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Original Article

On conformal transformation of Ξ-curvature

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ABSTRACT: In this paper, we study the conformal transformation of recent defined non-Riemannian curvature in Finsler Geometry, namely Ξ-curvature. Indeed, we obtain the necessary and sufficient condition under which the conformal transformation preserves the Ξ-curvature.

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1. Introduction

The theory of conformal transformations plays an important role in Riemannian geometry for its applications in Cartography, Image Processing, Geometrical Optics, General Relativity and Quantum Mechanics. The theory of conformal transformation of the class of Finsler metrics has been studied by many Finsler geometers [1, 3, 4, 5, 7, 9, 11]. The well-known Weyl theorem reported that the projective and conformal properties of a Finsler metric characterize the metric properties uniquely. Thus, studying the conformal properties of a Finsler metric needs extra consideration. In [4], M. Hashiguchi formulated the properties of conformal change of Finsler metrics and gave a meaningful geometrical criterion for the fundamental tensors of F and $\tilde{F}$. Indeed, the fundamental tensors g_{ij} and $\tilde{g}_{ij}$ related to F and $\tilde{F}$, respectively, are conformal on a manifold M if satisfy $g_{ij} = e^{\kappa}\tilde{g}_{ij}$, where $\kappa = \kappa(x, y)$ is a scalar function on tangent bundle TM. In [6], Knebelman proved that this criterion implies that $\kappa = \kappa(x)$ is a function of position only. Let $F = F(x, y)$ and $\tilde{F} = \tilde{F}(x, y)$ be two Finsler metrics on a manifold M. Then F is conformal to $\tilde{F}$ if and only if there exists a scalar function $\kappa = \kappa(x)$ on M such that

$$F(x, y) = e^{\kappa(x)}\tilde{F}(x, y), \tag{1}$$

where the scalar function $\kappa = \kappa(x)$ is called the conformal factor.

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In conformal geometry, it is one important problem how to characterize the conformally change of Riemannian and non-Riemannian curvatures for Finsler metrics. In [2], Bácsó-Cheng characterized the conformal transformations which preserve Riemann curvature, Ricci curvature, Landsberg curvature and S-curvature respectively. In particular, they proved that, if the conformal transformation (1) preserves the geodesics, then it must be a homothety.

There is another non-Riemannian quantity that obtained from the S-curvature. Indeed, the Ξ-curvature $\mathbf{\Xi} = \Xi_i dx^i$ on the tangent bundle TM is defined by

$$\Xi_i := \mathbf{S}_{.i|m} y^m - \mathbf{S}_{|i},$$

where "." and "|" denote the vertical and horizontal covariant derivative with respect to the Berwald connection of F, respectively [8]. Here, we give the necessary and sufficient condition under which the conformal transformation preserves the Ξ-curvature. More precisely, we prove the following.

Theorem 1.1. *Let F and $\bar{F}$ be two Finsler metrics on a manifold M. If $\bar{F}(x,y) = e^{\sigma} F(x,y)$, then the conformal transformation preserves the Ξ-curvature if and only if the conformal factor $\sigma = \sigma(x)$ satisfies following equation:*

$$\begin{aligned}
Q^m\mathbf{S}_{.i.m} - \mathbf{S}_{.m}Q^m_i + 2y_i\Big[\big[\nabla_0\sigma^r + \sigma^m(P\delta^r_m - Q^r_m - 4Q^jC^r{}_{mj})\big]I_r + \sigma^r\big[J_r + PI_r + I_mQ^m_r + 2I_{r.m}Q^m\big]\Big]\\
+ F^2\Big[\Big[\nabla_0\sigma^r - \sigma^p(4Q^jC^r{}_{pj} - P\delta^r_p - P_py^r + Q^r_p)\Big]I_{r.i} + \sigma^r\Big[J_{r.i} - I_{r|i} + P_iI_r + 2PI_{r.i}\\
+ I_iP_r + 2I_{r.p.i}Q^p + I_{r.p}Q^p_i + I_{p.i}Q^p_r\Big] - 2\Big[\nabla_0\sigma^p - \sigma^s(4Q^jC^p{}_{sj} - P\delta^p_s + Q^p_s)\Big]C^r{}_{pi}I_r\\
- 2\sigma^p\Big[L^r{}_{pi} + PC^r{}_{pi} + 2Q^sC^r{}_{pi.s} + P_sC^s{}_{pi}y^r - Q^r_sC^s{}_{pi} + C^r{}_{si}Q^s_p + C^r{}_{ps}Q^s_i\Big]I_r\\
- 2\sigma^pC^r{}_{pi}\Big[J_r + PI_r + 2I_{r.s}Q^s + I_sQ^s_r\Big]\Big] - F^2\Big[\Big[\sigma^r{}_{|i} + \sigma^m(2PC^r{}_{mi} - 2Q^j_iC^r{}_{mj}\\
+ P_i\delta^r_m + P_m\delta^r_i - Q^r_{im})\Big]I_r + \sigma^r\Big[I_{r|i} - PI_{r.i} - P_rI_i + I_{r.m}Q^m_i + I_mQ^m_{ri}\Big]\Big] = 0.
\end{aligned}$$

