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Original Article

Finding the extreme efficient solutions of multi-objective pseudo-convex programming problems

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ABSTRACT: In this paper, we present two methods to find the strictly efficient and weakly efficient points of multi-objective programming (MOP) problems in which their objective functions are pseudo-convex and their feasible sets are polyhedrons. The obtained efficient solutions in these methods are the extreme points. Since the pseudo-convex functions are quasi-convex as well, therefore the presented methods can be used to find efficient solutions of the (MOP) problem with the quasi-convex objective functions and the polyhedron feasible set. Two experimental examples are presented.

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1. Introduction

Multi-objective programming (MOP) problems are a well-known and applicable research field in optimization and operations research. The multi-objective optimization problems are included several objective functions and a set of feasible spaces. An important class of them is included multi-objective problems with several pseudo-convex objective functions and a polyhedron as its feasible set; we call it MOPP problem. It is worth mentioning that the multi-objective problems with the convex and quasi-convex objective functions and the polyhedron feasible set are also MOPP problems. For example, the multi-objective linear fractional programming (MOLFP) problem is a kind of MOPP problem.

There are many methods to find efficient solutions of multi-objective optimization problems which are constructed based on iterative, scalarization, interactive and etc. methods. (see for instance [3, 7]). One of the well-known methods to find the efficient solution of a MOP is the weighted sum method; by this method, the MOP is converted to an optimization problem with a single objective function. The weights of MOP objective

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functions usually are determined according to the opinion of the decision maker. If the weights are non-negative then the optimal solutions are weakly efficient and especially, if the weights are positive then the optimal solutions are efficient. Besides the weighted sum approach, the constraint method is also a well-known technique to solve MOP problems. In this technique; there is no aggregation of objective functions, instead, only one of the original objectives is optimized, while the others are transformed into constraints, this approach was introduced in [9] and an extensive discussion can be found in [5].

With regards to the structure of MOPP problems [10], sometimes, it may not be possible to find an efficient solution of MOPP problems with the mentioned methods. Researchers have provided some methods to find efficient solutions for specific cases of the MOPP problems such as MOLFP or MOP problems with quasi-convex objective functions. Some researchers used iterative methods to solve multiobjective linear fractional programming (MOLFP) problems [11, 14]. Transforming the MOLFP problems to linear programming problems for finding the efficient solution of the MOLFP problems is another technique presented by researchers [12, 13]. Also, methods for solving the MOP problems with the quasi-convex objective functions are provided [2, 15]. but in general, no method has been provided for finding the efficient solution of MOP problems with the pseudo-convex functions.
In this paper, two methods for finding efficient solutions (Pareto) and weakly efficient solutions are presented. In these methods, the extreme points of MOPP problems are identified and the efficiency status (Pareto and weak) of these points are determined. In this study, for presenting our methods, we use the constraint and the weighted sum methods. In Sections 2 and 3 some notions, definitions, and properties which are required in the main discussion are introduced. In Section 4, the main discussions and properties are presented. Two examples to illustrate our approach are presented in Section 5 and the final section is the conclusion section.

2. The single objective MOPP problem

Consider the following single-objective problem.

$$\begin{array}{ll} \min & f_0(x); \\ \text{s.t.} & x \in X = \{x \in R^n : \, Ax = b\}, \end{array} \tag{1}$$

where, $X \subseteq \mathbf{R}^n$ is the feasible set and $Ax = b$ is the affine condition, $A \in R^{m\times n}$, $b \in R^m$, $rankA = m < n$, also the objective function f_0 is pseudo-convex.

Definition 2.1. *For a convex set $X \subset R^n$ the function $f : X \longrightarrow R$ is called quasiconvex if for each $x_1, x_2 \in X$ and $\lambda \in (0, 1)$, then, $f(x_1 + \lambda(x_2 - x_1)) \leqslant \max\{f(x_1), f(x_2)\}$.*

Definition 2.2. *The differentiable function $f : R^n \longrightarrow R$ is pseudo-convex if and only if dom f is convex and for all $x, y \in domf$ if $f(y) < f(x)$, then, $\bigtriangledown f(x)^t(y - x) < 0$.*

For example, linear fractional programming (LFP) problems are pseudo-convex optimization problems ([1, 6]).

