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Original Article

A remark on the metric dimension in Riemannian manifolds of constant curvature

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ABSTRACT: We compute the metric dimension of Riemannian manifolds of constant curvature. We define the edge weghited metric dimension of the geodesic graphs in Riemannian manifolds and we show that each complete geodesic graph $G = (V, E)$ embedded in a Riemannian manifold of constant curvature resolves a totally geodesic submanifold of dimension $|V| - 1$.

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1. Introduction

In the Euclidean plane, any set of three non-colinear points $\{x_1, x_2, x_3\}$ is enough to uniquely distinguish all points in the space based on distances. More precisely, if a and b are two different points in the plane, the following vectors are different in R^3.

$$(d(a, x_1), d(a, x_2), d(a, x_3)), \quad (d(b, x_1), d(b, x_2), d(b, x_3)).$$

In general, let (X, d) be a metric space and let R be a non-empty subset of X with the following property:

$$d(x, b) = d(y, b), \forall b \in R \Rightarrow x = y.$$

It is said that R resolves X. The metric dimension $\dim_m(X)$ of (X, d) is, by definition, the cardinality of the smallest set which resolves X (small up to cardinality). First, the author of [2] introduced the concept of resolving set for metric spaces. But, because of many interesting applications in chemistry, biology, robotics, image processing networking problems (see [3, 4, 5, 10, 12]) mostly it attracted the attention of the people who were working on the graph theory. The definition of the metric dimension of a graph is similar, where $d(x, y)$ is the length of the smallest path between vertices x, y.
Returning to the original idea of the metric dimension of a metric space, Bau and Beardon in [1], among other

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results, computed the metric dimension for n-dimensional Euclidean space, spherical space, hyperbolic space and Riemannian surfaces. The authors of [7] presented the generalizations of [1] and computed the metric dimensions of n-dimensional geometric spaces which include the homogeneous Riemannian manifolds as a subclass. In section 2 of the present article, we compute the metric dimension of all Riemannian manifolds of constant curvature.
In Section 3, we consider the graphs embedded in Riemannian manifolds of constant curvature. In the classic graph theory, a graph is considered from set theory point of view. When the geometry is important, the graphs are usually considered as the union of points (vertices) and line segments (edges) joining the points in the Euclidean space $\mathbb{R}^n$. But, motivated by many new applications and also from mathematical points of view, study of graphs with emphasize on the ambient space, has been recently attractive for mathematicians. In such a problems possibility of the embedding of a graph in a manifold is noticeable (see [8, 9, 12]). We consider in the present article the graphs which are embedded in Riemannian manifolds of constant curvature. We introduce a dimension similar to the metric dimension, depended on the length of the edges which are considered to be geodesics of the ambient space. Then, we show that a complete geodesic graph $G = (V, E)$ embedded in a Riemannian manifold of constant curvature resolves and is included in a totally geodesic submanifold of dimension $|V| - 1$.

2. Metric dimension of Riemannian manifolds of constant curvature

We will use the following standard notations, definitions and criterions in differential geometry (see [6]) and topology (see [11]).

1) If M is a complete Riemannian manifold, we will denote by $\tilde{M}$ its universal Riemannian covering manifold (i.e, $\tilde{M}$ is simply connected) with the covering map $p : \tilde{M} \to M$. We recall that p is a local isometry and for each point $x \in M$, $p^{-1}(x)$ which is called the fiber on x, is a disjoint union of points in $\tilde{M}$. The decktransformation group of the covering $p : \tilde{M} \to M$ which we denote by Γ is a group of the isometries of $\tilde{M}$ such that for all $x \in M$, and all $\delta \in \Gamma$, $\delta(p^{-1}(x)) = p^{-1}(x)$. It is known that the fundamental group of M is isomorphic to Γ.
2) If M is a complete Riemannian manifold of constant curvature c, then $\tilde{M}$ is isometric to one of the following manifolds:
   I) If $c = 0$, $\mathbb{R}^n$.
   II) If $c > 0$, $\mathbb{S}^n(c) = \{x \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = \frac{1}{c}, \}$, with the metric induced from $\mathbb{R}^{n+1}$.
   III) If $c < 0$, $\mathbb{H}^n(c) = \{x \in \mathbb{R}^{n+1}, -x_1^2 + x_2^2 + \ldots x_{n+1}^2 = \frac{1}{c}, x_{n+1} > 0\}$, with the metric induced from $\mathbb{R}_1^{n+1}$. This model is called the Lorentzian model of $\mathbb{H}^n(c)$.
3) If a, b are different points on M, then there is a minimizing geodesic γ joining a to b (a curve $\gamma : [0, 1] \to M$, such that $\gamma(0) = a$, $\gamma(1) = b$, $d(a, b)$ is equal to the length of γ).