In particular, if $\sigma(x) = constant$, then $\bar{\Xi} = \Xi$.

2. Preliminaries

Let M be a n-dimensional C^∞ manifold and $TM = \bigcup_{x\in M} T_xM$ the tangent bundle. Let (M,F) be a Finsler manifold. The following quadratic form $\mathbf{g}_y$ on T_xM is called fundamental tensor

$$\mathbf{g}_y(u,v) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y+su+tv)\Big]|_{s=t=0}, \quad u,v \in T_xM.$$

Let $x \in M$ and $F_x := F|_{T_xM}$. To measure the non-Euclidean feature of F_x, define $\mathbf{C}_y : T_xM \otimes T_xM \otimes T_xM \to \mathbb{R}$ by

$$\mathbf{C}_y(u,v,w) := \frac{1}{2}\frac{d}{dt}\Big[\mathbf{g}_{y+tw}(u,v)\Big]_{t=0} = \frac{1}{4}\frac{\partial^3}{\partial r\partial s\partial t}\Big[F^2(y+ru+sv+tw)\Big]_{r=s=t=0},$$

where $u,v,w \in T_xM$. By definition, $\mathbf{C}_y$ is a symmetric trilinear form on T_xM. The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TM_0}$ is called the Cartan torsion. It is well known that $\mathbf{C} = 0$ if and only if F is Riemannian.

For $y \in T_xM_0$, define $\mathbf{I}_y : T_xM \to \mathbb{R}$ by $\mathbf{I}_y(u) = \sum_{i=1}^n g^{ij}(y)\mathbf{C}_y(u,\partial_i,\partial_j)$, where $\{\partial_i\}$ is a basis for T_xM at $x \in M$. The family $\mathbf{I} := \{\mathbf{I}_y\}_{y\in TM_0}$ is called the mean Cartan torsion. Thus, $\mathbf{I}_y(u) := I_i(y)u^i$, where $I_i := g^{jk}C_{ijk}$.

Given a Finsler manifold (M,F), then a global vector field $\mathbf{G}$ is induced by F on TM_0, which in a standard coordinate (x^i,y^i) for TM_0 is given by

$$\mathbf{G} = y^i\frac{\partial}{\partial x^i} - 2G^i\frac{\partial}{\partial y^i},$$

where $G^i = G^i(x,y)$ are local functions on TM_0 satisfying $G^i(x,\lambda y) = \lambda^2 G^i(x,y)$, $\lambda > 0$, and given by

$$G^i = \frac{1}{4}g^{il}\Big[\frac{\partial^2 F^2}{\partial x^k\partial y^l}y^k - \frac{\partial F^2}{\partial x^l}\Big].$$

The vector field $\mathbf{G}$ is called the associated spray to (M, F). The projection of an integral curve of the spray $\mathbf{G}$ is called a geodesic in M.

Define $\mathbf{B}_y : T_xM \otimes T_xM \otimes T_xM \to T_xM$ by $\mathbf{B}_y(u, v, w) := B^i{}_{jkl}(y)u^j v^k w^l \frac{\partial}{\partial x^i}|_x$, where

$$B^i{}_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

$\mathbf{B}_y(u, v, w)$ is symmetric in u, v and w. $\mathbf{B}$ is called the Berwald curvature.

