, C_1), (A_1, B_2, C_1)\},$$
$$\sigma_{p^{-3}} = \{(A_1, B_1, C_1), (A_1, B_1, C_2)\}.$$

Suppose $p = (p^m, p^{-m})$ is an arbitrary strategy profile of the strategy profiles set E, we define $\mathcal{K}_p$ as the collection of all

$$\sigma_{p^{-m}} = \{(q^m, p^{-m}) \mid q^m \in E^m\},$$

and its nonempty subsets, for every $m \in M$. Since $\mathcal{K}_p$ is satisfied in the following properties

1. For every $m \in M$ and $p = (p^m, p^{-m})$

$$\sigma_{p^{-m}} = \{(q^m, p^{-m}) \mid q^m \in E^m\},$$

   is an element (a facet) in $\mathcal{K}_p$.

2. Every nonempty subset of $\sigma_{p^{-m}} = \{(q^m, p^{-m}) \mid q^m \in E^m\}$ is an element (a face) in $\mathcal{K}_p$.
3. The vertex set of $\mathcal{K}_p$ consists of all comparable strategy profiles of p in E. (The vertex set of $\mathcal{K}_p$ is the union of the one-point elements of $\mathcal{K}_p$. We make no distinction between the vertex v in $\mathcal{K}_p$ and the 0-simplex $\{v\} \in \mathcal{K}_p$),

so it is clear that $\mathcal{K}_p$ is an abstract simplicial complex on all comparable strategy profiles of p in E. Let p and q be the m-comparable strategy profiles. The sets $\sigma_{p^{-m}}$ and $\sigma_{q^{-m}}$ are equal; $\sigma_{p^{-m}} = \sigma_{q^{-m}}$. So $\sigma_{p^{-m}}$ and $\sigma_{q^{-m}}$ are simplices in both $\mathcal{K}_p$ and $\mathcal{K}_q$. The dimension of $\mathcal{K}_p$ is the maximum dimension of any of its facets. Since for every $m \in M$, the simplex $\sigma_{p^{-m}}$ is a facet in the complex $\mathcal{K}_p$; we have

$$\dim \mathcal{K}_p = Max\{\dim \sigma_{p^{-m}} \mid \forall m \in M\}.$$

Example 3.2. *Let G be a given game in the example 3.1, we have*

$$\mathcal{K}_p = \{\{(A_1, B_1, C_1)\}, \{(A_2, B_1, C_1)\}, \{(A_1, B_2, C_1)\}, \{(A_1, B_1, C_2)\},$$
$$\sigma_{p^{-1}}, \sigma_{p^{-2}}, \sigma_{p^{-3}}\},$$

and

$$\dim \sigma_{p^{-1}} = \dim \sigma_{p^{-2}} = \dim \sigma_{p^{-3}} = 1,$$
$$\dim \mathcal{K}_p = 1.$$

Let $\mathcal{K}_E$ be the collection of all elements of $\mathcal{K}_p$ for every $p \in E$, the collection $\mathcal{K}_E = \cup_{p \in E} K_p$ is an abstract simplicial complex and every $\sigma_{p^{-m}}$ is a facet of $\mathcal{K}_E$. Furthermore, we define $\mathcal{K}_U$ as follows,

1. Every element of $U = \{u_p^m \mid \forall \sigma_{p^{-m}} \in K_E\}$ is a vertex of $\mathcal{K}_U$;
2. $\{u_p^m, u_p^{m+1}\}$ is a 1-simplex of $\mathcal{K}_U$ if

$$u^m(p) - u^m(q) \geq 0 \quad \forall q \in \sigma_{p^{-m}}, \quad where \quad p = (p^m, p^{-m}),$$

   and

$$u^{m+1}(p) - u^{m+1}(q) \geq 0 \quad \forall q \in \sigma_{p^{-(m+1)}}, \quad where \quad p = (p^{m+1}, p^{-(m+1)}).$$

3. $\{u_p^1, u_p^n\}$ is a 1-simplex of $\mathcal{K}_U$ if

$$u^1(p) - u^1(q) \geq 0 \quad \forall q \in \sigma_{p^{-1}}, \quad where \quad p = (p^1, p^{-1}),$$

and

$$u^n(p) - u^n(q) \geq 0 \quad \forall q \in \sigma_{p^{-n}}, \quad where \quad p = (p^n, p^{-n}).$$

If p and q are m-comparable strategy profiles, that is, $\sigma_{p^{-m}} = \sigma_{q^{-m}}$, then there is no distinct between u_p^m and u_q^m. Clearly, $\mathcal{K}_U$ is an abstract simplicial complex, we call it **Utility simplicial complex**. All simplices of $\mathcal{K}_U$ are of dimension at most one, so each simplex σ in $\mathcal{K}_U$ is either a vertex or a 1-simplex .

