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Original Article

Analyzing of approximate symmetry and new conservation laws of perturbed generalized Benjamin-Bona-Mahony equation

Mehdi Jafari*[a], Razie Darvazebanzade[a]

[a] *Department of Mathematics, Payame Noor University, PO BOX 19395-4697, Tehran, Iran*

ABSTRACT: In this paper, we prove that the perturbed generalized Benjamin-Bona-Mahony (BBM) equation with a small parameter is approximately nonlinear self-adjoint. It's important for constructing approximate conservation laws associated with approximate symmetries.

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1. Introduction

In physics, laws that keep the qualitative characteristics of a system unchanged for a certain period of time are called conservation laws. For example, the law of conservation of matter and energy, the law of conservation related to electrical changes, or the law of conservation related to linear momentum. It is necessary to pay attention to the fact that some features are preserved only in certain cases. For example, the value of a quantity may be maintained at only one point, or it may not change inside a volume, while changes are taking place outside that volume. In this case, we have the laws of local consistency. Also, we propose the approximate conservation laws for systems with small disturbance parameters, which of course is related to this article. We need these rules to increase our knowledge and understanding of the surrounding world. Conservation laws have an important role in analyzing and checking basic characteristics for solutions of an equation [10].

**Corresponding author.*
E-mail addresses: m.jafarii@pnu.ac.ir, razie.ddarvazeban@yahoo.com

According to Noether's research in 1918, for systems that basically apply changes, it can be concluded that every law of consistency is derived from a symmetry [20]. In fact, Noether's theorem establishes a relationship in the middle of the symmetries of the differential equations system and constancy laws for equations that apply changes in principle. This class of equations is called Euler-Lagrange's equations and of course, it is possible to find the Lagrangian for certain types of equations [18]. A large part of the phenomena that we deal with in various sciences of mathematics, physics, and engineering are described by partial differential equations. Therefore, obtaining the exact solutions to this class of equations plays a very important role in their analysis, although in many cases it is very difficult or even impossible to find such exact solutions.

In recent years, special and practical methods for constructing exact solutions of this category of equations have been discovered. The most famous of which is the method of classical symmetries or, in other words, Lie symmetry groups which is attributed to Sophos Lie [16].

We have a type of PDEs in which there is a small disturbance parameter like ε [6, 23, 26, 27, 28]. This small disturbance parameter affects the solutions of the system. In other words, the symmetry group changes in these equations. For this purpose, people like Ibragimov et al., Baikov, and Gazizov proposed a new method to determine the symmetries of these equations and also to determine their conservation laws when the conservation laws cannot be obtained to help of Noether's method. In this method, the concept of approximate conservation laws is proposed with the help of Lagrangian for perturbed equations [1, 2, 3, 6, 12]. There are other methods for determining approximate symmetries, which can be seen in the words of researchers such as Fushchich and Shtelen [9].

In this article, we are going to establish the approximate conservation laws for the generalized and perturbed BBM equation as:

$$2uu_t + 3u^2u_x + u_{3x} + \varepsilon(u_{2x} + u_{4x}) = 0, \tag{1}$$

which was first proposed by Benjamin-Bona-Mahony in order to improve the Kortewage-de-Vries (KdV) equation for short-amplitude wavelengths with 1+1 dimension [4, 5, 8]. Derks and Gils examine the uniqueness of traveling waves of (1) in 1993 [7]. Ogawa [21] investigates the existence of periodic waves and solitary waves of (1) in 1994 and presents the connection between the wavelength and the amplitude. If we remove terms u_{2x} and u_{4x} in (1), the KdV equation is obtained, which is widely used in the study of water waves [13, 15]. Perturbed generalized KdV equation was studied by Yan et al. in 2014. Utilizing the regular perturbation analysis for a Hamiltonian system and the geometric singular perturbation theory, they have shown that solitary wave solutions and periodic wave solutions stay for enough small perturbation parameter [25]. Also, Wazwaz has studied some nonlinear dispersive generalized forms of the BBM equation in [24]. Recently, many studies have been done on BBM equation and exact solutions and conservation laws for this equation and its different types have been provided [14]. But regarding the perturbed equation, limited studies have been done and the present work will obtain valuable results using the Ibragimov method on this equation.