For $y \in T_xM$, define the Landsberg curvature $\mathbf{L}_y : T_xM \times T_xM \times T_xM \to \mathbb{R}$ by

$$\mathbf{L}_y(u, v, w) := -\frac{1}{2}\mathbf{g}_y\big(\mathbf{B}_y(u, v, w), y\big).$$

In local coordinates, $\mathbf{L}_y(u, v, w) := L_{ijk}(y)u^i v^j w^k$, where

$$L_{ijk} := -\frac{1}{2}y_l B^l{}_{ijk}.$$

The quantity $\mathbf{L}$ is called the Landsberg curvature and F is called a Landsberg metric if $\mathbf{L} = 0$.

For $y \in T_xM$, define $\mathbf{J}_y : T_xM \to \mathbb{R}$ by $\mathbf{J}_y(u) := J_i(y)u^i$, where

$$J_i := g^{jk} L_{ijk}.$$

The quantity $\mathbf{J}$ is called the mean Landsberg curvature. A Finsler metric F is called a weakly Landsberg metric if $\mathbf{J} = 0$.

For a Finsler metric F on an n-dimensional manifold M, the Busemann-Hausdorff volume form $dV_F = \sigma_F(x)dx^1 \cdots dx^n$ is defined by

$$\sigma_F(x) := \frac{Vol(B^n(1))}{Vol\Big[(y^i) \in R^n \;\Big|\; F\Big(y^i \frac{\partial}{\partial x^i}|_x\Big) < 1\Big]}.$$

Let G^i denote the geodesic coefficients of F in the same local coordinate system. The S-curvature is defined by

$$\mathbf{S}(\mathbf{y}) := \frac{\partial G^i}{\partial y^i}(x, y) - y^i \frac{\partial}{\partial x^i}\Big[\ln \sigma_F(x)\Big],$$

where $\mathbf{y} = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$.

The non-Riemannian quantity Ξ-curvature $\mathbf{\Xi} = \Xi_i dx^i$ on the tangent bundle TM is defined by

$$\Xi_i := \mathbf{S}_{.i|m} y^m - \mathbf{S}_{|i}, \tag{2}$$

where "." and "|" denote the vertical and horizontal covariant derivative with respect to the Berwald connection of F, respectively.

3. Proof of Theorem 1.1

Let F and $\bar{F}$ be two Finsler metrics on a manifold M. By using the Rapcsák's identity, the following relationship between G^i and $\bar{G}^i$ holds

$$\bar{G}^i = G^i + \frac{\bar{F}_{;m} y^m}{2\bar{F}} y^i + \frac{\bar{F}}{2}\bar{g}^{il}\Big\{\bar{F}_{;k,l} y^k - \bar{F}_{;l}\Big\}, \tag{3}$$

where ";" and "," denote the horizontal and vertical derivations with respect to the Berwald connection of F. Suppose that F is conformally related to a $\bar{F}$, namely, $\bar{F} = e^\sigma F$, where $\sigma = \sigma(x)$ is a scalar function on M. Since $F_{;m} = 0$, then the following hold

$$\bar{F}_{;m} = \sigma_m e^\sigma F, \quad \bar{F}_{,i} = e^\sigma F_{,i}, \quad \bar{F}_{;m,l} = \sigma_m e^\sigma F_{,l}, \quad \bar{g}_{ij} = e^{2\sigma} g_{ij}, \quad \bar{g}^{ij} = e^{-2\sigma} g^{ij}. \tag{4}$$

where $\sigma_m := \partial\sigma/\partial x^m$. By putting (4) in (3), we get

$$\bar{G}^i = G^i + \sigma_0 y^i - \frac{1}{2}F^2 \sigma^i, \tag{5}$$

where

$$\sigma_0 := \sigma_i y^i, \quad \sigma^i := g^{im}\sigma_m.$$

(5) can be written as follows

$$\bar{G}^i = G^i + Py^i - Q^i,$$

where

$$P := \sigma_k y^k, \quad Q^i := \frac{1}{2}F^2\sigma^i.$$

Let us define

$$G^i_j := \frac{\partial G^i}{\partial y^j}, \quad G^i_{jk} := \frac{\partial G^i_j}{\partial y^k}, \quad \bar{G}^i_j := \frac{\partial \bar{G}^i}{\partial y^j}, \quad \bar{G}^i_{jk} := \frac{\partial \bar{G}^i_j}{\partial y^k},$$
$$Q^i_j := \frac{\partial Q^i}{\partial y^j}, \quad Q^i_{jk} := \frac{\partial Q^i_j}{\partial y^k}, \quad Q^i_{jkl} := \frac{\partial Q^i_{jk}}{\partial y^l},$$
$$P_j := \frac{\partial P}{\partial y^j}, \quad P_{jk} := \frac{\partial P_j}{\partial y^k}.$$

