AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 5(4) (2024) 329-336
https://doi.org/10.22060/AJMC.2023.22528.1166

Original Article

K-contact generalized square Finsler manifolds

Hannaneh Faraji[a], Behzad Najafi*[a], Tayebeh Tabatabaeifar[a]

[a] *Department of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), Iran*

ABSTRACT: We study almost contact generalized square Finsler manifolds and introduce the notion of K-contact Finsler structures. Then, we characterize generalized square K-contact almost contact manifolds. As an application, we show that every 3-dimensional Lie group admits a left-invariant generalized square Finsler structure.

Review History:

Received:04 July 2023
Revised:30 September 2023
Accepted:30 September 2023
Available Online:01 October 2024

Keywords:

Almost contact Finsler manifolds
Randers metrics
K-contact metrics
Unimodular Lie groups
Non-unimodular Lie groups

MSC (2020):

53B40; 53C60

1. Introduction

Contact geometry was first defined in 1872 by the well-known mathematician Sophus Lie while solving partial differential equations. It has been used in thermodynamics, mechanics, optics, control theory, and low-dimensional topology [4]. Contact structure corresponds to the symplectic structure. Contact geometry and Riemannian geometry are related by considering the compatibility metric condition. In other words, the contact manifold (M^{2n+1}, η) is equipped with the Riemannian metric g if it satisfies $d\eta(S,T) = g(S, \varphi T)$, where φ is a $(1,1)$-tensor field. Contact geometry with a compatible Riemannian metric is called Riemannian contact geometry [1, 9, 10].

In [12], Tabatabaeifar, Najafi, and Rafie-Rad introduced almost contact Finsler manifolds (briefly, ACF-manifolds). They characterized almost contact Randers metrics. Generalized square Finsler manifolds are a natural generalization of two important classes of Finsler manifolds, namely, Randers manifolds and square Finsler manifolds [11]. First, we characterize ACF-generalized square manifolds.

*Corresponding author.
E-mail addresses: hanaaa.faraji@aut.ac.ir, behzad.najafi@aut.ac.ir, t.tabatabaeifar@aut.ac.ir

Theorem 1.1. *Suppose N is a manifold with an AC-structure (ξ, η, φ) and $F = \alpha + \varepsilon\beta + \kappa\beta^2/\alpha$ is a generalized square Finsler metric on N, where ε and κ are constants. Then $(N, F, \xi, \eta, \varphi)$ is an ACF-manifold if and merely if $(N, \alpha, \xi, \eta, \varphi)$ is an ACR-manifold and $\beta = \lambda\eta$, where $\lambda(x)$ is determined by*

$$\forall x \in N, \quad \kappa\lambda^2(x) + \varepsilon\lambda(x) + 1 - F\big(x, \xi(x)\big) = 0. \tag{1}$$

In [12], the authors considered an important class of ACF-manifolds, i. e., cosymplectic Finsler manifold and proved that such a manifold of constant flag curvature has vanishing flag curvature. It is natural to think of studying other classes of almost contact Riemannain manifolds (briefly, ACR-manifolds) and developing these classes in the setting of ACF-manifolds.

Let N be an odd-dimensional manifold. The AC-structure consists of a triplet (ξ, η, φ), where ξ, η and φ, are a vector field, a 1-form and a $(1, 1)$-tensor on N, respectively, such that they satisfy the specific conditions. Two important classes of ACR-manifolds are the class of Sasakian manifolds and the class of K-contact manifolds. It is known that the former class is a proper subclass of the latter one. These two classes are the same on 3-dimensional manifolds [5]. Due to this generality of the class of K-contact ACR-manifolds, we decide to generlaize this notion to the setting of ACF-manifolds in this paper.

In Section 4, we first define K-contact ACF-manifolds. Then, we characterize K-contact generalized square Finsler metrics as follows.

Theorem 1.2. *Suppose $(N, F = \alpha + \varepsilon\beta + \kappa\beta^2/\alpha, \xi, \eta, \varphi)$ is a ACF-manifold, where ε and κ are constants. Then $(N, F, \xi, \eta, \varphi)$ is a K-contact ACF-manifold if and merely if $(N, \alpha, \xi, \eta, \varphi)$ is a K-contact ACR-manifold and λ given by (1) is constant along the integral curves of ξ.*

D. Perrone proved that every 3-dimensional simply connected non-compact homogeneous contact Riemannian manifold is a Lie group with a left-invariant contact Riemannian structure [8]. Milnor classified Riemannian contact structures [6]. Milnor's classification does not include all ACR- manifolds. G. Calvaruso replaced the condition $\eta \wedge d\eta^n \neq 0$, which is used in Milnor's classification, by $d\eta(\cdot, \xi) = 0$ and extended Milnor's classification to all ACR-manifolds [2].