This article is classified as follows: In Section 2, we discuss important and practical concepts. In Section 3, with the Ibragimov method, we will calculate the approximate symmetries for (1). In Section 4, we obtain the approximate conservation laws for (1).

2. Definitions and concepts

In this section, we will express some necessary and important definitions and theorems.

Definition 2.1. *Let $z = (z^1, ..., z^N)$ be a variable of arbitrary dimension N. A category of first-order differential operators,*

$$X = \xi^i(z, \varepsilon)\frac{\partial}{\partial z^i},$$

such that

$$\xi^i(z, \varepsilon) \approx \xi_0^i(z) + \varepsilon\xi_1^i(z) + \cdots + \varepsilon^p\xi_p^i(z), \qquad i = 1, \dots, N,$$

that $\xi_0^i(z), \xi_1^i(z), \dots, \xi_p^i(z)$ for $i = 1, \dots, N$, are fixed functions and called approximate operators.

Definition 2.2. *Suppose $f(x, \varepsilon)$ is a function of n variables $x = (x^1, \dots, x^n)$ with parameter ε. If $f(x, \varepsilon)$ apply to that*

$$\lim_{\varepsilon \to 0} \frac{f(x, \varepsilon)}{\varepsilon^p} = 0,$$

then, $f(x, \varepsilon) = o(\varepsilon^p)$ and f is of lower order ε^p.

If

$$f(x,\varepsilon)-g(x,\varepsilon)=o(\varepsilon^p),$$

then we call g and f approximately equal and denoted by $f\approx g$ [11].

We consider,

$$\begin{aligned}\tilde{t}&=t+a\tau(t,x,u,\varepsilon)+o(a^2),\\ \tilde{x}&=x+a\xi(t,x,u,\varepsilon)+o(a^2),\\ \tilde{u}&=u+a\phi(t,x,u,\varepsilon)+o(a^2),\end{aligned}\tag{2}$$

such that

$$\xi(t,x,u,\varepsilon)\approx\xi_0(t,x,u)+\varepsilon\xi_1(t,x,u)+\cdots+\varepsilon^p\xi_p(t,x,u),$$

that a is the parameter of group and ε is small parameter as a one-parameter group of approximate transformations G. Functions τ and ϕ are defined similarly.

The infinitesimal generator of G is in the form,

$$X=\xi(t,x,u,\varepsilon)\frac{\partial}{\partial x}+\tau(t,x,u,\varepsilon)\frac{\partial}{\partial t}+\phi(t,x,u,\varepsilon)\frac{\partial}{\partial u}.$$

Let,

$$X=X_0+\varepsilon X_1,\tag{3}$$

be an arbitrary infinitesimal generator, where

$$\begin{aligned}X_0&=\xi_0(t,x,u)\frac{\partial}{\partial x}+\tau_0(t,x,u)\frac{\partial}{\partial t}+\phi_0(t,x,u)\frac{\partial}{\partial u},\\ X_1&=\xi_1(t,x,u)\frac{\partial}{\partial x}+\tau_1(t,x,u)\frac{\partial}{\partial t}+\phi_1(t,x,u)\frac{\partial}{\partial u}.\end{aligned}$$

Theorem 2.3. *If (3) is an approximate invariant which admits G, then*

$$X=X_0+\varepsilon X_1\approx\xi_0^i(z)\frac{\partial}{\partial z^i}+\varepsilon\xi_1^i(z)\frac{\partial}{\partial z^i},\tag{4}$$

is the generator of G if and only if

$$[X^{(k)}H(z,\varepsilon)]_{H\approx0}=o(\varepsilon),$$

or

$$[X_0^{(k)}H_0(z)+\varepsilon(X_1^{(k)}H_0(z)+X_0^{(k)}H_1(z))]_{(7)}=o(\varepsilon),\tag{5}$$

and on the contrary, where $X^{(k)}$ is the prolongation of X in order k [11].

If operator (4) satisfied the relation (5), we say, an infinitesimal approximate symmetry has been accepted by (3).

