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Original Article

Strong domination number of a modified graph

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ABSTRACT: Let $G = (V, E)$ be a simple graph. A set $D \subseteq V$ is a strong dominating set of G, if for every vertex $x \in V \setminus D$ there is a vertex $y \in D$ with $xy \in E(G)$ and $\deg(x) \leq \deg(y)$. The strong domination number $\gamma_{\rm st}(G)$ is defined as the minimum cardinality of a strong dominating set. In this paper, we study the effects on $\gamma_{\rm st}(G)$ when G is modified by operations on vertices and edges of G.

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1. Introduction

A dominating set of a graph $G = (V, E)$ is a subset D of V such that every vertex in $V \setminus D$ is adjacent to at least one member of D. The minimum cardinality of all dominating sets of G is called the domination number of G and is denoted by $\gamma(G)$. This parameter has been extensively studied in the literature and there are hundreds of papers concerned with domination. For a detailed treatment of domination theory, the reader is referred to [6]. Also, the concept of domination and related invariants has been generalized in many ways.

A set $D \subseteq V$ is a strong dominating set of G, if for every vertex $x \in V \setminus D$ there is a vertex $y \in D$ with $xy \in E$ and $\deg(x) \leq \deg(y)$ (in this case we say that the vertex y strong dominate the vertex x). The strong domination number $\gamma_{st}(G)$ is defined as the minimum cardinality of a strong dominating set. A strong dominating set with cardinality $\gamma_{st}(G)$ is called a γ_{st}-set. The strong domination number was introduced in [7] and some upper bounds on this parameter were presented in [7, 8]. Similar to strong domination number, a set $D \subset V$ is a weak dominating set of G, if every vertex $v \in V \setminus D$ is adjacent to a vertex $u \in D$ such that $\deg(v) \geq \deg(u)$ (see [5]). The minimum cardinality of a weak dominating set of G is denoted by $\gamma_w(G)$. Boutrig and Chellali proved that the relation $\gamma_w(G) + \frac{3}{\Delta+1}\gamma_{st}(G) \leq n$ holds for any connected graph of order $n \geq 3$ ([5]). Alikhani, Ghanbari and Zaherifar [3]

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examined the effects on $\gamma_{st}(G)$ when G is modified by edge deletion, edge subdivision and edge contraction. Also they studied the strong domination number of k-subdivision of G.

Motivated by counting of the number of dominating sets of a graph and domination polynomial (see e.g. [1, 4]), the number of the strong dominating sets for certain graphs has been studied in [9].

Let e be an edge of a connected simple graph G. The graph obtained by removing an edge e from G is denoted by $G - e$. The edge subdivision operation for an edge $\{u, v\} \in E$ is the deletion of $\{u, v\}$ from G and the addition of two edges $\{u, w\}$ and $\{w, v\}$ along with the new vertex w. A graph which has been derived from G by deleting a vertex v is denoted by $G - v$. The contraction of v in G denoted by G/v is the graph obtained by deleting v and putting a clique on the (open) neighbourhood of v. An edge contraction is an operation that removes an edge from G while simultaneously merging the two vertices that it previously joined. The obtained graph is denoted as G/e.

In this paper, we examine the effects on $\gamma_{st}(G)$ when G is modified by operations such as vertex deletion, vertex contraction and edge contraction

2. Main Results

In this section, we study the effects on $\gamma_{st}(G)$ when G is modified by some operations. Before we state our results, we start with a simple example.

Example 2.1. *Consider star graph $S_n = K_{1,n}$ as shown in Figure 1. Let $D = \{u\}$. Then D is a strong dominating set of S_n, because $\deg(u) \geq \deg(v_i)$ and u strong dominates v_i, for $i = 1, 2, \ldots, n$. Therefore $\gamma_{st}(S_n) = 1$. Also, for empty graph $\overline{K_n}$, since all vertices are isolated vertices, we have $\gamma_{st}(\overline{K_n}) = n$.*

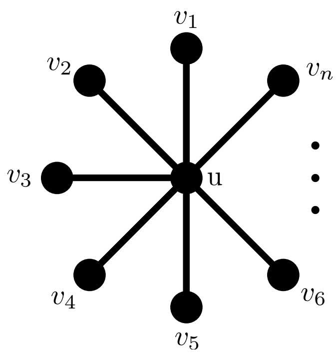

Figure 1: Star graph S_n

Now, we consider the vertex deletion.