In Section 5, we consider left-invariant ACF-square structures (F, ξ, η, φ) on 3-dimensional Lie groups, where the 1-form η is not contact. Then, we classify all 3-dimensional left-invariant ACF-generalized square Lie groups in both cases unimodular or non-unimodular.

Theorem 1.3. *Suppose $(F = \alpha + \varepsilon\beta + \kappa\frac{\beta^2}{\alpha}, \xi, \eta, \varphi)$ is a left-invariant generalized square ACF-structure on a unimodular Lie group G described by (21) with $\sigma_1 = 0$. If $d\eta(\cdot, \xi) = 0$, then G is one of the following*

(a) *If $I > 0$, then G is $\tilde{E}(2)$,*
(b) *If $I < 0$, then G is $E(1, 1)$,*
(c) *If $I = 0$ and $\mathfrak{g}$ is not abelian, then G is H,*
(d) *If $I = 0$ and $\mathfrak{g}$ is abelian, then G is $\mathbb{R}^3$,*

where $I = \sigma_2\sigma_3$, and $\tilde{E}(2)$, $E(1, 1)$, and H are the universal covering group of rigid motions of Euclidean 2-space, the group of rigid motions of Minkowski 2-space and the Heisenberg group, respectively.

As a result, any 3-dimensional unimodular Lie algebra $\mathfrak{g}$ admits a left-invariant generalized square ACF-structure (F, ξ, η, φ) with $d\eta(\cdot, \xi) = 0$.

2. Preliminaries

Let N be an n-dimensional C^∞ manifold, $TN = \bigcup_{x\in N} T_xN$ the tangent bundle and $TN_0 := TN - \{0\}$ the slit tangent bundle. Let (N, F) be a Finsler manifold. The following quadratic form $\mathbf{g}_y : T_xN \times T_xN \to \mathbb{R}$ is called the fundamental tensor of F

$$\mathbf{g}_y(u, v) = \frac{1}{2}\frac{\partial^2}{\partial s\partial t}\Big[F^2(y + su + tv)\Big]_{s=t=0}, \quad u, v \in T_xN.$$

Let $x \in N$ and $F_x := F|_{T_xN}$. Non-Euclidean feature of F_x is measured by $\mathbf{C}_y : T_xN \times T_xN \times T_xN \to \mathbb{R}$ defined by

$$\mathbf{C}_y(u, v, w) := \frac{1}{2}\frac{d}{dt}\left[\mathbf{g}_{y+tw}(u, v)\right]_{t=0}, \quad u, v, w \in T_xN.$$

The family $\mathbf{C} := \{\mathbf{C}_y\}_{y\in TN_0}$ is called the Cartan torsion.

Assume N is a $(2n+1)$-dimensional manifold. The AC-structure consists of a triplet (ξ, η, φ), where ξ, η and φ, are a 1-form, a vector field, and a $(1,1)$-tensor on N, respectively, with the following conditions:

$$\varphi(\xi) = 0, \qquad \eta(\xi) = 1, \quad \varphi^2(S) = -S + \eta(S)\xi,$$

where $S \in T_xN$. The manifold N with AC-structure (ξ, η, φ) is an *AC-manifold*. For any AC-structure, conditions are established

a) $\eta(\varphi) = 0$,
b) The rank of linear mapping φ is equal to $2n$,
c) $\varphi^3 = -\varphi$,

An AC-manifold (N, ξ, η, φ) is said to be compatible with a Riemannian metric α on N if the following holds good

$$a(\varphi S, \varphi T) = a(S,T) - \eta(S)\eta(T), \quad S,T \in T_xN, \ x \in N, \tag{2}$$

where $a(\cdot,\cdot)$ is the fundamental tensor of α. In this case, $(\alpha, \xi, \eta, \phi)$ is named an *ACR-structure* on N. If we replace φS with S in equation (2), we obtain

$$a(\varphi(S), S) = 0.$$

Moreover, if we put $S = T = \xi$ in (2), we have $\alpha(\xi) = 1$, or equivalently ξ is a unit vector field.

Let (N, F) be a Finsler manifold and AC-structure (ξ, η, φ) on N and $\mathbb{S}^1$ be the unit circle in $\mathbb{R}^2$ and define

$$\begin{cases} \psi : \mathbb{S}^1 \times T_xN \longrightarrow T_xN \\ \qquad\quad (\theta, y) \longmapsto \theta \cdot y, \end{cases}$$

where $\theta \cdot y := \sin(\theta)\varphi_x(y) + \cos(\theta)y$.

In [12], the authors define almost contact Finsler manifolds (briefly ACF-manifolds) as follows.