Definition 3.1. *We say that the collection $\mathcal{K}_G$ is a game complex if the following conditions are satisfied,*

1. *Every element of $\mathcal{K}_E$ and $\mathcal{K}_U$ is an element (a simplex) in $\mathcal{K}_G$.*
2. *$\{u_p^m\} \cup \sigma_{p^{-m}}$ and all of its nonempty subsets are elements (faces) of $\mathcal{K}_G$ provided the degree of u_p^m is greater than zero.*

The simplex $\{u_p^m\} \cup \sigma_{p^{-m}}$ of $\mathcal{K}_G$ is called the **cone** on $\sigma_{p^{-m}}$ with vertex u_p^m, and

$$\dim \mathcal{K}_G = \dim \mathcal{K}_E + 1.$$

Since $\mathcal{K}_U$ is a subcollection of $\mathcal{K}_G$ and it is itself a complex. So, $\mathcal{K}_U$ is a subcomplex of $\mathcal{K}_G$. We have an inclusion map from the underlying space of $\mathcal{K}_U$ to that of $\mathcal{K}_G$ and therefore we have an induced homomorphism, $f : H_1(\mathcal{K}_U) \to H_1(\mathcal{K}_G)$.

Proposition 3.2. *Suppose that γ be a nontrivial homology class in $H_1(\mathcal{K}_U)$, then γ does not merge with another homology class in $H_1(\mathcal{K}_G)$ and stay persistent through the induced homomorphism, $f : H_1(\mathcal{K}_U) \to H_1(\mathcal{K}_G)$.*

Proof. Since there are no 2-simplices in $\mathcal{K}_U$, according to the definitions of the first Homology group of a complex and the subgroups, $B_1(\mathcal{K}_U)$ is trivial. Therefore,

$$H_1(\mathcal{K}_U) = Z_1(\mathcal{K}_U).$$

The structure of the simplicial complex $\mathcal{K}_G$ requires that 1-simplices in $\mathcal{K}_U$ are not the face of any 2-simplex in $\mathcal{K}_G$, and every 1-cycle c which belongs to $Z_1(\mathcal{K}_U)$ is an element of $Z_1(\mathcal{K}_G)$, and does not merge with the elements in $B_1(\mathcal{K}_G)$. We easily conclude that if $c + B_1(\mathcal{K}_U)$ is a homology class in $H_1(\mathcal{K}_U)$, then $c + B_1(\mathcal{K}_G)$ is a homology class in $H_1(\mathcal{K}_G)$. □

Example 3.3. *Let G be a game in the example 3.1. Consider the following payoff tables.*

	A1	
Strategies	C1	C2
B1	0,2.1,0	-1,1.1,0.1
B2	1,0,-1	0,1,1.1

	A2	
Strategies	C1	C2
B1	0.1,1.1,1	1.1,0.1,-0.9
B2	-0.9,1,0	0.1,2,0.1

In figure 1, we illustrate $\mathcal{K}_E$ associated to strategy profiles $q = (A_2, B_1, C_1)$ and $p = (A_1, B_1, C_1)$. The strategy profile $q = (A_2, B_1, C_1)$ is one of the equilibrium points, so there is a 1-cycle formed in the $\mathcal{K}_U$ related to q.

In figure 2, we show how vertices u_q^1, u_q^2, u_q^3 connect by red edges to each other in $\mathcal{K}_U$. Also, a view of the simplicial complex $\mathcal{K}_G$ is shown.

The homology group $H_1(\mathcal{K}_U)$ of the Utility complex is of a great importance in our studying, because it helps us to find the NE points.