Theorem 2.3. *Suppose $X \subset R^n$ is a convex set, and $f : X \longrightarrow R$ is pseudo-convex function, then f is a quasi-convex function as well.*

Proof. See [1]. □

Theorem 2.4. *The differentiable function $f : R^n \longrightarrow R$ is quasiconvex if and only if dom f is convex and for all $x, y \in domf$ if $f(y) \leqslant f(x)$, then, $\bigtriangledown f(x)^t(y - x) \leqslant 0$.*

Proof. See [1, 4]. □

Definition 2.5. *In Problem (1), a vector $d \neq 0$ is called an improving feasible direction in $x_0 \in X$ if for each $\lambda \in (0, \delta)$, we have $(x_0 + \lambda d) \in X$, and $f_0(x_0 + \lambda d) < f_0(x)$ for some $\delta > 0$.*

Theorem 2.6. *For each feasible direction vector $d \neq 0$ of (1) in $x \in X$, if $\bigtriangledown f_0(x)^t d \geqslant 0$, then x is an optimal solution of (1) and $x \in X$ is a unique optimal solution of (1) if $\bigtriangledown f_0(x)^t d > 0$.*

Proof. Let $d \neq 0$ be a feasible direction in $x \in X$; therefore, there is $\delta > 0$ such that for each $\lambda \in (0, \delta)$, we have $(x + \lambda d) \in X$. Because f_0 is pseudo-convex, according to the Definition 2.2, for each feasible direction vector $d \neq 0$ by choosing $y = x + \lambda d$, if $\bigtriangledown f_0(x)^t(x + \lambda d - x) = \lambda \bigtriangledown f_0(x)^t d \geqslant 0$ then $f_0(x + \lambda d) \geqslant f(x)$; since $\lambda > 0$, it can be concluded that, if $\bigtriangledown f_0(x)^t d \geqslant 0$, then $f_0(x + \lambda d) \geqslant f_0(x)$, hence x is an optimal solution of (1). Because a pseudo-convex function is quasi-convex as well, and according to the assumption $\bigtriangledown f_0(x)^t d > 0$, therefore, according to Theorem 2.4, $f_0(x + \lambda d) > f_0(x)$, so, x is a unique optimal solution of (1). □

Corollary 2.7. *By considering Theorem 2.6, we can conclude that if* $\bigtriangledown f_0(x)^t d < 0$*, then* x *is not an optimal solution and must be improved.*

Definition 2.8. *In Problem (1), vector* $d \neq 0$ *is called an improving feasible direction in* $x \in X$ *if* $Ad = 0, -d_j \leqslant 0$ *and* $\bigtriangledown f_0(x)^t d < 0$*. (See [1]).*

Now we can obtain an improving feasible direction vector in $x \in X$ as follows;
in Problem (1), assuming that $rankA = m < n$ and for each vector x that applies to constraints we partition x as $x = (x_B, x_N)$, where, $x_B \geqslant 0$ is the basic variables of vector x with dimension m and x_N is the non-basic variables of vector x with dimension $n - m$, and also, Matrix A can be partitioned as $A = [B, N]$, and d is partitioned as $d^t = [d_B^t, d_N^t]$, where, B and N are martix of coefficients of the basic variables (x_B) and the non-basic variables (x_N) respectively. Similar to [8] we can obtain vector $d \neq 0$ as follows.

$$Ad = [B, N] \begin{bmatrix} d_B \\ d_N \end{bmatrix} = Bd_B + Nd_N = 0,$$

consequently,

$$d_B = -B^{-1} N d_N.$$

Now, we have

$$\bigtriangledown f_0(x) = \bigtriangledown f_0(x_B, x_N)^t =$$

$$= [\bigtriangledown_B f_0(B^{-1}b - B^{-1}Nx_N, x_N)^t, \bigtriangledown_N f_0(B^{-1}b - B^{-1}Nx_N, x_N)^t] = \left(0, \frac{\partial f}{\partial x_B}\frac{\partial x_B}{\partial x_N} + \frac{\partial f}{\partial x_N}\right) =$$

$$(0, \bigtriangledown_N f_0(x)^t - \bigtriangledown_B f_0(x)^t B^{-1} N);$$

vector $d \neq 0$ is an improving feasible direction, therefore,

$$0 > \bigtriangledown f_0(x)^t d = (0, \bigtriangledown_N f_0(x)^t - \bigtriangledown_B f_0(x)^t B^{-1} N)^t \begin{bmatrix} d_B \\ d_N \end{bmatrix} = [\bigtriangledown_N f_0(x)^t - \bigtriangledown_B f_0(x)^t B^{-1} N] d_N.$$

Denote $r_N^t = (r_B^t, r_N^t) = (0, \bigtriangledown_N f(x)^t - \bigtriangledown_B f(x)^t B^{-1} N)$; therefore $\bigtriangledown f(x)^t d = \sum_{j \notin I} r_j d_j < 0$, where, I is the set of the basic variables indices of vextor x.
To find the components $d_{j \notin I}$ of vector d, if $r_v = \max\{-r_j \mid r_j \leqslant 0, \{j \notin I\}$, we define $d_v = 1$ and for all $j \notin I$, $j \neq v$, $d_j = 0$; according to this definition, $\bigtriangledown f(x)^t d = r_v < 0$. With increasing the non-basic variable x_v, vector x is improved in the direction of the vector $d = \begin{bmatrix} -B^{-1} a_v \\ e_v \end{bmatrix}$ to vector $x + \lambda d$, where, a_v is the column corresponding to the non-basic variable x_v, and λ is obtained as follows.

$$\lambda = \min_{1 \leqslant j \leqslant m} \{ \frac{\overline{b_j}}{y_{jv}} : y_{jv} > 0 \}$$

where, $\bar{b} = B^{-1} b$ and $y_v = B^{-1} a_v$, see [8].