Definition 2.1. *Let (ξ, η, φ) be an AC-structure and F be a Finsler metric on a manifold N. Then the quadruplet (F, ξ, η, φ) is called an ACF-structure on N if F is a compatible Finsler metric, i.e.,*

$$\forall \theta \in \mathbb{S}^1, \ \forall y \in \ker(\eta_x), \quad F(x, \theta \cdot y) = F(x, y), \tag{3}$$

$$\forall S \in T_xN, \qquad\qquad g_\xi(\xi, S) = F^2(\xi)\eta(S). \tag{4}$$

In this case, the quintuple $(N, F, \xi, \eta, \varphi)$ is called an ACF-manifold.

In [12], the authors proved the following.

Theorem 2.2. *Let (F, ξ, η, φ) be an ACF-structure on a manifold N. Then for every $y \in \ker(\eta_x)$ and $S, T \in T_xN$, the following statements are equivalent.*

(a) $g_{ij}\varphi^i_k y^k y^j = 0$, *or* $\mathbf{g}_y(y, \varphi(y)) = 0$,
(b) $g_{im}\varphi^m_j + g_{jm}\varphi^m_i + 2C_{ijm}\varphi^m_r y^r = 0$, *or* $\mathbf{g}_y(\varphi S, T) + \mathbf{g}_y(\varphi T, S) + 2\mathbf{C}_y(\varphi(y), S, T) = 0$.

3. Generalized square ACF-metrics

Let N be a manifold. An (α, β)-metric is a scalar function on the tangent space TN defined by $F := \alpha\varrho(s)$, $s = \beta/\alpha$, in which $\varrho = \varrho(s)$ is C^∞ on $(-b_0, b_0)$ with certain regularity, $\alpha = \sqrt{a_{ij}y^iy^j}$ is a Riemannian metric, $\beta = b_i(x)y^i$ is a 1-form on a manifold N. Here, we consider the class of the generalized square metrics given by $\varrho(s) = 1 + \varepsilon s + \kappa s^2$, i.e.,

$$F = \alpha + \varepsilon\beta + \kappa\frac{\beta^2}{\alpha}, \tag{5}$$

where ε and κ are constants. First, we characterize generalized square ACF-manifolds.

Theorem 3.1. *Suppose (ξ, η, φ) is an AC-structure and F is a generalized square metric on a manifold N given by (5). Then, $(N, F, \xi, \eta, \varphi)$ is an ACF-manifold if and merely if $(N, \alpha, \xi, \eta, \varphi)$ is an almost contact Riemannian manifold (briefly ACR-manifold) and $\beta = \lambda\eta$, where $\lambda(x)$ is determined by*

$$\kappa\lambda^2(x) + \varepsilon\lambda(x) + 1 - F(\xi(x)) = 0, \tag{6}$$

for all $x \in N$.

Proof. Let F be an ACF-metric. By (3) for every tangent vector $y \in \ker(\eta_x)$ and every $\theta \in \mathbb{S}^1$, we have

$$\alpha(x, \theta \cdot y) + \varepsilon\beta(\theta \cdot y) + \kappa \frac{\beta^2(\theta \cdot y)}{\alpha(x, \theta \cdot y)} = \alpha(x, y) + \varepsilon\beta(y) + \kappa \frac{\beta^2(y)}{\alpha(x, y)}. \tag{7}$$

Taking the irrational and rational parts of (7), we get

$$\alpha(x, \theta \cdot y) + \kappa \frac{\beta^2(\theta \cdot y)}{\alpha(x, \theta \cdot y)} = \alpha(x, y) + \kappa \frac{\beta^2(y)}{\alpha(x, y)}, \tag{8}$$

and

$$\beta(\theta \cdot y) = \beta(y). \tag{9}$$

Letting $\theta = \frac{\pi}{2}$ in (9), we obtain

$$\beta(\varphi_x(y)) = \beta(y). \tag{10}$$

Applying $\frac{d}{d\theta}$ on (9), we get

$$-\sin\theta\ \beta(y) + \cos\theta\ \beta(\varphi_x(y)) = 0. \tag{11}$$

By putting $\theta = 0$ in (11) and using (10), one can conclude $\beta(y) = 0$ for every $y \in \ker(\eta_x)$. Thus $\ker(\eta) = \ker(\beta)$, which implies that η and β are linearly dependent at each point. Thus, for some scalar function $\lambda = \lambda(x)$ on N, we have

$$\beta = \lambda\eta. \tag{12}$$

It follows from (8), (10), and (12) that

$$\alpha(x, \theta \cdot y) = \alpha(x, y), \qquad \forall y \in \ker(\eta_x).$$

It is well-known that the fundamental tensor of a generalized square metric F given by (5) is in the following form [11]

$$\begin{aligned}\mathbf{g}_y(S, T) = {} & \frac{\left(\alpha^2(y) - \kappa\beta^2(y)\right) F(y)}{\alpha^3(y)} a(S, T) + \frac{6\kappa F(y) + \left(\varepsilon^2 - 4\kappa\right) \alpha(y)}{\alpha(y)} \beta(S)\beta(T) \\ & + \frac{\varepsilon\alpha^3(y) - 3\varepsilon\kappa\alpha(y)\beta^2(y) - 4\kappa^2\beta^3(y)}{\alpha^4(y)} \left\{\beta(S)a(y, T) + \beta(T)a(y, S) - \frac{\beta(y)}{\alpha^2(y)} a(y, S)a(y, T)\right\}.\end{aligned} \tag{13}$$