Theorem 3.3. *Let $\mathcal{K}_U$ be our Utility complex and assume the number of players in the game is greater than two. If the rank of $H_1(\mathcal{K}_U)$ is not zero, then the generators of $H_1(\mathcal{K}_U)$ are 1-cycles of length n as follows:*

$$\{u_p^1, u_p^2\} + \{u_p^2, u_p^3\} + \cdots + \{u_p^{n-1}, u_p^n\} + \{u_p^n, u_p^1\}.$$

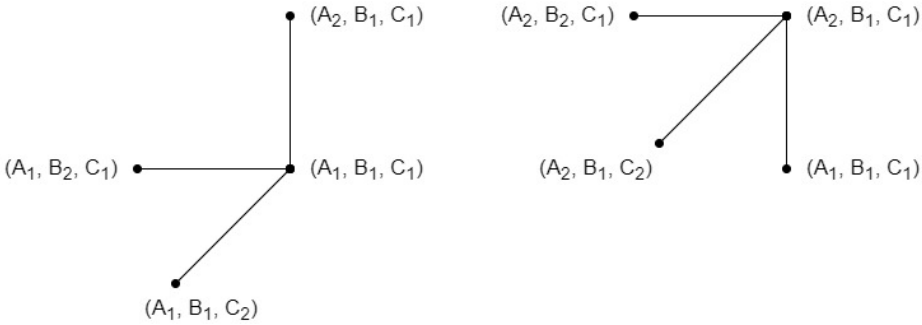

Figure 1: $\mathcal{K}_p$ and $\mathcal{K}_q$

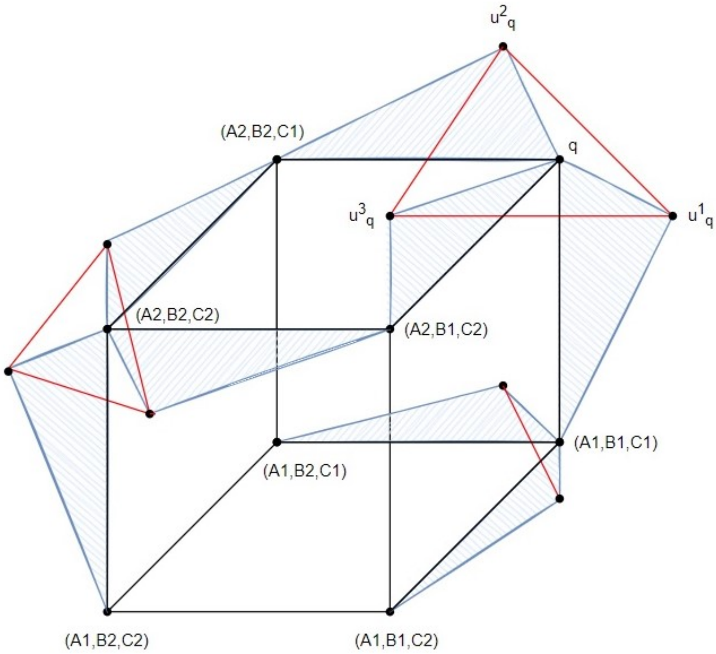

Figure 2: Black and red edges belong to $\mathcal{K}_E$ and $\mathcal{K}_U$, respectively. Blue triangles are related to the cones with vertices in $\mathcal{K}_U$ which are defined in definition 3.1.

Proof. Since there are no 2-simplices in $\mathcal{K}_U$, $B_1(\mathcal{K}_U)$ is trivial. Thus, by the definition of the first homology group, we have:

$$H_1(\mathcal{K}_U) = Z_1(\mathcal{K}_U).$$

Furthermore, according to the definition of the Utility complex, we conclude that $Z_1(\mathcal{K}_U)$ is generated by the 1-cycles:

$$\{u_p^1, u_p^2\} + \{u_p^2, u_p^3\} + \cdots + \{u_p^{n-1}, u_p^n\} + \{u_p^n, u_p^1\}.$$

□

Corollary 3.4. *If the number of players in the game is greater than two and the rank of $H_1(\mathcal{K}_U)$ is not zero, then there exists at least one Nash Equilibrium (NE) in the game. Moreover, the number of Nash Equilibria is equal to the rank of $H_1(\mathcal{K}_U)$.*