Taking vertical derivations of (6) imply that

$$\bar{G}^i_j = G^i_j + P_j y^i + P\delta^i_j - Q^i_j, \tag{6}$$
$$\bar{G}^i_{jk} = G^i_{jk} + P_{jk}y^i + P_j\delta^i_k + P_k\delta^i_j - Q^i_{jk}. \tag{7}$$

The following hold

$$P_i = \sigma_i, \quad P_{ij} = P_{ijk} = 0. \tag{8}$$

By (7) and (8), we get

$$\bar{B}^i_{jkl} = B^i_{jkl} - Q^i_{jkl}.$$

The following holds

$$(g^{im})_{,j} = -2C^{im}_{\ \ j}, \quad (\sigma^i)_{,j} = -2\sigma_m C^{mi}_{\ \ j} = -2\sigma^m C^i_{\ mj}. \tag{9}$$

Definition 3.1. *Let F and $\bar{F}$ be two conformal Finsler metrics on a manifold M, namely $\bar{F}(x,y) = e^{\sigma}F(x,y)$. For $y \in T_xM_0$, define $\mathbf{Z}_y : T_xM \to T_xM$ by $\mathbf{Z}_y(u) := Z^i_j(y)u^j \frac{\partial}{\partial x^i}|_x$, where*

$$Z^i_{\ j} := \frac{\partial \sigma^i}{\partial y^j}.$$

Then, we get the following.

Proposition 3.2. *Let F and $\bar{F}$ be two non-homothty conformal Finsler metrics on a manifold M. Then the quantity $\mathbf{Z}$ is vanishing if and only if F reduces to a Riemannian metric.*

Proof. By (9), we have

$$(\sigma^i)_{,j} = -2\sigma_m C^{mi}_{\ \ j} = -2\sigma^m C^i_{\ mj}.$$

A conformal transformation is called C-conformal if the conformal factor σ satisfies that $\sigma_m C^m_{\ ij} = 0$. On the other hand, Theorem 1 in [10] says that such a transformation must be a homothety unless the manifold is Riemannian. This completes the proof. □

Lemma 3.3. *Let F and $\bar{F}$ be two Finsler metrics on a manifold M. If $\bar{F}(x,y) = e^{\sigma}F(x,y)$, then the conformal transformation preserves the Ξ-curvature if and only if the conformal factor $\sigma = \sigma(x)$ satisfies following equation:*

$$Q^m\mathbf{S}_{.i.m} - \mathbf{S}_{.m}Q^m_i + 2FF_i(\sigma^r I_r)_{||m}y^m + F^2(\sigma^r I_r)_{.i||m}y^m - F^2(\sigma^r I_r)_{||i} = 0, \tag{10}$$

where "$||$" denotes the horizontal derivation with respect to the Berwald connection of $\bar{F}$.

Proof. The relation between S-curvatures of two conformal Finsler metrics F and $\bar{F}$ is given by

$$\bar{\mathbf{S}} = \mathbf{S} + F^2\sigma^r I_r. \tag{11}$$

By (2) and (11), we get

$$\bar{\Xi}_i := \bar{\mathbf{S}}_{.i||m}y^m - \bar{\mathbf{S}}_{||i} = \Big[\mathbf{S} + F^2\sigma^r I_r\Big]_{.i||m} y^m - \Big[\mathbf{S} + F^2\sigma^r I_r\Big]_{||i}, \tag{12}$$

By (12) and considering $F_{||i} = 0$, we get

$$\bar{\Xi}_i = \mathbf{S}_{.i||m}y^m - \mathbf{S}_{||i} + 2FF_i(\sigma^r I_r)_{||m}y^m + F^2(\sigma^r I_r)_{.i||m}y^m - F^2(\sigma^r I_r)_{||i}, \tag{13}$$

where $F_i := F_{y^i}$. By (6), we have

$$\begin{aligned}\mathbf{S}_{||i} = \frac{\partial \mathbf{S}}{\partial x^i} - \bar{G}^j_i\frac{\partial \mathbf{S}}{\partial y^j} &= \frac{\partial \mathbf{S}}{\partial x^i} - \Big(G^j_i + P_iy^j + P\delta^j_i - Q^j_i\Big)\frac{\partial \mathbf{S}}{\partial y^j}\\ &= \mathbf{S}_{|i} - \mathbf{S}P_i - \mathbf{S}_{.i}P + \mathbf{S}_{.j}Q^j_i.\end{aligned} \tag{14}$$