Putting $y = S = T = \xi(x)$ in (13) infer

$$\begin{aligned}F^2(\xi(x)) = {} & \frac{\left(\alpha^2(\xi(x)) - \kappa\lambda^2(x)\right) F(\xi(x))}{\alpha^3(\xi(x))} \alpha^2(\xi(x)) + \frac{6\kappa F(\xi(x)) + \left(\varepsilon^2 - 4\kappa\right) \alpha(\xi(x))}{\alpha(\xi(x))} \lambda^2(x) \\ & + \frac{\varepsilon\alpha^3(\xi(x)) - 3\varepsilon\kappa\lambda^2(x)\alpha(\xi(x)) - 4\kappa^2\lambda^3(x)}{\alpha^4(\xi(x))} \alpha^2(\xi(x))\lambda(x).\end{aligned} \tag{14}$$

Similarly, by putting $y = \xi(x)$ and $T = \xi(x)$ in (13), we obtain

$$\begin{aligned}\mathbf{g}_{\xi(x)}(\xi(x), S) = {} & \frac{\left(\alpha^2(\xi(x)) - \kappa\lambda^2(x)\right) F(\xi(x))}{\alpha^3(\xi(x))} a(\xi(x), S) + \frac{6\kappa F(\xi(x)) + \left(\varepsilon^2 - 4\kappa\right) \alpha(\xi(x))}{\alpha(\xi(x))} \lambda^2(x)\eta(S) \\ & + \frac{\varepsilon\alpha^3(\xi(x)) - 3\varepsilon\kappa\lambda^2(x)\alpha(\xi(x)) - 4\kappa^2\lambda^3(x)}{\alpha^4(\xi(x))} \alpha^2(\xi(x))\lambda(x)\eta(S).\end{aligned} \tag{15}$$

By (4), (14), and (15), we have

$$a(\xi(x), S) = \alpha^2(\xi(x))\eta(S),$$

which means that $(N, \alpha, \xi, \eta, \varphi)$ is an ACR-manifold. Hence, we get

$$\alpha(x, \xi(x)) = \eta\left(\xi(x)\right) = 1. \tag{16}$$

From (14) and (16), we conclude

$$F^2(\xi(x)) = (\kappa\lambda^2(x) + \varepsilon\lambda(x) + 1)^2,$$

from which, we get (6). □

Corollary 3.2. *Letting $\varepsilon = 2$ and $\kappa = 1$ in (5), we reach to the class of square Finsler metrics. Thus, a square Finsler metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ with an AC-structure (ξ, η, φ) on a manifold N is an ACF-manifold if and merely if $(N, \alpha, \xi, \eta, \varphi)$ is an ACR-manifold and $\beta = \lambda\eta$, where $(\lambda(x) + 1)^2 = F(\xi(x))$ for all $x \in N$.*

If we put $\varepsilon = 1$ and $\kappa = 0$ in (5), then the generalized square metric F becomes a Randers metric. Consequently, we revisit Theorem 1.3 of [12] with a slight improvement.

Theorem 3.3. *Suppose $F = \alpha + \beta$ is a Randers metric, and (ξ, η, φ) is an AC-structure on a manifold N. Then $(N, F, \xi, \eta, \varphi)$ is an ACF-manifold if and merely if $(N, \alpha, \xi, \eta, \varphi)$ is an ACR-manifold and $\beta = \lambda\eta$, where $\lambda(x) = F(x, \xi(x)) - 1$ for all $x \in N$.*

Proof. In [12], it is proved that (F, ξ, η, φ) is an ACF-structure on N if and only if $(\alpha, \xi/\alpha(\xi), \alpha(\xi)\eta, \varphi)$ is an ACR-structure on N, and $\beta = \lambda\eta$, where $\lambda(x) = F(x, \xi(x)) - 1$ for all $x \in N$ (see Theorem 1.3 of [12]). By a direct calculation, we have

$$\lambda(x) = F(x, \xi(x)) - 1 = \alpha(\xi(x)) + \beta(\xi(x)) - 1 = \alpha(\xi(x)) + \lambda(x) - 1.$$

Hence, we have

$$\alpha(\xi(x)) = 1.$$

This completes the proof. □

Indeed, Theorem 3.3 proposes a way to construct more examples of ACF-structures using the standard changings in the Finslerian world, such as Randers β-change.