Finding the value of λ in the above relation is the same as doing the minimum test operation in the simplex table related to the linear programming problem.

Corollary 2.9. *Since* $\bigtriangledown f_0(x)^t d = r_{0N}^t(x) d_N$ *and* $d_N = e_v$*, therefore, according to Theorem 2.6, if* $r_{0N}^t(x) \geqslant 0$*, then* x *is an optimal solution of (1), and* x *is a unique optimal solution of (1), if* $r_{0N}^t(x) > 0$ *.*

Theorem 2.10. *If the objective function of (1) is quasi-convex, then Problem (1) has the extreme optimal solution.*

Proof. See [1]. □

Corollary 2.11. *According to theorems 2.3 and 2.10, Problem (1) has the extreme optimal solution.*

3. Concepts of MOP problems

In the real world, problems usually are expressed as multi-objective programming (MOP). The multi-objective optimization problems include several objective functions and a set of constraints. Indeed, there are several special structures in multi-objective optimization problems. One class of them is multi-objective pseudo-convex programming problems which contain several pseudo-convex objective functions and the feasible set like (1) (MOPP); multi-objective linear fractional programming (MOLFP) problems are a sample of MOPP problems. Usually, there exists conflict among objective functions in targets of the MOP problem, and usually, there doesn' t exist any feasible solution of an MOP problem that optimizes all objective functions. In this regard, in MOP the notions of

efficient solutions and weak efficient solutions are introduced instead of optimal solutions.
Consider the following multi-objective programming problem:

$$\begin{array}{ll} \min & f(x) = (f_1(x), \dots, f_p(x))^t \\ \text{s.t.} & x \in S, \end{array} \tag{2}$$

where, $S \subseteq \mathbf{R}^n$ is the feasible set and $f_i(x)$, $i = 1, 2, \dots, p$, are the objective functions.

Definition 3.1. *A point $x \in S$ is an efficient solution or Pareto-optimal solution of Problem (2) if there is no other $y \in S$ such that $f_k(y) \leq f_k(x)$ for all $k = 1, \dots, p$ and $f_j(y) < f_j(x)$ for at least one $j \in \{1, \dots, p\}$.*

Definition 3.2. *A point $x \in S$ is a weakly efficient solution of Problem (2) if there is no other $y \in S$ such that $f_k(y) < f_k(x)$ for all $k = 1, \dots, p$.*

Definition 3.3. *A point $x \in S$ is a strictly efficient solution of Problem (2) if there is no other $y \in S$, $y \neq x$ such that $f_k(y) \leq f_k(x)$ for all $k = 1, \dots, p$.*

Definitions 3.1, 3.2, and 3.3 imply that each strictly efficient solution is an efficient solution and each efficient solution is a weakly efficient solution but the reverses, necessarily, are not true.

Definition 3.4. *The point $y^I = (y_1^I, y_2^I, \dots, y_p^I)$ given by $y_k^I = \min_{x \in X} f_k(x)$ is called the ideal point of MOP.*

There are many methods to find the efficient solutions in MOP. One of the most well-known methods to identify the efficiency status of a feasible solution was proposed by Haimes and Lasdon (1971) [9]. In this method, only one of the original objectives is minimized, while the others are transformed into constraints. The ε- constraint problem associated with the MOP Problem (2) is formulated as:

$$\begin{array}{ll} \min & f_j(x) \\ \text{s.t.} & f_k(x) \leq \varepsilon_k, \quad k = 1, 2, \dots, p, \ k \neq j, \\ & x \in X, \end{array} \tag{3}$$

where, $\varepsilon \in R^p$.

Theorem 3.5. *Let x^* be an optimal solution of (3) for some j, then x^* is a weakly efficient solution of (2), and if x^* be a unique optimal solution of (3) for some j, then x^* is a strictly efficient solution of (2), and therefore x^* is efficient.*

Proof. See [9]. □

Another technique to solve the multi-objective optimization problems is the weighted sum method. By this method, the MOP is converted to an optimization problem with a single objective function. The weights of the objective functions of the MOP are specified by the decision maker's point of view. If the weights are non-negative then the optimal solutions are weakly efficient and especially, if the weights are positive then the optimal solutions are efficient. See the theorem below.