By (6), (7) and (13), we get

$$\begin{aligned}\mathbf{S}_{.i||m} &= \frac{\partial \mathbf{S}_{.i}}{\partial x^m} - \bar{G}^j_m\frac{\partial \mathbf{S}_{.i}}{\partial y^j} - \mathbf{S}_{.j}\bar{G}^j_{im}\\ &= \frac{\partial \mathbf{S}_{.i}}{\partial x^m} - \Big(G^j_m + P_my^j + P\delta^j_m - Q^j_m\Big)\frac{\partial \mathbf{S}_{.i}}{\partial y^j} - \mathbf{S}_{.j}\Big(G^j_{im} + P_{im}y^j + P_i\delta^j_m + P_m\delta^j_i - Q^j_{im}\Big)\\ &= \mathbf{S}_{.i|m} - P\mathbf{S}_{.i.m} + Q^j_m\mathbf{S}_{.i.j} - P_{im}\mathbf{S} - P_i\mathbf{S}_{.m} - P_m\mathbf{S}_{.i} + \mathbf{S}_{.j}Q^j_{im}.\end{aligned} \tag{15}$$

Contracting (15) with y^m implies that

$$\mathbf{S}_{.i||m}y^m = \mathbf{S}_{.i|m}y^m + 2Q^m\mathbf{S}_{.i.m} - P_i\mathbf{S} - P\mathbf{S}_{.i} + \mathbf{S}_{.m}Q^m_i. \tag{16}$$

By (13), (14) and (16), we obtain

$$\bar{\Xi}_i = \Xi_i + Q^m\mathbf{S}_{.i.m} - \mathbf{S}_{.j}Q^j_i + 2FF_i(\sigma^r I_r)_{||m}y^m + F^2(\sigma^r I_r)_{.i||m}y^m - F^2(\sigma^r I_r)_{||i}. \tag{17}$$

By (17), we get the proof. □

Proof of Theorem 1.1: In order to simplifying (10), we should compute three elements $(\sigma^r I_r)_{||i}$, $(\sigma^r I_r)_{||m}y^m$ and $(\sigma^r I_r)_{.i||m}y^m$. We remark from (9) the useful relation $(\sigma^i)_{,j} = -2\sigma^m C^i{}_{mj}$. Also, we get

$$\begin{aligned}\sigma^r{}_{||i} &= \frac{\partial \sigma^r}{\partial x^i} - \bar{G}^j_i\frac{\partial \sigma^r}{\partial y^j} + \sigma^m\bar{G}^r_{im}\\ &= \frac{\partial \sigma^r}{\partial x^i} - (G^j_i + P_iy^j + P\delta^j_i - Q^j_i)\frac{\partial \sigma^r}{\partial y^j} + \sigma^m(G^r_{im} + P_{im}y^r + P_i\delta^r_m + P_m\delta^r_i - Q^r_{im})\\ &\overset{(9)}{=} \sigma^r{}_{|i} + \sigma^m Y^r_{mi},\end{aligned} \tag{18}$$

where

$$Y^r_{mi} := 2PC^r{}_{mi} - 2Q^j_iC^r{}_{mj} + P_{im}y^r + P_i\delta^r_m + P_m\delta^r_i - Q^r_{im}.$$

We have

$$\begin{aligned}I_{r||i} &= \frac{\partial I_r}{\partial x^i} - \bar{G}^j_i\frac{\partial I_r}{\partial y^j} - I_m\bar{G}^m_{ri}\\ &= \frac{\partial I_r}{\partial x^i} - (G^j_i + P_iy^j + P\delta^j_i - Q^j_i)\frac{\partial I_r}{\partial y^j} - I_m(G^m_{ri} + P_{ri}y^m + P_r\delta^m_i + P_i\delta^m_r - Q^m_{ri})\\ &= I_{r|i} + Z_{ri},\end{aligned} \tag{19}$$