Proposition 3.4. *Let N be a manifold with an ACF-structure (F, ξ, η, φ). Then, the Randers β-change $(N, \bar{F} = F + \beta, \xi, \eta, \varphi)$ is an ACF-manifold provided that $\beta = \lambda\eta$ for some scalar function on N.*

Proof. We want to show that $\bar{F}$ is an ACF-metric. For every tangent vector $y \in \ker(\eta_x)$, we have $\bar{F}(x, y) = F(x, y)$. Therefore, $\bar{F}$ satisfies (3).

It suffices to prove that $\bar{F}$ satisfies (4). By Theorem 2.2, for every $y \in \ker(\eta_x)$, we have

$$\mathbf{g}_y(\varphi y, y) = 0.$$

The fundamental tensor $\bar{\mathbf{g}}_y$ is as follows [3]

$$\bar{\mathbf{g}}_y(S,T) = \frac{\bar{F}(y)}{F(y)}\mathbf{g}_y(S,T) + \left(1 - \frac{\bar{F}(y)}{F(y)}\right)\frac{\mathbf{g}_y(y,S)\mathbf{g}_y(y,T)}{F^2(y)} + \frac{\mathbf{g}_y(y,S)\beta(T)}{F(y)} + \frac{\mathbf{g}_y(y,T)\beta(S)}{F(y)} + \beta(S)\beta(T). \tag{17}$$

Putting $y = \xi(x)$ and $S = T = \xi(x)$ in (17) yield

$$\bar{F}^2(\xi(x)) = (F(\xi(x)) + \lambda(x))^2.$$

Also, by putting $y = \xi(x)$ and $T = \xi(x)$ in (17) and taking into account (4) for F, we have

$$g_{\xi(x)}(S, \xi(x)) = (F(\xi(x)) + \lambda(x))^2\,\eta(S) = \bar{F}^2(\xi(x))\eta(S),$$

which means that $\bar{F}$ is an ACF-metric. □

4. K-contact ACF-manifolds

The class of K-contact manifolds is a wealthy class of Riemannian manifolds. It is natural o study K-contact Finsler manifolds. In this section, we first introduce K-contact ACF-manifolds and characterize generalized square K-contact ACF-manifolds.

Definition 4.1. *Suppose $(N, F, \xi, \eta, \varphi)$ is an ACF-manifold such that the Reeb vector field ξ is a Killing vector filed with respect to F, i.e., $\mathcal{L}_{\hat{\xi}}F = 0$, where $\hat{\xi}$ is the complete lift of ξ. In this case, we say $(N, F, \xi, \eta, \varphi)$ is a K-contact ACF-manifold.*

In [7], X. Mo proves that the Lie derivative of an (α, β) metric $F = \alpha\varrho(\beta/\alpha)$ is obtained as follows

$$\mathfrak{L}_{\hat{\xi}}F = (\varrho - s\varrho')\mathfrak{L}_{\hat{\xi}}(\alpha) + \varrho'\mathfrak{L}_{\hat{\xi}}(\beta). \tag{18}$$

Theorem 4.2. *Let (N, F, ξ, η, ϕ) be a generalized square ACF-manifold. Then F is a K-contact ACF-metric if and merely if $(N, \alpha, \xi, \eta, \phi)$ is a K-contact Riemannian manifold and $\xi^i \lambda_i = 0$, where $\lambda_i = \frac{\partial \lambda}{\partial x^i}$ and $\xi = \xi^i \frac{\partial}{\partial x^i}$.*

Proof. First, we assume $(N, \alpha, \xi, \eta, \varphi)$ is a K-contact Riemannian manifold and $\xi^i \lambda_i = 0$. Since $(N, \alpha, \xi, \eta, \varphi)$ is an ACR-manifold, then according to Theorem 3.1, (F, ξ, η, φ) is an ACF-structure on N. Moreover, it follows from (18) and $\xi^i \lambda_i = 0$ that F is a K-contact Finsler metric. Hence, $(N, F, \xi, \eta, \varphi)$ is a K-contact ACF-manifold.

Suppose $(N, F, \xi, \eta, \varphi)$ is a K-contact ACF-manifold. By definition, we have $\mathcal{L}_{\hat{\xi}} F = 0$. According to Theorem 3.1, $(N, \alpha, \xi, \eta, \varphi)$ is an ACR-manifold. We know that

$$\mathcal{L}_{\hat{\xi}} F = \mathcal{L}_{\hat{\xi}} \alpha + \varepsilon \mathcal{L}_{\hat{\xi}} \beta + \kappa \mathcal{L}_{\hat{\xi}} (\beta^2/\alpha).$$

Then we have

$$\mathcal{L}_{\hat{\xi}} \alpha + \varepsilon \mathcal{L}_{\hat{\xi}} \beta + \kappa \mathcal{L}_{\hat{\xi}} (\beta^2/\alpha) = 0. \tag{19}$$