Definition 2.4. *If we consider G as a group of one-parameter approximate transformations with*

$$\tilde{z}^i\approx f(z,a,\varepsilon)\equiv f_0^i(z,a)+\varepsilon f_1^i(z,a),\qquad i=1,\cdots,N,\tag{6}$$

then

$$H(z,\varepsilon)\equiv H_0(z)+\varepsilon H_1(z)\approx0,\tag{7}$$

is called an approximate equation and if $z=(x,u,u_{(1)},\cdots u_{(m)})$, then (7) is an approximate equation of order m and G becomes a group of approximate transformations for the differential equation [18].

Theorem 2.5. *If it's possible to consider the operator $X=X_0+\varepsilon X_1$ which is $X_0\neq0$, for the relation (7), then*

$$X_0=\xi_0^i\frac{\partial}{\partial z^i},\tag{8}$$

considered an exact symmetry for the equation of

$$H_0(z)=0.\tag{9}$$

Definition 2.6. *We call equation (7) the perturbed part and equation (9) the unperturbed part of the equation. According to what was stabled in Theorem 2.5, the operator X_0 obtained from equation (8) is the exact symmetry for the unperturbed equation (9) and the approximate symmetry generator that is obtained from the relation $X = X_0 + \varepsilon X_1$ is also called the transformation of infinitesimal symmetry X of (9) and is obtained with the help of $\varepsilon F_1(z)$ if these symmetries differ in only one ε, then the perturbed equation (7) inherits the symmetries of the unperturbed equation.*

Definition 2.7. *Suppose*

$$L \equiv vE, \tag{10}$$

with the dependent variable $v = v(x,t)$, in this case, the adjoint equation is defined as follows,

$$E^* = \frac{\delta L}{\delta u} = \frac{\partial L}{\partial u} - D_i \frac{\partial L}{\partial u_i} + D_i D_j \frac{\partial L}{\partial u_{ij}} - D_i D_j D_k \frac{\partial L}{\partial u_{ijk}} + \cdots = 0, \qquad i,j,k = 1,2, \tag{11}$$

which $\frac{\delta L}{\delta u}$, be as the variational derivative and D_i, be as the total differential operator.

Definition 2.8. *If the same initial equation is obtained by putting $v = u$ in (11), then $E = 0$ is called self-adjoint.*

Definition 2.9. *If a function like $v = \phi(u)$ where $\phi'(u) \neq 0$ exists so that*

$$E^*|_{v=\phi(u)} = \lambda E,$$

(λ is an undetermined coefficient) then $E = 0$ is quasi-self-adjoint.

Definition 2.10. *For equation (1), if exists a vector in form of $C = (C^1, C^2)$ which applies in the condition*

$$D_t(C^1) + D_x(C^2) = 0, \tag{12}$$

to all the solutions of the equation E and E^*, then has a non-local conservation law and has a local conservation law if C applies only to the solutions E.

Definition 2.11. *In the method of Ibragimov that is apply the Lagrangian for perturbed equations [11], first, you need to calculate the conserved vectors with the help of point symmetries using the following formula [6],*

$$v = \xi^1(t,x,u)\partial_x + \xi^2(t,x,u)\partial_t + \phi(t,x,u)\partial_u,$$

and

$$C^i = W[\frac{\partial L}{\partial u_i} - D_j(\frac{\partial L}{\partial u_{ij}}) + D_j D_k(\frac{\partial L}{\partial u_{i,j,k}})] + D_j(W)[\frac{\partial L}{\partial u_{i,j}} - D_k(\frac{\partial L}{\partial u_{i,j,k}})] + D_j D_k(W)\frac{\partial L}{\partial u_{i,j,k}}, \tag{13}$$

where $i,j,k = 1,2$ and $W = \phi - \xi^i u_i$.