Remark 2.1. *(1) Let $A = \{a_1, \ldots, a_n\} \subset \mathbb{S}^n(c)$, $c > 0$, and let o be the origin of $\mathbb{R}^{n+1}$. Consider the hyperplane P of $\mathbb{R}^{n+1}$ containing the set $\{o, a_1, \ldots, a_n\}$. Then, $P \cap \mathbb{S}^n(c)$ is a totally geodesic hypersurface in $\mathbb{S}^n(c)$ containing A (a subset of $\mathbb{S}^n(c)$ isometric to $\mathbb{S}^{n-1}(c)$).*
(2) Consider the lorentzian model for $\mathbb{H}^n(c)$. In this model, each totally geodesic hypersurface of $\mathbb{H}^n(c)$ is equal to $P \cap \mathbb{H}^n(c)$ for a hyperplane P of $\mathbb{R}^{n+1}$ which contains the origin. Then, similar to $\mathbb{S}^n(c)$, we can show that each set of n points of $\mathbb{H}^n(c)$ is included in a totally geodesic hypersurface (a subset isometric to $\mathbb{H}^{n-1}(c)$).

Remark 2.2. *If M is a Riemannian manifold of constant curvature and $\dim M = n$, then each subset $A = \{a_1, \ldots, a_n\}$ of M is included in a totally geodesic hypersurface.*

Proof. Denote by $\tilde{M}$ the universal covering manifold of M. Then, $\tilde{M}$ is one of the spaces $\mathbb{R}^n, \mathbb{H}^n(c), c < 0$ or $\mathbb{S}^n(c), c > 0$. Consider the covering map $p : \tilde{M} \to M$ and choose the points $b_i \in p^{-1}(a_i)$. By the above remark, there is a totally geodesic hypersurface $\tilde{K}$ in $\tilde{K}$ such that $\tilde{K} \subset \tilde{M}$. Clearly, $K = p(\tilde{K})$ is the desired hypersurface in M. □

Remark 2.3. *Let $p : \tilde{M} \to M$ be the covering map and let $x, y \in M$ be different points and $\tilde{x} \in p^{-1}(x)$. Choose $\tilde{y}(\tilde{x}) \in p^{-1}(y)$ with the property that $d(\tilde{x}, \tilde{y}(\tilde{x}))$ be the minimum number along the set $\{d(\tilde{x}, z) : z \in p^{-1}(y)\}$. By continuity of the distance function, if $\epsilon > 0$ is sufficiently small, then for each $\tilde{q}$ with $d(\tilde{x}, \tilde{q}) < \epsilon$, $d(\tilde{q}, \tilde{y}(\tilde{x}))$ is minimum along all numbers $d(\tilde{q}, z), z \in p^{-1}(y)$. Thus, we have*

$$d(x, y) = d(\tilde{x}, \tilde{y}(\tilde{x})), \quad d(q, y) = d(\tilde{q}, \tilde{y}(\tilde{x})), q = p(\tilde{q}) \tag{1}$$

Now, if $\{y_1, \ldots, y_m\}$ is a finite subset of M, for each i, there is $\epsilon_i > 0$ such that for all $\tilde{q}$ with $d(\tilde{x}, \tilde{q}) < \epsilon_i$ the property similar to $(*)$ is true (y replaced by y_i). If $\delta = \min\{\epsilon_i : 1 \leq i \leq m\}$, then for all $\tilde{q}$, with $d(\tilde{x}, \tilde{q}) < \delta$ the relations (1) is true (y replaces by each y_i). We denote the set of all points $z \in \tilde{M}$, with the property $d(\tilde{x}, z) < \delta$, by $O(\tilde{x}, \delta, y_1, \ldots, y_m)$.