Theorem 3.6. *Suppose that x is an optimal solution of the weight sum optimization problem $\min_{x \in X} \sum_{k=1}^p \lambda_k f_k(x)$ with $\sum_{k=1}^p \lambda_k = 1$ and for all k, $\lambda_k \geqslant 0$. Then the following statements hold.*

1) *If for all k, $\lambda_k \geqslant 0$, then x is the weakly efficient solution.*
2) *If for all k, $\lambda_k > 0$, then x is the efficient solution.*
3) *If for all k, $\lambda_k \geqslant 0$ and x is uniqe optimal solution, then x is the strictly efficient solution.*

Proof. See [7]. □

4. The proposed method to find the extreme efficient solutions of MOPP

Consider the following multi-objective problem.

$$\begin{array}{ll} \min & f(x) = (f_1(x), \dots, f_p(x))^t; \\ \text{s.t.} & x \in X = \{x \in R^n : \ Ax = b, \ -x \leqslant 0\} \end{array} \tag{4}$$

where, $X \subseteq \mathbf{R}^n$ is the nonempty feasible set, $Ax = b$ is affine, $A \in R^{m \times n}$, $b \in R^m$, $rankA = m < n$ and also the objective f_k for all $k = 1, 2, \dots, p$ are pseudo-convex. $D = (\bigcap_{i=k}^p D_{f_k})$ is domain of problem (4). Note that according to Theorem 2.3, the objective functions of (4) are quasi-convex.
Now, in this section, we propose two methods for finding the extreme efficient solutions and the extreme weakly efficient solutions of (4).

4.1. The ε-constraint method

First, we express the ε-constraint method for Problem (4). Let $\varepsilon_k = \infty$, $k = 1, 2, \ldots, j-1, j+1, \ldots, p$. Therefore, the ε-constraint Problem (3) corresponding to Problem (4) is as follows.

$$\begin{array}{lll} \min & f_j(x) & j \in \{1, 2, \ldots, p\} \\ \text{s.t.} & f_k(x) \leq \infty, & k = 1, 2, \ldots, p, \ k \neq j, \\ & Ax = b, & \\ & -x \leqslant 0. & \end{array} \tag{5}$$

According to the constraints of (5), the feasible area of (5) is the same as the feasible area of (4). Therefore in (5), we can remove the additional constraints $f_k(x) \leq \infty, \ k = 1, 2, \ldots, p, \ k \neq j$, and the above problem is transformed into the following problem:

$$\begin{array}{lll} \min & f_j(x) & j \in \{1, 2, \ldots, p\} \\ \text{s.t.} & Ax = b, & \\ & -x \leqslant 0. & \end{array} \tag{6}$$

Any feasible solution of (6) is a feasible solution of (5), and also, the optimal solution of (6) is equal to the optimal solution of (5), and according to theorem 3.5, is a weakly efficient solution of (4), and also, the unique optimal solution of (6) is a strictly efficient and efficient solution of (4). According to what was said, we state the following theorem.

Theorem 4.1. *In Problem (6), let $x \in X$, and for $j \in \{1, 2, \ldots, p\}$, $r^t_{jN}(x) = \bigtriangledown_N f_j(x)^t - \bigtriangledown_B f_j(x)^t B^{-1} N$, if $r^t_{jN}(x) \geqslant 0$, then x is a weakly efficient solution of (4), and x is a strictly efficient and efficient solution of (4), if $r^t_N(x) > 0$.*

Proof. It can be proved according to Corollary 2.9. □

Since, according to Corollary 2.11, all f_j have the optimal extreme points, for this reason, in the presented method, the efficiency status of the extreme points of the feasible set X is checked.

Theorem 4.2. *Suppose x_1 and x_2 are the adjacent extreme points and x_1 is an efficient solution. If for all k, $r^t_{kN}(x_2) = \bigtriangledown_N f_k(x)^t - \bigtriangledown_B f_k(x)^t B^{-1} N < 0$, then all the points between x_1 and x_2 are efficient solutions.*

Proof. For each $\alpha \in (0, 1)$, $x_1 + \alpha(x_2 - x_1)$ is located on the connecting line between x_1 and x_2. Now, contradiction, suppose that $x_1 + \alpha(x_2 - x_1)$ is not an efficient solution of MOPP problem. Then, there exists a feasible solution $y \in X$ such that $f_k(y) \leqslant f_k(x_1 + \alpha(x_2 - x_1))$ for all k and $f_j(y) < f_j(x_1 + \alpha(x_2 - x_1))$ for some j. Because f_k for all k are the pseudo-convex and quasi-convex functions, therefore $f_k(y) \leqslant f_k(x_1 + \alpha(x_2 - x_1)) \leqslant \max\{f_k(x_1), f_k(x_2)\}$ for all k and $f_j(y) < f_j(x_1 + \alpha(x_2 - x_1)) \leqslant \max\{f_j(x_1), f_j(x_2)\}$ for some j.
On the other hand x_1 and x_2 are the adjacent extreme points, we can show $x_2 = x_1 + \lambda d$ for some $\lambda > 0$. Therefore, for all k, $\bigtriangledown f_k(x_2)(x_1 - x_2) = -\lambda \bigtriangledown f_k(x_2) d = -\lambda r^t_{kN}(x_2) d_N$, since for all k, $r^t_{kN}(x_2) < 0$, $\lambda > 0$, and $d_N = e_v$, therefore, for all k, $\bigtriangledown f_k(x_2)(x_1 - x_2) > 0$, and since that all f_k are pseudo-convex and quasi-convex functions, therefore according to Theorem 2.4, for all k, $f_k(x_1) > f_k(x_2)$, and $\max\{f_k(x_1), f_k(x_2)\} = f_k(x_1)$.
These relations imply that $f_k(y) \leqslant f_k(x_1)$ for all k and $f_j(y) < f_j(x_1)$ for some j which contradicts with the assumption of the efficiency of x_1. □

To find the efficient solution of (4), we design a table similar to simplex table as Table 1.