Taking the rational and irrational parts of the equation (19), we get

$$\mathcal{L}_{\hat{\xi}} \alpha + \kappa \mathcal{L}_{\hat{\xi}} (\beta^2/\alpha) = 0,$$

and

$$\mathcal{L}_{\hat{\xi}} \beta = 0. \tag{20}$$

From (20), we conclude $\xi^i \lambda_i = 0$. A direct computation shows that

$$\mathcal{L}_{\hat{\xi}} \alpha + \kappa \mathcal{L}_{\hat{\xi}} (\beta^2/\alpha) = (1 - s^2) \mathcal{L}_{\hat{\xi}} \alpha + 2\kappa s \mathcal{L}_{\hat{\xi}} \beta),$$

which, with the help of (20), implies that $\mathcal{L}_{\hat{\xi}} \alpha = 0$. Therefore, $(N, \alpha, \xi, \eta, \varphi)$ is a K-contact ACR- manifold. □

Since every square Finsler metric and Randers metric is a generalized square metric, we have the following two immediate results.

Corollary 4.3. *A square ACF- manifold $(N, \frac{(\alpha+\beta)^2}{\alpha}, \xi, \eta, \varphi)$ is K-contact if and merly if $(N, \alpha, \xi, \eta, \varphi)$ is K-contact and $\xi^i \lambda_i = 0$.*

Corollary 4.4. *A Randers ACF- manifold $(N, \alpha + \beta, \xi, \eta, \varphi)$ is K-contact if and merely if $(N, \alpha, \xi, \eta, \varphi)$ is K-contact and $\xi^i \lambda_i = 0$.*

Remark 4.5. *It is worth mentioning that for a Randers ACF- manifold $(N, \alpha + \beta, \xi, \eta, \varphi)$, the condition $\xi^i \lambda_i = 0$ is equivalent to $(\mathfrak{L}_{\hat{\xi}} F)(x, \xi(x)) = 0$ for all $x \in N$.*

Example 4.1. *Let $h(x, y)$ be a differentiable function on $\mathbb{R}^3$ satisfying the following:*

$$|h(x, y)| < \frac{1}{\sqrt{1 + 3y^2 + y^4}}.$$

Let $N = \{(x, y, z) \in \mathbb{R}^3 \mid |h(x, y)| < \dfrac{1}{\sqrt{1 + 3y^2 + y^4}}\}$. Let us consider the 1-form, vector field and Riemannian metric on N given by

$$\xi = \frac{\partial}{\partial z}, \qquad \eta = -y dx + dz, \qquad \alpha = \eta \otimes \eta + dx \otimes dx + dy \otimes dy,$$

respectively. If the $(1, 1)$-tensor field φ on N is defined as

$$\varphi = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & y & 0 \end{bmatrix},$$

then $(N, \alpha, \xi, \eta, \varphi)$ is a K-contact Riemannian manifold. Suppose $(N, F = \alpha + \beta)$ is a Randers manifold, where $\beta = h(x, y)\eta$. One can easily see that $(N, \alpha, \xi, \eta, \varphi)$ is an ACR-manifold. Thus, according to Theorem 3.3, $(N, F, \xi, \eta, \varphi)$ is an ACF-manifold. We can see $\mathfrak{L}_\xi \alpha = 0$ and $\mathfrak{L}_\xi \beta = 0$, and consequently $\mathfrak{L}_{\hat{\xi}} F = 0$. Hence ξ is a Killing vector field of F. Therefore, $(N, F, \xi, \eta, \varphi)$ is a K-contact ACF-manifold.

Inspired by Corollary 4.4, we close this section, by proposing a natural way to construct new examples of K-contact ACF-manifolds using the Randers β-changes. It suffices to observe that if $\bar{F} = F + \beta$, then $\mathcal{L}_{\hat{\xi}} \bar{F} = \mathcal{L}_{\hat{\xi}} F + \mathcal{L}_{\hat{\xi}} \beta$.

Proposition 4.6. *Let $(N, F, \xi, \eta, \varphi)$ and the Randers β-change $(N, \bar{F} = F + \beta, \xi, \eta, \varphi)$ be an ACF-manifold. The Randers β-change $(N, \bar{F} = F + \beta, \xi, \eta, \varphi)$ is a K-contact ACF-manifold if and merely if $(N, F, \xi, \eta, \varphi)$ is a K-contact ACF-manifold and $\xi^i \lambda_i = 0$.*

5. 3-Dimensional Left Invariant K-Contact Finsler Structures

Let (F, ξ, η, φ) be an ACF-structure on a 3-dimensional Lie group G. We say that (F, ξ, η, φ) is left-invariant, if for every $a \in G$, the left translation along a, $L_a : G \longrightarrow G$ is an ACF-isomorphism, i.e.,

$$L_{a_*} \circ \varphi = \varphi \circ L_{a_*}, \qquad L_a^*(F) = F, \qquad L_{a_*}\xi = \xi, \qquad L_a^*(\eta) = \eta.$$