Proof. The dimension of the Utility complex $\mathcal{K}_U$ is at most one, and each 1-cycle is created by the concept of Nash Equilibrium. According to the definition of the Utility complex, every profile p in a 1-cycle of $H_1(\mathcal{K}_U)$ represents a Nash Equilibrium of the game. On the other hand, every 1-cycle is a generator of $H_1(\mathcal{K}_U)$. Hence, the rank of $H_1(\mathcal{K}_U)$ is equal to the number of 1-cycles in the Utility complex $\mathcal{K}_U$ and the number of N.E. profiles as well. □

Example 3.4. *In figure 2 of the example 3.3, we draw $\mathcal{K}_{\mathcal{G}}$ and illustrate that red edges are 1-simplices of $\mathcal{K}_U$. Since there are not any 2-simplices in $\mathcal{K}_U$, $B_1(\mathcal{K}_U)$ is trivial and,*

$$H_1(\mathcal{K}_U) = Z_1(\mathcal{K}_U).$$

$Z_1(\mathcal{K}_U)$ is consisting of two 1-cycles, $\{u_q^1, u_q^2\} + \{u_q^2, u_q^3\} + \{u_q^3, u_q^1\}$ and $\{u_s^1, u_s^2\} + \{u_s^2, u_s^3\} + \{u_s^3, u_s^1\}$, where $q = (A_2, B_1, C_1)$ and $s = (A_2, B_2, C_2)$ are N.E profiles of the game; these two 1-cycles are the generator of $H_1(\mathcal{K}_U)$. As can be seen, rank of $H_1(\mathcal{K}_U)$ is equal to the number of N.E profiles of the game. □

4. The game decomposition

In non-cooperative games, for any given m-comparable strategy profiles p and q, the difference $[u^m(p) - u^m(q)]$ can be identified as their pairwise comparison. According to the definition of d-chains as a function from d-simplices to real numbers, a 1-chain can be considered as the pairwise comparison of the game. Based on the value of the difference $[u^m(p) - u^m(q)]$, we can attribute the value of zero or non-zero to the 1-simplex $\{p, q\}$ of the simplicial complex $\mathcal{K}_E$. Accordingly, we form a 1-chain in $\mathcal{K}_E$ which may serve as a representation of any game.

Definition 4.1. *Let G be a game and $\mathcal{K}_E$ be its simplicial complex. We define an edge flow on $\mathcal{K}_E$ as a 1-chain in the group of $C_1(\mathcal{K}_E)$.*

The pairwise comparisons are uniquely defined for any given game. However, the converse is not true in the sense that there are infinitely many games that correspond to given pairwise comparisons. Games with identical pairwise comparisons share the same equilibrium sets. Thus, we refer to games with identical pairwise comparisons as **strategically equivalent games**. Now, we can apply the above description to the simplicial complex $\mathcal{K}_G$.

Definition 4.2. *Two games are equivalent if and only if they have the same flow representation and simplicial complex $\mathcal{K}_G$.*

The subcomplex $\mathcal{K}_U$ and its connection to the subcomplex $\mathcal{K}_E$ contain information such as the pairwise comparison and the equilibrium set of the game. One can correspond an edge flow to each game by considering the zero or non-zero values of the pairwise comparison of the game. When two games are equivalent, they have identical pairwise comparisons and the same equilibrium sets. Therefore, they present the same edge flows and simplicial complex $\mathcal{K}_G$. Since we define an edge flow on $\mathcal{K}_E$ as a 1-chain in the group of $C_1(\mathcal{K}_E)$ where a 1-chain represents the pairwise comparison of the game and simplicial complex $\mathcal{K}_G$ contains the game information such as comparable strategy profiles, N.E profiles, the number of players, and strategies of each player, then two games can be strategically equivalent. We state below a basic flow-decomposition theorem of a noncooperative game.

Theorem 4.3. *The vector space of edge flows $C_1(\mathcal{K}_E)$ admits an orthogonal decomposition*

$$C_1(\mathcal{K}_E) = U_1 \oplus V_1 \oplus W_1,$$

where $\partial_1(U_1) \subset W_0$ and $\partial_1(V_1) = 0$ and $\partial_1(W_1) = 0$.