Corollary 2.4 ([2]). *If M is a Riemannian manifold and y, z are two different points of M, then the following set which is called the bisector of y, z, is a totally geodesic hypersurface of M.*

$$B(y, z) = \{x \in M : d(x, y) = d(x, z)\}.$$

Now, we can prove the following theorem about the metric dimension of Riemannian manifolds of constant curvature.

Theorem 2.5. *If M is a Riemannian manifold of constant curvature, then*

$$\dim_m M = \dim M + 1.$$

Proof. Step 1. $\dim_m M \leq n+1$, $n = \dim M$.
It is enough to show that there is a subset $R = \{x_1, \ldots, x_{n+1}\}$ of different $n+1$ points such that R is not included in any totally geodesic hypersurface of M. Then, by Corollary 2.4, for all different points y, z of M at least one point of R is out of $B(y, z)$ and consequently, R resolves M.
We use induction on $n = \dim M$. If $n = 1$, then the claim is trivial. Suppose that the claim is true for all $k < n$ (each Riemannian manifold of dimension k has a subset of $k+1$ points which is not included in any totally geodesic hypersurface).
Consider a totally geodesic hypersurface E in M. Since $\dim(E) = n - 1 < n$, there is a set of different points $x_1, \ldots, x_n$ of E which has a point out of each totally geodesic hypersurface of E. Let γ be a geodesic starting at $\gamma(0) = x_1$ which is normal to E at x_1. Choose a non zero number t such that $\gamma(t) \neq \gamma(0)$ and put $x_{n+1} = \gamma(t)$. Clearly, $x_{n+1} \notin E$. Because, if $x_{n+1} \in E$, then γ is a geodesic joining two points x_1 and x_{n+1} of E. Since E is totally geodesic, then γ is included in E and this is a contradiction (since γ is normal to E). Now, let D be a totally geodesic hypersurface of M. If $\{x_1, \ldots, x_n, x_{n+1}\} \subset D$, then $\{x_1, \ldots, x_n\} \subset E \cap D$. But, $E \cap D$ is a totally geodesic hypersurface of E and this is a contradiction.
Step 2. $\dim_m M > n$. We show that no set of n points resolves M.
Let $\tilde{M}$ be the universal covering manifold of M and $p : \tilde{M} \to M$ be the covering map. Let $A = \{x, y_1, \ldots, y_{n-1}\} \subset M$. Fix a point $\tilde{x} \in p^{-1}(x)$ and for all $i \in \{1, \ldots, n-1\}$ consider the points $\tilde{y}_i(\tilde{x}) \in p^{-1}(y_i)$ such that $d(\tilde{x}, \tilde{y}_i(\tilde{x}))$ be the minimum number along the set $\{d(\tilde{x}, z) : z \in p^{-1}(y_i)\}$. By Remark 2.2, there is a totally geodesic hypersurface D in $\tilde{M}$ such that

$$\{\tilde{x}, \tilde{y}_1(\tilde{x}), \ldots, \tilde{y}_{n-1}(\tilde{x})\} \subset D.$$