Table 1: The structure of the basic and non-basic variables of MOPP problem

	x_B	x_N	RHS
r^t_1, r^t_2, $\vdots$, r^t_p	0	$\bigtriangledown_N f_k(x)^t - \bigtriangledown_B f_k(x)^t B^{-1} N$	$f_k(x_B, 0)$
x_B	I	$B^{-1}N$	$B^{-1}b$

Since, all the objective functions in (4) are pseudo-convex, with regarding to corollary 2.11, each function has an extreme optimal solution in the polyhedron set X. Therefore, in Table 1, if $r^t_{kN} > 0$ for $k \in \{1, 2, \ldots, p\}$, the obtained solution is an extreme optimal solution of f_k for $k \in \{1, 2, \ldots, p\}$, and according to the Theorem 4.1, it is an efficient solution of (4). Also, if $r^t_{kN} \geqslant 0$ for $k \in \{1, 2, \ldots, p\}$, the obtained solution is an extreme optimal

solution for Problem (4), and according to Theorem 4.1 is a weakly efficient solution of (4).
Now, in the following, we present an algorithm that can be used to obtain an extreme efficient solution and also an extreme weakly efficient solution of (4).

4.2. Algorithm

Step 1. In Problem (4), we choose a extreme feasible solution $x = (x_B, 0)$.
Step 2. Calculate the vector $r_k^t(x) = (r_{kB}^t(x), r_{kN}^t(x)) = (0, \bigtriangledown_N f_k(x)^t - \bigtriangledown_B f_k(x)^t B^{-1} N)$ for all $k = 1, 2, \ldots, p$ and we put it in the Table 1.
Step 3. If $r_{kN}^t(x) > 0$ for all $k = 1, 2, \ldots, p$, then x is an extreme strictly efficient solution and efficient solution, and also $y^I = (f_1(x), f_2(x), \ldots, f_p(x))$ is ideal point of (4).
Step 4. If for all k, $r_{kN}^t(x) \ngtr 0$, and for some j, $r_{jN}^t(x) > 0$, then x is an extreme strictly efficient solution and efficient solution, and if for some j, $r_{jN}^t(x) \geqslant 0$, then x is an extreme weakly efficient solution, otherwise, x is inefficient. To find and check the other vertex of the feasible space, go to Step 5.
Step 5. Suppose $r_{ik}(x)$ is i_{th} component of vector $r_k^t(x)$, and I is set of basic variables indices of vector x, if $r_v(x) = \max\{-r_{ik}(x) \mid r_{ik}(x) \leqslant 0, i \notin I, k = 1, 2, \ldots, p\}$, then the nonbasic variable x_v becomes the candidate to enter the basic. Also, using the minimum test, the output variable of the base is determined. Then go to Step 2.

It is noteworthy, according to Theorem 4.2, if x_1 and x_2 be the adjacent extreme points and x_1 be efficient solution, if for all k, $r_{kN}^t(x_2) < 0$, then $y = \alpha x_1 + (1-\alpha)x_2$ for $\alpha \in (0, 1)$ is efficient solution of (4).

4.3. The weighted sum method

The weight sum model of Problem (4) with weights $\lambda_k \geqslant 0$ and $\sum_{k=1}^p \lambda_k = 1$, is as follows.

$$\begin{array}{ll} \min & F_0(x) = \sum_{k=1}^p \lambda_k f_k(x) \\ \text{s.t.} & Ax = b, \\ & -x \leqslant 0. \end{array} \tag{7}$$

According to the corollary 2.9 and Theorem 3.6; suppose x is a feasible solution of (4) and for all k, $\lambda_k \geqslant 0$ and $\sum_{k=1}^p \lambda_k = 1$, if $r_{0N}^t(x) = (\bigtriangledown_N F_0(x)^t - \bigtriangledown_B F_0(x)^t B^{-1} N) \geqslant 0$, then x is the weakly efficient solution of (4), and for all k, $\lambda_k > 0$ and $\sum_{k=1}^p \lambda_k = 1$, if $r_{0N}^t(x) = (\bigtriangledown_N F_0(x)^t - \bigtriangledown_B F_0(x)^t B^{-1} N) \geqslant 0$, then x is efficient soluation of (4). Also, for all k, $\lambda_k \geqslant 0$ and $\sum_{k=1}^p \lambda_k = 1$, if $r_{0N}^t(x) = (\bigtriangledown_N F_0(x)^t - \bigtriangledown_B F_0(x)^t B^{-1} N) > 0$, then x is strictly efficient solution of (4).