From [2], the Lie group G is called unimodular if, for every $u \in \mathfrak{g}$, we have $tr(ad_u) = 0$, where $\mathfrak{g}$ is the Lie algebra of G. In this case, it is well-known that the Lie algebra $\mathfrak{g}$ admits an orthonormal basis $\{\xi, e, \varphi e\}$, such that

$$[\xi, e] = \sigma_3 \varphi e, \qquad [e, \varphi e] = \sigma_1 \xi, \qquad [\varphi e, \xi] = \sigma_2 e. \tag{21}$$

Next, consider a left-invariant ACF-structure of the form (21). In [2], the authors prove that η is not a contact form if and only if $\sigma_1 = 0$. We get the following classification result.

Theorem 5.1. *Suppose $(F = \alpha + \varepsilon\beta + \kappa\frac{\beta^2}{\alpha}, \xi, \eta, \varphi)$ is a left-invariant generalized square ACF-structure on a unimodular Lie group G described by (21) with $\sigma_1 = 0$. If $d\eta(\cdot, \xi) = 0$, then G is one of the following*

(a) *If $I > 0$, then G is $\tilde{E}(2)$,*
(b) *If $I < 0$, then G is $E(1,1)$,*
(c) *If $I = 0$ and $\mathfrak{g}$ is not abelian, then G is H,*
(d) *If $I = 0$ and $\mathfrak{g}$ is abelian, then G is $\mathbb{R}^3$,*

where $I = \sigma_2\sigma_3$, and $\tilde{E}(2)$, $E(1,1)$, and H are the universal covering group of rigid motions of Euclidean 2-space, the group of rigid motions of Minkowski 2-space and the Heisenberg group, respectively.

Proof. By Theorem 3.1, we conclude that $(\alpha, \xi, \eta, \varphi)$ is a left-invariant ACR-structure on the 3-dimensional unimodular Lie group G. Now by Theorem 3.4 of [2], we get the proof. □

Proposition 5.2. *Any 3-dimensional unimodular Lie algebra $\mathfrak{g}$ admits a left-invariant generalized square ACF-structure (F, ξ, η, φ) with $d\eta(\cdot, \xi) = 0$.*

The Lie group G is non-unimodular if there exists a tangent vector $u \in \mathfrak{g}$ such that $tr(ad_u) \neq 0$. In this case, there exists an orthonormal basis $\{e_1, e_2, e_3\}$ of $\mathfrak{g}$, such that

$$[e_1, e_2] = \nu e_2 + \mu e_3, \qquad [e_1, e_3] = \delta e_2 + \gamma e_3, \qquad [e_2, e_3] = 0, \tag{22}$$

for some real constants ν, μ, γ, and δ satisfying conditions

$$\nu + \gamma \neq 0, \qquad \nu\delta + \mu\gamma = 0. \tag{23}$$

It is easy to see that the quantity

$$Q = \frac{4(\nu\gamma - \mu\delta)}{(\nu + \gamma)^2},$$

is an invariant of $\mathfrak{g}$. Indeed, if we denote $\mathfrak{g}$ by $\mathfrak{g}_{(\nu,\mu,\delta,\gamma)}$, then $\mathfrak{g}_{(\nu',\mu',\delta',\gamma')}$ and $\mathfrak{g}_{(\nu,\mu,\delta,\gamma)}$ are isomorphic Lie algebras if and only if $Q = Q'$ (see [2] for more details).

Suppose $(F = \alpha + \varepsilon\beta + \kappa\frac{\beta^2}{\alpha}, \xi, \eta, \varphi)$ is a left-invariant generalized square ACF-structure on a non-unimodular Lie group G described by (22) with $d\eta(\cdot, \xi) = 0$. If $\xi = \cos\theta e_2 + \sin\theta e_3$, then an orthonormal basis (and so, a φ-basis) of $\ker\eta$ is given by $\{E_1 = e_1, E_2 = -\sin\theta e_2 + \cos\theta e_3\}$. From (23), we then easily obtain

$$\begin{aligned}
&[\xi, E_1] = (\mu\cos^2\theta + (\gamma - \nu)\sin\theta\cos\theta - \delta\sin^2\theta)E_2 - (\nu\cos^2\theta + (\mu + \delta)\sin\theta\cos\theta + \gamma\sin^2\theta)\xi,\\
&[\xi, E_2] = 0,\\
&[E_1, E_2] = (\gamma\cos^2\theta - (\mu + \delta)\sin\theta\cos\theta + \nu\sin^2\theta)E_2 - (\delta\cos^2\theta + (\gamma - \nu)\sin\theta\cos\theta - \mu\sin^2\theta)\xi.
\end{aligned}$$

We put

$$\begin{aligned}
A &:= \gamma\cos^2\theta - (\mu + \delta)\sin\theta\cos\theta + \nu\sin^2\theta,\\
B &:= -(\delta\cos^2\theta + (\gamma - \nu)\sin\theta\cos\theta - \mu\sin^2\theta),\\
C &:= \mu\cos^2\theta + (\gamma - \nu)\sin\theta\cos\theta - \delta\sin^2\theta.
\end{aligned}$$

By Theorem 3.1 and Theorem 4.3 of [2], we get the following.