where

$$Z_{ri} := -PI_{r.i} - P_rI_i + I_{r.m}Q^m_i + I_mQ^m_{ri}.$$

By (18), (19) and $y^r I_r = 0$, one can get

$$(\sigma^r I_r)_{||i} = \sigma^r{}_{||i} I_r + \sigma^r I_{r||i} = \Big[\sigma^r{}_{|i} + \sigma^m(2PC^r{}_{mi} - 2Q^j_i C^r{}_{mj} + P_i\delta^r_m + P_m\delta^r_i - Q^r_{im})\Big] I_r + \sigma^r\Big[I_{r|i} - PI_{r.i} - P_r I_i + I_{r.m}Q^m_i + I_m Q^m_{ri}\Big] \tag{20}$$

and

$$(\sigma^r I_r)_{||i} y^i = \Big[\nabla_0\sigma^r + \sigma^m(P\delta^r_m - Q^r_m - 4Q^j C^r{}_{mj})\Big] I_r + \sigma^r\Big[J_r + PI_r + 2I_{r.m}Q^m + I_m Q^m_r\Big], \tag{21}$$

where

$$\nabla_0\sigma^r := \sigma^r{}_{|i} y^i.$$

The following holds

$$(\sigma^r I_r)_{.i} = (\sigma^r)_{.i} I_r + \sigma^r I_{r.i} = -2\sigma^m C^r{}_{mi} I_r + \sigma^r I_{r.i}$$

which yields

$$\begin{aligned}(\sigma^r I_r)_{.i||m} &= \sigma^r{}_{||m} I_{r.i} + \sigma^r I_{r.i||m} - 2\sigma^p{}_{||m} C^r{}_{pi} I_r - 2\sigma^p C^r{}_{pi||m} I_r - 2\sigma^p C^r{}_{pi} I_{r||m}\\ &= (\sigma^r{}_{|m} + \sigma^p Y^r_{pm}) I_{r.i} + \sigma^r I_{r.i||m} - 2(\sigma^p{}_{|m} + \sigma^s Y^p_{sm}) C^r{}_{pi} I_r - 2\sigma^p C^r{}_{pi||m} I_r - 2\sigma^p C^r{}_{pi} I_{r||m}.\end{aligned}$$

Then

$$(\sigma^r I_r)_{.i||m} y^m = (\sigma^r{}_{|m} + \sigma^p Y^r_{pm}) y^m I_{r.i} + \sigma^r I_{r.i||m} y^m - 2(\sigma^p{}_{|m} + \sigma^s Y^p_{sm}) y^m C^r{}_{pi} I_r - 2\sigma^p C^r{}_{pi||m} y^m I_r - 2\sigma^p C^r{}_{pi} I_{r||m} y^m. \tag{22}$$

In order to simplifying (22), we need to compute $I_{r||m}y^m$, $I_{r,i||m}y^m$ and $C^r{}_{pi||m}y^m$. First, we have

$$\begin{aligned}I_{r||m} &= \frac{\partial I_r}{\partial x^m} - \bar{G}^j_m \frac{\partial I_r}{\partial y^j} - I_p \bar{G}^p_{rm}\\ &= \frac{\partial I_r}{\partial x^m} - (G^j_m + P_m y^j + P\delta^j_m - Q^j_m)\frac{\partial I_r}{\partial y^j} - I_p(G^p_{rm} + P_{rm}y^p + P_r\delta^p_m + P_m\delta^p_r - Q^p_{rm})\\ &= I_{r|m} - PI_{r.m} - I_m P_r + I_{r.p}Q^p_m + I_p Q^p_{rm}\end{aligned} \tag{23}$$

which implies that

$$I_{r||m} y^m = J_r + PI_r + 2I_{r.p}Q^p + I_p Q^p_r. \tag{24}$$

The following Bianchi identity holds

$$I_{r.i||m} = I_{r||m.i} + I_s B^s{}_{rim}. \tag{25}$$

Also, (23) yields

$$I_{r||m.i} = I_{r|m.i} - P_i I_{r.m} - PI_{r.m.i} - I_{m.i}P_r - I_m P_{r.i} + I_{r.p.i}Q^p_m + I_{r.p}Q^p_{mi} + I_{p.i}Q^p_{rm} + I_p Q^p_{rmi}. \tag{26}$$

By (25) and (26), we get

$$I_{r.i||m} = I_{r|m.i} - P_i I_{r.m} - PI_{r.m.i} - I_{m.i}P_r - I_m P_{r.i} + I_{r.p.i}Q^p_m + I_{r.p}Q^p_{mi} + I_{p.i}Q^p_{rm} + I_p Q^p_{rmi} + I_s B^s{}_{rim}$$

which by considering

$$I_{m.i} y^m = -I_i$$

implies that

$$I_{r.i||m} y^m = I_{r|m.i} y^m + P_i I_r + 2PI_{r.i} + I_i P_r + 2I_{r.p.i}Q^p + I_{r.p}Q^p_i + I_{p.i}Q^p_r. \tag{27}$$