Proof. In the previous section, the simplicial complex $\mathcal{K}_E$ is constructed on the strategy profile set E, then $\{C_d(\mathcal{K}_E), \delta_d\}$ is the chain complex associated to $\mathcal{K}_E$. By using theorem 2.3, W_d consists of all elements $c_d \in C_d(\mathcal{K}_E)$ such that some nonzero multiple of c_d belongs to B_d, and $\partial_d(U_d) \subset W_{d-1}$ and $\partial_d(V_d) = 0$ and $\partial_d(W_d) = 0$. Since the equation $mc_d = \partial_{d+1}c_{d+1}$ implies that $\partial c_d = 0$, W_d is a subgroup of Z_d. Suppose that $f_1 + W_d, f_2 + W_d, \dots, f_k + W_d$ is a basis for Z_d/W_d, and $e_1, \dots, e_l$ is a basis for W_d, then $Z_d = V_d \oplus W_d$ and $f_1, \dots, f_k$ is a basis for V_d. Based on the fact that there is a matrix related to boundary map and the normal form of that, there exist the desired bases for U_d and W_{d-1}; therefore, $C_d = U_d \oplus Z_d$. For each d there are subgroups U_d, V_d, W_d of $C_d(\mathcal{K}_E)$ such that

$$C_d(\mathcal{K}_E) = U_d \oplus V_d \oplus W_d.$$

Moreover, in this section, an edge flow is defined as a 1-chain in the group of $C_1(\mathcal{K}_E)$ and we can decompose the edge flow set by theorem 2.3 into three components with which we can examine the game relatively accurate. □

Now, we consider the boundary map ∂ and the coboundary map δ to have another flow-decomposition for a noncooperative game. Notice that δ_d can be simply interpreted as the linear operator $\delta_d : C_d \to C_{d+1}$, which is defined for all d-chains belong to C_d and $(d+1)$-chains in C_{d+1}. The domains and codomains of the operators δ_0, δ_1, ∂_1, ∂_2 are summarize below.

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0,$$

$$C_2 \xleftarrow{\delta_1} C_1 \xleftarrow{\delta_0} C_0.$$

The Laplacian operator Δ_0 and the vector Laplacian Δ_1, which will be used to decompose the edge flows in the game is defined as follows, see [1, 7],

$$\Delta_0 : C_0 \to C_0,$$
$$\Delta_0 = \partial_1 o \delta_0.$$

And,

$$\Delta_1 : C_1 \to C_1,$$
$$\Delta_1 = \partial_2 o \delta_1 + \delta_0 o \partial_1.$$

We can use the above defined operators for the simplicial complex $\mathcal{K}_E$ associated to the game where each vertex corresponds to a strategy profile, and each edge connects two comparable strategy profiles. An edge flow $c_1 \in C_1(\mathcal{K}_E)$ is said to be globally consistent if c_1 corresponds to some $c_0 \in C_0$, that is, $c_1 = \delta_0 c_0$; the 0-chain c_0 is referred to as the potential chain corresponding to c_1. Equivalently, the set of globally consistent edge flows can be represented as $\text{im}(\delta_0)$, and called **potential flows**. By the closeness of δ_0, observe that $\delta_1 c_1 = 0$ for every globally consistent edge flow c_1. We define locally consistent edge flows as those satisfying $\delta_1(c_1) = 0$. Note that $\ker(\delta_1)$ is the set of locally consistent edge flows. The latter subset is generally not equivalent to $\text{im}(\delta_0)$, as there may exist edge flows that are globally inconsistent but locally consistent. We refer to such flows as **harmonic flows**. Note that the operators δ_0, δ_1 are linear operators, thus their image spaces are orthogonal to the kernels of their dual operators ∂_1, ∂_2, that is, $\text{im}(\delta_0) \perp \ker(\partial_1)$ and $\text{im}(\delta_1) \perp \ker(\partial_2)$. The following theorem implies that edge flows of the simplicial complex $\mathcal{K}_E$ of a game can be decomposed into three orthogonal flows, which we refer to as potential, harmonic and nonstrategic flows.

Theorem 4.4. *The vector space of edge flows $C_1(\mathcal{K}_E)$ admits an orthogonal decomposition*

$$C_1(\mathcal{K}_E) = \text{im}(\delta_0) \oplus \ker(\Delta_1) \oplus \text{im}(\partial_2),$$

where $\ker(\Delta_1) = \ker(\delta_1) \cap \ker(\partial_1)$.