Theorem 5.3. *Suppose* $(F = \alpha + \varepsilon\beta + \kappa\frac{\beta^2}{\alpha}, \xi, \eta, \varphi)$ *is a left-invariant generalized square ACF-structure on a* 3*-dimensional non-unimodular Lie group* G *with the Lie algebra* $\mathfrak{g}$ *described by* (22)*,* (23)*. Then, up to isomorphisms, the following are all and the ones left-invariant ACF-structures* (F, ξ, η, φ) *on* $\mathfrak{g}$*, satisfying* $d\eta(\cdot, \xi) = 0$*:*

(a) $[\xi, e] = \nu e + \mu\varphi e, \quad [\xi, \varphi e] = \delta e + \gamma\varphi e, \quad [e, \varphi e] = 0,$
for any value of ν*,* μ*,* δ*, and* γ *satisfying* (23)*.*

(b) $[\xi, e] = C\varphi e, \quad [\xi, \varphi e] = 0, \quad [e, \varphi e] = A\varphi e + B\xi$*, with* $A \neq 0$
for any value of α*,* μ*,* γ*, and* δ *satisfying* (23)*.*

(c) $[\xi, e] = \nu\cos\theta e, \quad [\xi, \varphi e] = 0, \quad [e, \varphi e] = \nu\sin\theta e$*, with* $(\nu\cos\theta)^2 + (\nu\sin\theta)^2 \neq 0$
only when $\nu \neq 0$*,* $\mu = \gamma = \delta = 0$*.*

Corollary 5.4. *Any* 3*-dimensional non-unimodular Lie algebra* $\mathfrak{g}$ *admits a left-invariant generalized square ACF-structure* (F, ξ, η, φ) *with* $d\eta(\cdot, \xi) = 0$*.*

References

[1] D. E. Blair, *Riemannian geometry of contact and symplectic manifolds*, vol. 203 of Progress in Mathematics, Birkhäuser Boston, Ltd., Boston, MA, second ed., 2010.

[2] G. Calvaruso, *Three-dimensional homogeneous almost contact metric structures*, J. Geom. Phys., 69 (2013), pp. 60–73.

[3] S.-S. Chern and Z. Shen, *Riemann-Finsler geometry*, vol. 6 of Nankai Tracts in Mathematics, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2005.

[4] H. Geiges, *A brief history of contact geometry and topology*, Expo. Math., 19 (2001), pp. 25–53.

[5] J.-B. Jun, I. B. Kim, and U. K. Kim, *On* 3*-dimensional almost contact metric manifolds*, Kyungpook Math. J., 34 (1994), pp. 293–301.

[6] J. Milnor, *Curvatures of left invariant metrics on Lie groups*, Advances in Math., 21 (1976), pp. 293–329.

[7] X. Mo, *On some Finsler metrics of constant (or scalar) flag curvature*, Houston J. Math., 38 (2012), pp. 41–54.

[8] D. Perrone, *Homogeneous contact Riemannian three-manifolds*, Illinois J. Math., 42 (1998), pp. 243–256.

[9] E. Peyghan, A. Naderifard, and A. Tayebi, *Almost paracontact structures on tangent sphere bundle*, Int. J. Geom. Methods Mod. Phys., 10 (2013), pp. 1320015, 11.

[10] E. Peyghan and A. Tayebi, *Almost paracontact structure in Finslerian indicatrix*, An. Ştiinţ. Univ. "Ovidius" Constanţa Ser. Mat., 19 (2011), pp. 151–162.

[11] Z. Shen, *Differential geometry of spray and Finsler spaces*, Kluwer Academic Publishers, Dordrecht, 2001.

[12] T. Tabatabaeifar, B. Najafi, and M. Rafie-Rad, *On almost contact Finsler structures*, Int. J. Geom. Methods Mod. Phys., 17 (2020), pp. 2050126, 15.

Please cite this article using:

Hannaneh Faraji, Behzad Najafi, Tayebeh Tabatabaeifar, K-contact generalized square Finsler manifolds, AUT J. Math. Comput., 5(4) (2024) 329-336
https://doi.org/10.22060/AJMC.2023.22528.1166