In the following, we explain the method of obtaining Lie point symmetries for an arbitrary system of differential equations. We have considered a nonlinear system of differential equations with partial derivatives of order n, including p independent variable and q dependent variable along with its derivatives [17, 22],

$$\Delta_v(x, u^n) = 0, \qquad v = 1, \ldots, L, \quad x = (x^1, \ldots, x^p), \quad u = (u^1, \ldots, u^q). \tag{14}$$

In this way, we consider the one-parameter Lie group transformations that act on (14) as follows:

$$\begin{aligned} \bar{x}_i &= x^i + \varepsilon \xi^i(x,u) + o(\varepsilon^2), \\ \bar{u}_j &= u^j + \varepsilon \phi^j(x,u) + o(\varepsilon^2), \end{aligned}$$

where $i = 1, \ldots, p, j = 1, \ldots, q$, and ξ^i as well as ϕ^j are the infinitesimal of system [19]. By choosing the vector field v in the following form,

$$v = \sum_{i=1}^{p} \xi^i(x,u)\partial_{x^i} + \sum_{j=1}^{q} \phi^j(x,u)\partial_{u^j},$$

the symmetries of the system are obtained. The application of these symmetries to the infinitesimal transformation causes the invariance conditions

$$v^n[\Delta_v(x, u^{(n)})] = 0, \qquad \Delta_v(x, u^{(n)}) = 0, \qquad v = 1, \ldots, r,$$

where v^n is called the infinitesimal generator prolongation of order n that compute by

$$v^n = v + \sum_{j=1}^{q}\sum_{k}\phi_k^j(x,u^{(n)})\partial_{u_k^j},$$

where

$$\phi_j^k(x,u^{(n)}) = D_k(\phi_j - \sum_{i=1}^{p}(\xi^i u_i^j)) + \sum_{i=1}^{p}\xi^i u_{k,i}^j,$$

and $k=(i_1,\dots,i_\alpha), 1\le i_\alpha\le p, 1\le\alpha\le n$.

3. Approximate symmetry analysis for the perturbed generalized BBM equation

In this section, we intend to calculate the group of approximate symmetries for the perturbed and generalized BBM equation, for this purpose, we find the vector field $X = X_0+\varepsilon X_1$ as an approximate symmetry group for the following equation,

$$\Delta_0 := 2uu_x + 3u^2u_x + u_{3x} + \varepsilon(u_{2x}+u_{4x}) = 0, \qquad 0\le\varepsilon<1.$$

In this way, the infinitesimal generator for the equation is obtained in the following form,

$$X_0 = (-2C_1t+C_2)\partial_t + (-C_1x+C_3)\partial_x + C_1u\partial_u, \tag{15}$$

where C_1, C_2, C_3 are arbitrary constants. To determine the approximate symmetries of (1), we must first calculate the auxiliary function I according to the following command,

$$I = \frac{1}{\varepsilon}\left[pr^{(k)}X_0(H_0(z)+\varepsilon H_1(z))\Big|_{H_0(z)+\varepsilon H_1(z)=0}\right].$$

By putting X_0 from (15) in the above equation, we have,

$$I = -C_1(u_{2x}-u_{4x}).$$

To calculate X_1, its enough to use the relation,

$$pr^{(k)}X_1(H_0(z))\Big|_{H_0(z)} + I = 0. \tag{16}$$

By placing the above results in (16) we have,

$$pr^{(3)}X_1(\Delta_1)\Big|_{\Delta_1=0} - C_1(u_{2x}-u_{4x}) = 0. \tag{17}$$

By solving (17), the coefficients ξ_1, τ_1, and ϕ_1 are obtained. After substituting them in X we have,

$$\begin{aligned} X &= X_0+\varepsilon X_1 \\ &= (-2C_1t+C_2)\partial_t + (-C_1x+C_3)\partial_x + C_1u\partial_u + \varepsilon\Big((-2A_1t+A_2)\partial_t + (-A_1x+A_3)\partial_x + A_1u\partial_u\Big), \end{aligned}$$

and the infinitesimal approximate symmetry spanned by the following independent operators:

$$\begin{array}{lll} V_1 = \varepsilon\partial_x, & V_2 = \varepsilon(\frac{1}{4}x\partial_x + u\partial_u), & V_3 = \varepsilon\partial_t, \\ V_4 = \varepsilon(t\partial_t + \frac{1}{4}x\partial_x), & V_5 = \partial_x, & V_6 = \frac{1}{4}x\partial_x + u\partial_u, \\ V_7 = \partial_t, & V_8 = t\partial_t + \frac{1}{4}x\partial_x. & \end{array} \tag{18}$$