We can choose $a, b \in \tilde{M}$ in such a way that $\tilde{x}$ belongs to the mid point of a minimal geodesic γ parameterized by the arc length joining a to b ($\gamma(0) = a, \gamma(\frac{1}{2}) = \tilde{x}, \gamma(1) = b$). Choose $t > 0$, so small that $\gamma(\frac{1}{2} - t), \gamma(\frac{1}{2} + t)$ belong to $O(\tilde{x}, \delta, y_1, \ldots, y_{n-1})$ (see Remark 2.3). Put $\tilde{c} = \gamma(\frac{1}{2} - t), \tilde{d} = \gamma(\frac{1}{2} + t)$. Clearly, $B(\tilde{c}, \tilde{d}) = D$. (this is because $\tilde{M} \in \{\mathbb{S}^n, \mathbb{R}^n, \mathbb{H}^n\}$). Thus,

$$\{\tilde{x}, \tilde{y}_1(\tilde{x}), \ldots, \tilde{y}_{n-1}(\tilde{x})\} \subset B(\tilde{c}, \tilde{d}) \tag{2}$$

Let $c = p(\tilde{c}), d = p(\tilde{d})$. We show that

$$A = \{x, y_1, \ldots, y_{n-1}\} \subset B(c, d) \tag{3}$$

Then, A will not resolve M and we are done. We get from Remark 2.3, that $d(y_i, c) = d(\tilde{y}_i(\tilde{x}), \tilde{c})$, $1 \leq i \leq n-1$ and similarly,

$$d(y_i, d) = d(\tilde{y}_i(\tilde{x}), \tilde{d}).$$

Since by (2), $d(\tilde{y}_i(\tilde{x}), \tilde{c}) = d(\tilde{y}_i(\tilde{x}), \tilde{d})$, then $d(y_i, c) = d(y_i, d)$. Also, it is clear that $d(x, c) = d(x, d)$. Consequently, we get (3). □

3. Metric dimension of the edge weighted graphs embedded in Riemannian manifolds

An embedding of a graph $G = (V, E)$ on a Riemannian manifold M is a representation G' of G on M, associating the vertices of G with the points of M and the edges of G with the geodesic segments of M. More precisely, there is a set E' of geodesic segments in M and a set V' of points of M and bijections $\sigma_1 : V \to V'$, $\sigma_2 : E \to E'$ such that v_1, v_2 are endpoints of $e \in E$ iff $\sigma_1(v_1), \sigma_1(v_2)$ are endpoints of $\sigma_2(e)$, and for all different edges e_1, e_2 in E, $\sigma_2(e_1)$, $\sigma_2(e_2)$ have no intersection except (possibly) in the endpoints. In what follows, we identify G by G' and we say that G is a geodesic graph in M.
For two vertices x, y, consider a sequence $v_1 = x, v_2, v_3, \ldots, v_k = y$ of vertices such that for all $i \in \{1, \ldots, k-1\}$, v_i is adjacent to v_{i+1}. In the general graph theory, $k - 1$ is the length of the path $x, v_2, v_3, \ldots, y$ and the distance between x and y, $d(x, y)$, is by definition the length of the smallest path joining x to y. The metric dimension of G is defined similar to the metric dimension of metric spaces using the distance function d. Thus, the usual metric dimension of a graph embedded in M is independent to the length of the geodesics. But, from geometric

points of view and because of many possible applications, it will be useful to refine the definition in such a way that the length of the edges be effective. In this direction, we will define a new dimension for the graphs embedded in Riemannian manifolds, similar to the classic definition of the metric dimension, which we call it "the edge weighted metric dimension".

Definition 3.1. *We say that a graph $G = (V, E)$ is edge weighted with the edge weight function Ω if $\Omega : V \times V \to R^+$ is a symmetric function such that $\Omega(u, v) = 0$ when there is no edge between u, v. We use the symbol (G, Ω) to denote a graph G with an edge weight function Ω.*

Definition 3.2. *If (G, Ω) is an edge weighted graph and $L = \{u = u_1, \ldots, u_n = v\}$ is a path from u to v, then the weighted length of L is defined by*

$$\Omega(L) = \sum_{i=1}^{n-1} \Omega(u_i, u_{i+1}).$$

The weighted distance between u, v is by definition

$$d_\Omega(u, v) = \min\{\Omega(L) : L \ \ is \ \ a \ \ path \ \ between \ \ u, v\}.$$

Definition 3.3. *If (G, Ω) is an edge weighted graph, then its weighted metric dimension m is the minimal number of vertexes $\{v_1, \ldots, v_m\}$ such that the following vectors are different*

$$\{(d_\Omega(u, v_1), \ldots, d_\Omega(u, v_m)) : u \in V\}.$$

We use the symbol $\dim_w(G, \Omega)$ for the weighted metric dimension of (G, Ω).