5. Example

In this section, two examples are presented and their extreme efficient points are determined using the presented methods.

5.1. Example 1

As mentioned, the linear fractional programming (LFP) problems are a sample of the pseudo-convex programming problem. Here, we calculate the extreme efficient solutions and the extreme weakly efficient solutions of the multi-objective problem with the fractional functions and the polyhedron solution set (MOLFP). Consider the following multi-objective linear fractional programming (MOLFP) problem

$$\begin{array}{ll} \min & f_1(x) = \frac{-x_1 - x_2}{x_1 + 2}; \\ \min & f_2(x) = \frac{-x_1}{x_2 + 3}; \\ \text{s.t.} & \frac{-3}{2} x_1 + x_2 \leqslant 4; \\ & x_1 + x_2 \leqslant 11; \\ & 2x_1 + x_2 \leqslant 16; \\ & x_1, x_2 \geqslant 0. \end{array} \tag{8}$$

The feasible solution set of (8) is shown in Figure 1.

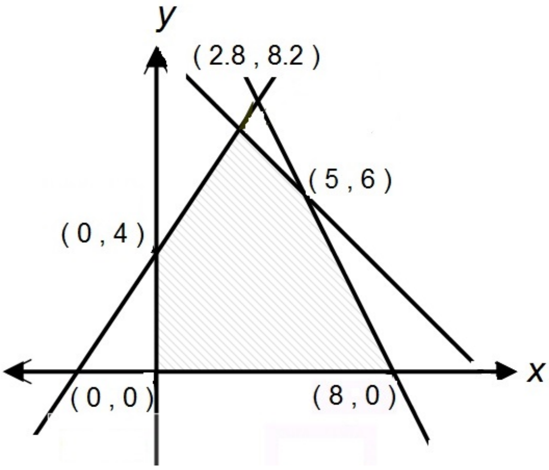

Figure 1: The possible area of the Problem (8)

For finding efficient solutions of Problem (8), we first start with the extreme point $x^1 = (0, 0, 4, 11, 16)$, and check its efficiency; the requested items are calculated from Table 1 and the results are displayed in Table 2 (We have used MATLAB 2017 software). By checking the steps of the presented algorithm, since $r^t_{1N}(x^1) < 0$ and $r^t_{2N}(x^1) \leqslant 0$, and according to Step 4 of algorithm, therefore the extreme point $x^1 = (0, 0, 4, 11, 16)$ is inefficient.

Table 2: Checking the efficiency of the extreme point $x^1 = (0, 0, 4, 11, 16)$

	x_1	x_2	s_1	s_2	s_3	RHS
r_1	-.5	-.5	0	0	0	
r_2	-.33	0	0	0	0	
s_1	-1.5	1	1	0	0	4
s_2	1	1	0	1	0	11
s_3	2	1	0	0	1	16

Step 5 of the algorithm; by entering the non-base variable x_2 to the base, the extreme $x^2 = (0, 4, 0, 7, 12)$ is obtained, and by forming Table 3 and checking it with the presented algorithm, we conclude that this point is also inefficient.

Table 3: Checking the efficiency of the extreme point $x^2 = (0, 4, 0, 7, 12)$

	x_1	x_2	s_1	s_2	s_3	RHS
r_1	-0.25	0	0.50	0	0	
r_2	-0.14	0	0	0	0	
x_2	-1.5	1	1	0	0	4
s_2	2.5	0	-1	1	0	7
s_3	3.5	0	-1	0	1	12

By continuing this process, the extreme point $x^3 = (2.8, 8.2, 0, 0, 2.2)$ is obtained (see Table 4), and by examining the steps of the presented algorithm, since $r^t_{1N}(x^3) > 0$, therefore according to Step 4 of the presented algorithm the extreme point $x^3 = (2.8, 8.2, 0, 0, 2.2)$ is a strictly efficient and efficient solution of (8).

Table 4: Checking the efficiency of the extreme point $x^3 = (2.8, 8.2, 0, 0, 2.2)$

	x_1	x_2	s_1	s_2	s_3	RHS
r_1	0	0	0.19	0.017	0	
r_2	0	0	-0.04	0.02	0	
x_2	0	1	0.4	0.6	0	8.2
x_1	1	0	-0.4	0.4	0	2.8
s_3	0	0	0.4	-1.4	1	2.2

Next, by calculating $r_v(x) = \max\{-r_{ik}(x) \mid r_{ik}(x) \leqslant 0, i \notin I, k = 1, 2\}$, the non-basic variable s_1 enters the base, and by the minimum test variable s_3 leaves the base, and the extreme point $x^4 = (5, 6, 5.5, 0, 0)$ is obtained, and since $r^t_{1N}(x^4) \ngeqslant 0$ and $r^t_{2N}(x^4) \ngeqslant 0$, therefore according to Step 4 of algorithm, x^4 is the inefficient solution of (8).