4. Adjoint equation and computing conserved laws

Formal Lagrangian for (1) is

$$L = vE = v(2uu_t + 3u^2u_x + u_{3x} + \varepsilon(u_{2x}+u_{4x})),$$

and the adjoint equation (E^*) to (1) is

$$E^* = v(2u_t+6uu_x) - D_x(3u^2v) - D_t(2uv) + D_x^2(\varepsilon v) - D_x^3(v) + D_x^4(\varepsilon v) = 0.$$

With setting $u = v$ it becomes the original equation, hence (1) is self-adjoint. With substitution

$$v(t,x,u,\varepsilon) \simeq \phi(t,x,u) + \varepsilon\psi(t,x,u),$$

that is satisfying in nonlinear self-adjoint condition

$$E^*|_{(4)} \simeq \lambda E,$$

we have,

$$\begin{aligned}
&-3u^2\phi_x - 3u^2\phi_u u_x - 3u^2\varepsilon\psi_x - 3u^2\varepsilon\psi_u u_x - 2u\phi_t - 2u\phi_u u_t - 2u\varepsilon\psi_t - 2u\varepsilon\psi_u u_t + \\
&\varepsilon\phi_{2x} + 2\varepsilon\phi_{xu}u_x + \varepsilon\phi_{2u}u_x^2 + \varepsilon\phi_u u_{2x} - \phi_{3x} - 3\phi_{2xu}u_x - 3\phi_{2ux}u_x^2 - 3\phi_{xu}u_{2x} - \\
&\phi_{3u}u_x^3 - 3\phi_{2u}u_x u_{2x} - \phi_u u_{3x} - \psi_{3x} - 3\psi_{2xu}u_x - 3\psi_{2ux}u_x^2 - 3\phi_{xu}u_{2x} - \phi_{3u}u_x^3 - \\
&3\psi_{2u}u_x u_{2x} - \psi_u u_{3x} + \phi_{4x} + 4\phi_{3xu}u_x + 6\phi_{2x2u}u_x^2 + 6\phi_{2xu}u_{2x} + 3\phi_{3ux}u_x^3 + \\
&9\phi_{x2u}u_x u_{2x} + 3\phi_{2x2u}u_{2x}u_x + 4\phi_{xu}u_{3x} + \phi_{3ux}u_x^2 + \phi_{4u}u_x^4 + 6\phi_{3u}u_x^2 u_{2x} + \\
&3\phi_{2u}u_{2x}^2 + 9\phi_{2u}u_x u_{3x} + \phi_u u_{4x} = \lambda(2uu_t + 3u^2 u_x + u_{3x} + \varepsilon(u_{2x} + u_{4x})).
\end{aligned}$$

Then $v = \lambda u + \varepsilon$ and (1) is quasi-self-adjoint. By Approximate symmetry of (18), We obtain W_i and C^i for corresponding v_i.

For V_1 we have $W = -\varepsilon u_x$ and

$$\begin{aligned}
&C_1 = -2\varepsilon u_x u^2, \\
&C_2 = -\varepsilon u_x(3u^2 + u_{2x}) + \varepsilon u_x u_{2x} - \varepsilon u_{3x}u, \\
&D_t^{C_1} + D_x^{C_2}\Big|_{(1)} = 0.
\end{aligned}$$

For V_2 we have $W = \varepsilon(u - \frac{1}{4}xu_x)$ and

$$\begin{aligned}
&C_1 = 2\varepsilon(u - \frac{1}{4}xu_x)u^2, \\
&C_2 = \varepsilon(u - \frac{1}{4}xu_x)(3u^3 + u_{2x}) + (\frac{3}{4}\varepsilon u_x - \frac{1}{4}\varepsilon xu_{2x})(-u_x) + (\frac{1}{2}\varepsilon u_{2x} - \frac{1}{4}\varepsilon xu_{3x})u, \\
&D_t^{C_1} + D_x^{C_2}\Big|_{(1)} \neq 0.
\end{aligned}$$

For V_3 we have $W = \varepsilon u_t$ and

$$\begin{aligned}
&C_1 = -2\varepsilon u_t u^2, \\
&C_2 = -\varepsilon u_t(3u^3 + u_{2x}) + \varepsilon u_{xt}u_x - \varepsilon u_{2xt}u, \\
&D_t^{C_1} + D_x^{C_2}\Big|_{(1)} = 0.
\end{aligned}$$