Clearly, the metric dimension of (G, Ω) depends on the weight function Ω. The following corollary shows that in finite graphs, we can choose Ω in such a way that $\dim_w(G, \Omega)$ be equal to 1.

Corollary 3.4. *If G is a connected finite graph, then there is a weight function Ω such that $\dim_w(G, \Omega) = 1$.*

Proof. We use induction on the number of vertices $|V| = n$. The claim is clear if $n = 2$. Suppose that the corollary is true for all graphs with n vertexes. Let $G = (V, E)$ be a graph with $n+1$ vertexes $V = \{v_1, \ldots, v_n, v_{n+1}\}$. Consider the subgraph $G' = (V', E')$ with $V' = \{v_1, \ldots, v_n\}$. By assumption, there is a weight function $g : V' \times V' \to R^+$ such that $\dim_w(G', g) = 1$. Put $\lambda = \max\{d_\Omega(v_i, v_j) : 1 \leq i, j \leq n\}$. Define the map $\Omega : V \times V \to R^+$ as follows: $\Omega(v_i, v_j) = g(v_i, v_j)$ for all $i, j \in \{1, \ldots, n\}$, $\Omega(v_{n+1}, v_j) = 1 + \lambda$ if v_{n+1} is connected to v_j and $\Omega(v_{n+1}, v_j) = 0$ otherwise. It is clear that $\dim_w(G, \Omega) = 1$. □

By the above corollary, if we let Ω be arbitrary, the weighted metric dimension is not useful (it is 1 for a suitable Ω). Thus, the weighted metric dimension is possibly noticeable only when we compute it for a fixed Ω . From a geometric point of view, one of the interesting cases is the case where we use the metric of the ambient space in the weight function. If $G = (V, E)$ is a graph embedded in a metric space (X, d) then the following weight function is interesting and useful.

$$\Omega : V \times V \to R^+, \Omega(u, v) = d(u, v).$$

If X is a complete Riemannian manifold, then the edge joining two vertexes u, v can suppose to be the minimal geodesic joining u, v, and the weight will be the length of the geodesic segments. The following theorem shows that in simply connected Riemannian manifolds of constant curvature the vertex set of such graphs resolve a totally geodesic submanifold.

Theorem 3.5. *If $G = (V, E)$ is a complete geodesic graph embedded in a simply connected Riemannian manifold M of constant curvature, then G is contained in a totally geodesic submanifold N of dimension $|V| - 1$ and V resolves N.*

Proof. If $|V| = 2$ the claim is trivial (V resolves a geodesic). Let $|V| > 2$ and consider the following cases separately:

Case 1. $M = R^n$.

If $|V| = p + 1$, then V is contained in a p-dimensional plane of M, which without lose of generality, we can suppose that it is R^p. V is contained in no hyperplane of dimension $p - 1$ (this comes from the completeness of G and the fact that, by definition, each pair of edges can intersect only in end points). Thus, by Corollary 2.4, it is not contained in any bisector in R^p. Consequently, it resolves R^p.

Case 2. $M = S^n$.