Table 5: Checking the efficiency of the extreme point $x^4 = (5, 6, 5.5, 0, 0)$

	x_1	x_2	s_1	s_2	s_3	RHS
r_1	0	0	0	-0.30	.22	
r_2	0	0	0	0.28	-0.17	
x_2	0	1	0	2	-1	6
x_1	1	0	0	-1	1	5
s_1	0	0	1	-3.5	2.5	5.5

In the same way, according to Table 6, another extreme point is found, and finally, based on the proposed algorithm and the observations of Table 6, since $r^t_{2N}(x^5) > 0$, therefore the extreme points $x^5 = (8, 0, 16, 3, 0)$ is strictly efficient and efficient solution of (8).

Table 6: Checking the efficiency of the extreme point $x^5 = (8, 0, 16, 3, 0)$

	x_1	x_2	s_1	s_2	s_3	RHS
r_1	0	-0.09	0	0	0.01	
r_2	0	1.05	0	0	0.16	
s_2	0	0.5	0	1	-0.5	3
x_1	1	0.5	0	0	0.5	8
s_1	0	1.75	1	0	-1	16

Table 7 presents the results of the efficiency status of the extreme feasible solutions of (8).

Table 7: Results of the efficiency of the extreme points of (8)

ROW	extreme feasible solutions	Efficiency status
1	$(0, 0, 4, 11, 16)$	Inefficient
2	$(0, 4, 0, 7, 12)$	Inefficient
3	$(2.8, 8.2, 0, 0, 2.2)$	Strictly efficient and efficient
4	$(5, 6, 5.5, 0, 0)$	Inefficient
5	$(8, 0, 16, 3, 0)$	Strictly efficient and efficient

5.2. Example 2

Consider the following two-objective problem.

$$\begin{aligned} \min \quad & f_1(x) = \tfrac{-1}{3}x_1^3 - x_2 - x_3; \\ \min \quad & f_2(x) = \tfrac{1}{x_1+x_2+1} - \tfrac{1}{3}x_3^3; \\ \text{s.t.} \quad & x_1 + x_2 + x_3 \leqslant 4; \\ & 3x_2 - x_3 \leqslant 6; \\ & x_1, x_2, x_3 \geqslant 0, \end{aligned} \tag{9}$$

where, $f_1(x)$ and $f_2(x)$ are pseudo-convex functions. The feasible solution set of (9) is shown in Figure 2. First, we check the efficiency status of the extreme point A=(0, 0, 0, 4, 6). The vectors $r^t_{1N}(A)$ and $r^t_{2N}(A)$ are calculated and are located in Table 8. Since $r^t_{1N}(A) \ngeqslant 0$ and $r^t_{2N}(A) \ngeqslant 0$, therefore, the extreme point A is an inefficient solution of (9).

In the following, since $r_{x_1} = 1$, therefore the non-basic variable x_1 enters the base and by the minimum test variable s_1 becomes exit from the base, and the extreme point $E = (4, 0, 0, 0, 6)$ is obtained. By calculating the vectors $r^t_{1N}(E)$ and $r^t_{2N}(E)$, Table 8 is transformed into Table 9.

Table 8: Checking the efficiency of the extreme point $A = (0, 0, 0, 4, 6)$

	x_1	x_2	x_3	s_1	s_2	RHS
r_1	0	-1	-1	0	0	
r_2	-1	-1	0	0	0	
s_1	1	1	1	1	0	4
s_2	0	3	-1	0	1	6

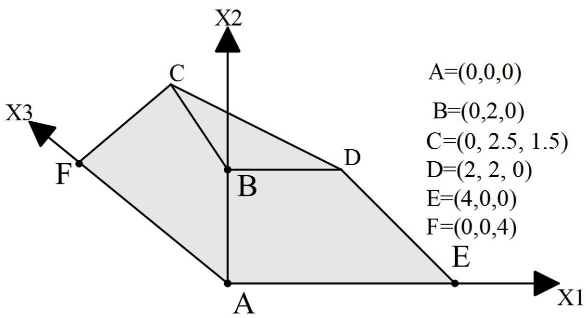

Figure 2: The possible area of the Problem (9)

Table 9: Checking the efficiency of the extreme point E=(4, 0, 0, 0, 6)

	x_1	x_2	x_3	s_1	s_2	RHS
r_1	0	15	15	16	0	
r_2	0	0	0.20	0.20	0	
x_1	1	1	1	1	0	4
s_2	0	3	-1	0	1	6

According to Table 9, since $r_{1N}^t(E) > 0$, therefore the extreme point E is the unique optimal solution of $f_1(x)$ and is a strictly efficient solution and an efficient solution of (9).
In the same way, the extreme points D, C, F, and B are found (see Tables 10-13), since $r_{2N}^t(F) > 0$, therefore the extreme point F is strictly efficient as well. therefore Problem (9) has two extreme points E and F that both are strictly efficient and efficient.