For V_4 we have $W = -\varepsilon(tu_t + \frac{1}{4}xu_x)$ and

$$\begin{aligned}
&C_1 = -2\varepsilon(tu_t + \frac{1}{4}xu_x)u^2, \\
&C_2 = -\varepsilon(tu_t + \frac{1}{4}xu_x)(3u^3 + u_{2x}) + \varepsilon u_x(-\frac{1}{4}u_x - \frac{1}{4}xu_{2x} - tu_{xt}) + \varepsilon(-\frac{1}{2}u_{2x} - \frac{1}{4}xu - 3x - tu_{2xt})u, \\
&D_t^{C_1} + D_x^{C_2}\Big|_{(1)} \neq 0.
\end{aligned}$$

For V_5 we have $W = -u_x$ and

$$\begin{aligned}
&C_1 = -2u_x u^2, \\
&C_2 = -u_x(3u^2 - \varepsilon u_x + u_{2x}) - u_{2x}(\varepsilon u - u_x) - uu_{3x}, \\
&D_t^{C_1} + D_x^{C_2}\Big|_{(1)} \neq 0.
\end{aligned}$$

For V_6 we have $W = u - \frac{1}{4}xu_x$ and

$$\begin{aligned}
&C_1 = 2(u - \frac{1}{4}xu_x)u^2,\\
&C_2 = (u - \frac{1}{4}xu_x)(3u^3 - \varepsilon u_x + u_{2x}) + (\frac{3}{4}u_x - \frac{1}{4}xu_{2x})(\varepsilon u - u_x) + (\frac{1}{2}u_{2x} - \frac{1}{4}xu_{3x})u,\\
&D_t^{C_1} + D_x^{C_2}\Big|_{(1)} \neq 0.
\end{aligned}$$

For V_7 we have $W = -u_t$ and

$$\begin{aligned}
&C_1 = -2u_tu^2,\\
&C_2 = -u_t(3u^3 - \varepsilon u_x + u_{2x}) - u_{xt}(\varepsilon u - u_x) - u_{2xt}u,\\
&D_t^{C_1} + D_x^{C_2}\Big|_{(1)} \neq 0.
\end{aligned}$$

For V_8 we have $W = -(tu_t + \frac{1}{4}xu_x)$ and

$$\begin{aligned}
&C_1 = -2(tu_t + \frac{1}{4}xu_x)u^2,\\
&C_2 = -(tu_t + \frac{1}{4}xu_x)(3u^3 - \varepsilon u_x + u_{2x}) + (\varepsilon u - u_x)(-\frac{1}{4}u_x - \frac{1}{4}xu_{2x} - tu_{xt}) + (-\frac{1}{2}u_{2x} - \frac{1}{4}xu - 3x - tu_{2xt})u,\\
&D_t^{C_1} + D_x^{C_2}\Big|_{(1)} \neq 0.
\end{aligned}$$

Therefore, 8 new conservation laws were obtained for the perturbed generalized Benjamin-Bona-Mahony equation. It should be noted that the effect of the Euler operator on all the above conservation laws is non-zero, which shows that these laws are non-trivial. Also, if we apply the Euler operator to the two-by-two difference of these laws, the result is still non-zero, which indicates that these conservation laws are not equivalent. On the other hand, the method of finding these laws in this study is different from the methods available in the previous literature, and this is also a confirmation of the new and valuable nature of these laws.

5. Conclusions

In this article, the approximate symmetry group of the perturbed generalized Benjamin-Bona-Mahony (BBM) equation was investigated. To find approximate symmetry, the power of the perturbation analysis and Lie symmetry theory have been combined. Two different theories of approximate symmetry and therefore two distinct derived methods have been created as a practical tool for computing the invariant solutions and symmetries of equations. Also, the Baikov and Gazizov method was used to find approximate symmetry and the concept of self-adjoint equations to obtain the conservation laws. The method used to find the conservation laws and the consequent results show that these conservation laws are not equivalent to each other or to the previous conservation laws, and therefore it can be concluded that these conservation laws are new and valuable.

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