Let $V = \{v_1, \ldots, v_p\}$. Consider the plane B of R^{n+1} generated by $A = \{\circ, v_1, \ldots, v_p\}$. $B \cap S^n$ is a $(p-1)$-dimensional totally geodesic submanifold of S^n containing V which, without lose of generality, we can suppose that it is S^{p-1}. We show by induction on $|V| = p$ that V resolves S^{p-1} . The claim is clear if $p = 3$ (use the fact that no great circle of S^2 contains a geodesic three angle). Now, let the criterion be true for all V with $|V| = p < q$. We show that it is true if $|V| = q$. Let $V = \{v_1, \dots, v_q\}$ and consider the subgraph (V', E') with vertices $V' = \{v_1, \dots, v_{q-1}\}$. (V', E') is a geodesic complete graph which similar to the above argument, it is contained in a $q-2$ dimensional totally geodesic submanifold which without lose of generality we can assume that it is equal to $S^{q-2} \subset S^{q-1}$. By assumption, A' resolves S^{q-2}. Then, for all different points $x, y \in S^{q-2}$ the vectors $V_x = (d(x, v_1), \dots, d(x, v_{q-1}))$ and $V_y = (d(y, v_1), \dots, d(y, v_{d-1}))$ are different. If x or y belongs to $S^{q-1} - S^{q-2}$ and the vectors V_x and V_y are equal, then S^{q-2} must be the bisector of the minimal geodesic joining x to y. Since, $v_q \notin S^{q-2}$, then $d(x, v_q) \neq d(y, v_q)$ and the following vectors are different

$$(d(x, v_1), \dots, d(x, v_q)) \quad ; \quad (d(y, v_1), \dots, d(y, v_q)).$$

Thus, V resolves S^{q-1}.
Case 3. $M = H^n$. The proof is similar. □

In the above theorem if M is not simply connected, then by a similar argument as Remark 2.2 and the proof of Theorem 2.5, by using of the covering map $p : \tilde{M} \to M$, we can omit the simple connectivity assumption and get the following more general result:

Theorem 3.6. *If $G = (V, E)$ is a complete geodesic graph embedded in a Riemannian manifold M of constant curvature, then G is included in a totally geodesic submanifold N of dimension $|V| - 1$ and V resolves N.*

References

[1] S. Bau and A. F. Beardon, *The metric dimension of metric spaces*, Comput. Methods Funct. Theory, 13 (2013), pp. 295–305.

[2] J. K. Beem, *Pseudo-Riemannian manifolds with totally geodesic bisectors*, Proc. Amer. Math. Soc., 49 (1975), pp. 212–215.

[3] D. L. Boutin, *Determining sets, resolving sets, and the exchange property*, Graphs Combin., 25 (2009), pp. 789–806.

[4] J. Cáceres, C. Hernando, M. Mora, I. M. Pelayo, and M. L. Puertas, *On the metric dimension of infinite graphs*, in LAGOS'09—V Latin-American Algorithms, Graphs and Optimization Symposium, vol. 35 of Electron. Notes Discrete Math., Elsevier Sci. B. V., Amsterdam, 2009, pp. 15–20.

[5] G. G. Chappell, J. Gimbel, and C. Hartman, *Bounds on the metric and partition dimensions of a graph*, Ars Combin., 88 (2008), pp. 349–366.

[6] M. P. a. do Carmo, *Riemannian geometry*, Mathematics: Theory & Applications, Birkhäuser Boston, Inc., Boston, MA, 1992. Translated from the second Portuguese edition by Francis Flaherty.

[7] M. Heydarpour and S. Maghsoudi, *The metric dimension of metric manifolds*, Bull. Aust. Math. Soc., 91 (2015), pp. 508–513.

[8] P. Hoffman and B. Richter, *Embedding graphs in surfaces*, J. Combin. Theory Ser. B, 36 (1984), pp. 65–84.

[9] S. K. Lando and A. K. Zvonkin, *Graphs on surfaces and their applications*, vol. 141 of Encyclopaedia of Mathematical Sciences, Springer-Verlag, Berlin, 2004. With an appendix by Don B. Zagier, Low-Dimensional Topology, II.

[10] R. A. Melter and I. Tomescu, *Metric bases in digital geometry*, Computer Vision, Graphics, and Image Processing, 25 (1984), pp. 113–121.

[11] J. R. Munkres, *Topology: a first course*, Prentice-Hall, Inc., Englewood Cliffs, NJ, 1975.

[12] A. Robles-Kelly and E. R. Hancock, *A Riemannian approach to graph embedding*, Pattern Recognition, 40 (2007), pp. 1042–1056.

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