Table 10: Checking the efficiency of the extreme point D=(2, 2, 0, 0, 0)

	x_1	x_2	x_3	s_1	s_2	RHS
r_1	0	0	3.99	4	-0.99	
r_2	0	0	-6.33	0.20	6.53	
x_1	1	0	1.33	1	-0.33	2
x_2	0	1	-0.33	0	0.33	2

Table 11: Checking the efficiency of the extreme point $C = (0, 2.5, 1.5, 0, 0)$

	x_1	x_2	x_3	s_1	s_2	RHS
r_1	1	0	0	1	0	
r_2	1.47	0	0	1.76	-0.49	
x_3	0.75	0	1	0.75	-0.25	1.5
x_2	0.25	1	0	0.25	0.25	2.5

Table 12: Checking the efficiency of the extreme point $F = (0, 0, 4, 0, 10)$

	x_1	x_2	x_3	s_1	s_2	RHS
r_1	1	1	0	0	0	
r_2	15	15	0	16	0	
x_3	1	1	1	1	0	4
s_2	1	4	0	1	1	10

Table 13: Checking the efficiency of the extreme point $B = (0, 2, 0, 2, 0)$

	x_1	x_2	x_3	s_1	s_2	RHS
r_1	0	0	-1.33	0	0.33	
r_2	-0.33	0	-0.11	0	0.11	
s_1	1	0	1.33	1	-0.33	2
x_2	0	1	-0.33	0	0.33	2

In the following, we find an efficient solution of (9), by the weight sum method. for this purpose, we consider $\lambda = (0.2, 0.8)$.
The weighted model of Problem (9) is as follows.

$$\begin{array}{ll} \min & F_0(x) = 0.2(\frac{-1}{3}x_1^3 - x_2 - x_3) + 0.8(\frac{1}{x_1+x_2+1} - \frac{1}{3}x_3^3); \\ \text{s.t.} & x_1 + x_2 + x_3 \leqslant 4; \\ & 3x_2 - x_3 \leqslant 6; \\ & x_1, x_2, x_3 \geqslant 0. \end{array} \tag{10}$$

The optimal solution of the single-objective Problem (10) is an efficient solution of (9), and its unique optimal solution is also a strictly efficient solution of (9). The obtained efficient solution from this method for (9) is the extreme point E=(4, 0, 0, 0, 6). To reach this efficient point, we start from point A, and by calculating the vector $r_{0N}^t(A) = (\bigtriangledown_N F_0(A)^t - \bigtriangledown_B F_0(A)^t B^{-1} N)$, we form Table 14.
Table 14 is not optimal, therefore A is inefficient. In Table 14, $r_v(x) = \max\{-r_{i0}(x) \mid r_{i0}(x) \leqslant 0, i \notin I, \} = 0.80$, therefore the non-basic variable x_1 enters the base and by the minimum test, variable s_1 exists from the base, and the extreme point $E = (4, 0, 0, 0, 6)$ is obtained, and for checking the optimality status E, we calculate the vector $r_{0N}^t(E) = (\bigtriangledown_N F_0(E)^t - \bigtriangledown_B F_0(E)^t B^{-1} N)$. (See Table 15).
Since $r_{0N}^t(E) > 0$, therefore E is the unique optimal solution of (10), and a strictly efficient solution and efficient solution of (10).

Table 14: Checking the efficiency of the extreme point $A = (0, 0, 0, 4, 6)$ for $\lambda = (0.2, 0.8)$

	x_1	x_2	x_3	s_1	s_2	RHS
r	-0.80	-1	-0.20	0	0	
s_1	1	1	1	1	0	4
s_2	0	3	-1	0	1	6

Table 15: Checking the efficiency of the extreme point $E = (4, 0, 0, 0, 6)$ for $\lambda = (0.2, 0.8)$

	x_1	x_2	x_3	s_1	s_2	RHS
r	0	3	3.16	3.36	0	
x_1	1	1	1	1	0	4
s_2	0	3	-1	0	1	6

6. Discussion and Results

The multi-objective optimization problem is one of the most important problems of optimizations in which finding an efficient solution of the MOP is of great importance. It is not easy to find an efficient solution of MOP problems with common methods such as the weighted or constraint methods. Sometimes, we come across a multi-objective problem in that their objective functions are the pseudo-convex functions and a polyhedron; we call them MOPP problems. In this type of problem, due to the type of objective function, the efficient solution may not be found by applying the methods that are usually used to determine the efficient solution. Therefore, in this paper, we present two methods to find the extreme weakly efficient and efficient solutions. Because the set of quasi-convex and convex functions are subsets of the set of pseudo-convex functions, therefore, the presented method can be used for MOPP problems with the quasi-convex and convex functions.

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