Since

$$I_{r|m.i} y^m = J_{r.i} - I_{r|i}$$

then (27) reduces to following

$$I_{r.i||m} y^m = J_{r.i} - I_{r|i} + P_i I_r + 2PI_{r.i} + I_i P_r + 2I_{r.p.i}Q^p + I_{r.p}Q^p_i + I_{p.i}Q^p_r. \tag{28}$$

Now, we are going to compute ${C^r}_{pi||m}y^m$. We have

$$
\begin{aligned}
{C^r}_{pi||m} &= \frac{\partial {C^r}_{pi}}{\partial x^m} - \bar{G}^j_m \frac{\partial {C^r}_{pi}}{\partial y^j} + {C^k}_{pi}\bar{G}^r_{km} - {C^r}_{si}\bar{G}^s_{pm} - {C^r}_{ps}\bar{G}^s_{im} \\
&= \frac{\partial {C^r}_{pi}}{\partial x^m} - (G^j_m + P_m y^j + P\delta^j_m - Q^j_m)\frac{\partial {C^r}_{pi}}{\partial y^j} + {C^k}_{pi}(G^r_{km} + P_{km}y^r + P_k\delta^r_m + P_m\delta^r_k - Q^r_{km}) \\
&\quad - {C^r}_{si}(G^s_{pm} + P_{pm}y^s + P_p\delta^s_m + P_m\delta^s_p - Q^s_{pm}) - {C^r}_{ps}(G^s_{im} + P_{im}y^s + P_i\delta^s_m + P_m\delta^s_i - Q^s_{im}) \\
&= {C^r}_{pi|m} - (P_m y^j + P\delta^j_m - Q^j_m)\frac{\partial {C^r}_{pi}}{\partial y^j} + {C^k}_{pi}(P_{km}y^r + P_k\delta^r_m + P_m\delta^r_k - Q^r_{km}) \\
&\quad - {C^r}_{si}(P_{pm}y^s + P_p\delta^s_m + P_m\delta^s_p - Q^s_{pm}) - {C^r}_{ps}(P_{im}y^s + P_i\delta^s_m + P_m\delta^s_i - Q^s_{im}).
\end{aligned} \tag{29}
$$

By considering $C_{ijk}y^i = 0$ and multiplying (29) with y^m, we get the following

$$
{C^r}_{pi||m}y^m = {L^r}_{pi} + P{C^r}_{pi} + 2Q^s{C^r}_{pi.s} + P_s{C^s}_{pi}y^r - Q^r_s{C^s}_{pi} + {C^r}_{si}Q^s_p + {C^r}_{ps}Q^s_i. \tag{30}
$$

Putting (24), (28) and (30) in (22) imply that

$$
\begin{aligned}
(\sigma^r I_r)_{.i||m}y^m =& \Big[\nabla_0\sigma^r - \sigma^p(4Q^j{C^r}_{pj} - P\delta^r_p - P_p y^r + Q^r_p)\Big]I_{r.i} \\
&+ \sigma^r\Big[J_{r.i} - I_{r|i} + P_i I_r + 2PI_{r.i} + I_i P_r + 2I_{r.p.i}Q^p + I_{r.p}Q^p_i + I_{p.i}Q^p_r\Big] \\
&- 2\Big[\nabla_0\sigma^p - \sigma^s(4Q^j{C^p}_{sj} - P\delta^p_s + Q^p_s)\Big]{C^r}_{pi}I_r \\
&- 2\sigma^p\Big[{L^r}_{pi} + P{C^r}_{pi} + 2Q^s{C^r}_{pi.s} + P_s{C^s}_{pi}y^r - Q^r_s{C^s}_{pi} + {C^r}_{si}Q^s_p + {C^r}_{ps}Q^s_i\Big]I_r \\
&- 2\sigma^p{C^r}_{pi}\Big[J_r + PI_r + 2I_{r.p}Q^p + I_p Q^p_r\Big].
\end{aligned} \tag{31}
$$

Now, we come back to (10). By putting (20), (21) and (31) in (10) at Lemma 3.3, we get (10). □

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