Proof. δ_1 is a linear operator on the vector space $C_1(\mathcal{K}_E)$ and $\ker(\delta_1)$ is the subspace of $C_1(\mathcal{K}_E)$, so we have $C_1(\mathcal{K}_E) = \ker(\delta_1) \oplus (\ker(\delta_1))^{\perp}$. Since we have $(\ker(\delta_1))^{\perp} = \text{im}(\partial_2)$, we get $C_1(\mathcal{K}_E) = \ker(\delta_1) \oplus \text{im}(\partial_2)$. On the other hand, we have

$$\begin{aligned}
\ker(\delta_1) &= C_1(\mathcal{K}_E) \bigcap \ker(\delta_1), \\
&= [\ker(\partial_1) \oplus \text{im}(\delta_0)] \bigcap \ker(\delta_1), \\
&= [\ker(\delta_1) \bigcap \ker(\partial_1)] \oplus [\ker(\delta_1) \bigcap \text{im}(\delta_0)].
\end{aligned}$$

Furthermore,

$$\begin{aligned}
[\ker(\delta_1) \bigcap \ker(\partial_1)] &= \ker[\partial_2 o \delta_1 + \delta_0 o \partial_1], \\
&= \ker(\Delta_1),
\end{aligned}$$

and $[\ker(\delta_1) \bigcap \text{im}(\delta_0)] = \text{im}(\delta_0)$, so we get a decomposition for $C_1(\mathcal{K}_E)$ as follows

$$C_1(\mathcal{K}_E) = \text{im}(\delta_0) \oplus \ker(\Delta_1) \oplus \text{im}(\partial_2).$$

□

5. Conclusion

In this paper, we introduced the concept of a Game Complex for finite non-cooperative games in strategic form and demonstrated its utility in identifying Nash equilibria through the rank of the first homology group. By constructing the Utility complex K_U, we were able to translate the problem of finding Nash equilibria into a topological framework.

The main theoretical contribution of our work is the establishment of a direct relationship between the rank of the first homology group $H_1(K_U)$ and the number of Nash equilibria in the game. This result provides a novel and efficient way to count Nash equilibria using algebraic topology. Moreover, our approach offers deeper insights into the structure of strategic games by decomposing the game into potential, harmonic, and non-strategic components, each revealing different aspects of the game dynamics.

Our examples illustrate the practical application of the theoretical results, showing how the Utility complex can be used to identify Nash equilibria in specific games. These examples highlight the simplicity and elegance of our approach, as well as its potential for broader application in various fields where game theory is relevant.

Future work may involve extending this framework to dynamic games and games with incomplete information, exploring the interplay between topological properties and game dynamics. Additionally, computational approaches for constructing and analyzing utility complexes for large-scale games could be investigated to enhance the practical applicability of our theoretical findings.

We believe that the integration of algebraic topology with game theory opens up promising avenues for further research and invites collaboration across disciplines to deepen our understanding of strategic interactions in various contexts.

References

[1] O. Candogan, I. Menache, A. Ozdaglar, and P. A. Parrilo, *Flows and decompositions of games: harmonic and potential games*, Math. Oper. Res., 36 (2011), pp. 474–503.

[2] H. Edelsbrunner and J. L. Harer, *Computational topology*, American Mathematical Society, Providence, RI, 2010. An introduction.

[3] F. Erhun and P. Keskinocak, *Game theory in business applications*, Management Science and Engineering, Stanford University, Stanford, (2003).

[4] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[5] A. Hayes, *Ultimate guide to game theory: Principles and applications*, 2020. `https://www.investopedia.com/terms/g/gametheory.asp`.

[6] M. Herlihy, D. Kozlov, and S. Rajsbaum, *Distributed computing through combinatorial topology*, Elsevier/Morgan Kaufmann, Waltham, MA, 2014.

[7] X. Jiang, L.-H. Lim, Y. Yao, and Y. Ye, *Statistical ranking and combinatorial Hodge theory*, Math. Program., 127 (2011), pp. 203–244.

[8] J. R. Munkres, *Elements of algebraic topology*, Addison-Wesley Publishing Company, Menlo Park, CA, 1984.

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