Theorem 2.1. *If $G = (V, E)$ is a connected graph and $v \in V$, then*

$$\gamma_{\rm st}(G) - \deg(v) \leq \gamma_{\rm st}(G - v) \leq \gamma_{\rm st}(G) + \deg(v) - 1.$$

Furthermore, these bounds are tight.

Proof. First we consider the upper bound. Suppose that D is a $\gamma_{\rm st}$-set of G. If $v \in D$, then $(D \cup N(v)) \setminus \{v\}$ is a strong dominating set of $G - v$ and we are done. If $v \notin D$, then there exists $u \in N(v)$ such that u strong dominate v. So $D \cup N(v)$ is a strong dominating set with size at most $\gamma_{\rm st}(G) + \deg(v) - 1$. Therefore we have $\gamma_{\rm st}(G - v) \leq \gamma_{\rm st}(G) + \deg(v) - 1$. The equality holds for the star graph, and v is the universal vertex. Now, we obtain the lower bound. First we consider $G - v$ and suppose that S is a $\gamma_{\rm st}$-set of $G - v$. We have two cases for v in G:

(i) $\deg(v) > \deg(u)$ for all $u \in N(v)$. Then clearly $S \cup \{v\}$ is a strong dominating set for G. So $\gamma_{\rm st}(G) \leq \gamma_{\rm st}(G - v) + 1$.

(ii) There exists $u \in N(v)$ such that $\deg(u) \geq \deg(v)$. So $S \cup N(v)$ is a strong dominating set for G and so $\gamma_{\rm st}(G) \leq \gamma_{\rm st}(G - v) + \deg(v)$.

Therefore we have $\gamma_{\rm st}(G - v) \geq \gamma_{\rm st}(G) - \deg(v)$. Now, we show that this bound is tight. Consider Figure 2. The set of black vertices is a $\gamma_{\rm st}$-set of G, say D, and we have $\gamma_{\rm st}(G) = 18$. Now, $D \setminus N(v)$ is a $\gamma_{\rm st}$-set of $G - v$, and $\gamma_{\rm st}(G - v) = 15$. Therefore we have the result. □

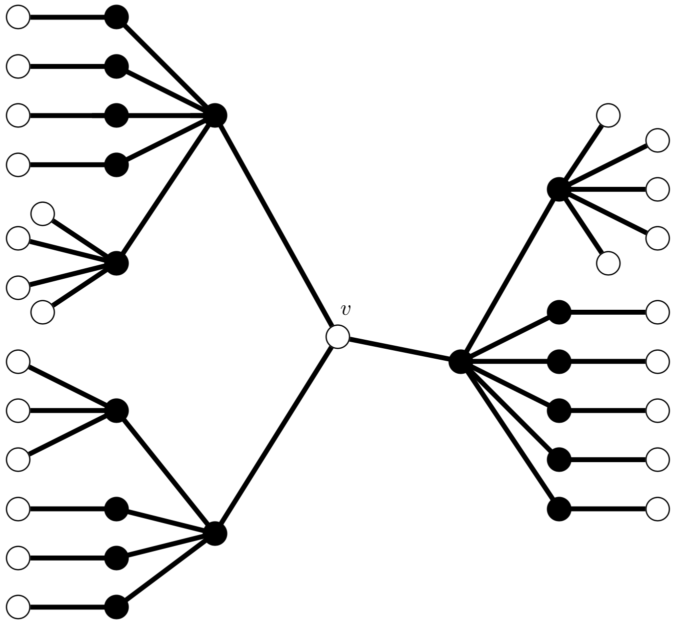

Figure 2: Graph G with $\gamma_{st}(G) = 18$ and $\gamma_{st}(G - v) = 15$.

The following theorem gives bounds for the strong domination number of a graph G/v, where G/v is a graph obtained by G and contraction of a vertex v. We recall that a vertex v is a pendant vertex, if $\deg(v) = 1$.

Theorem 2.2. *If $G = (V, E)$ is a connected graph and $v \in V$ is not a pendant vertex, then*

$$\gamma_{st}(G) - \deg(v) + 1 \leq \gamma_{st}(G/v) \leq \gamma_{st}(G) + 1.$$

Furthermore, these bounds are tight.

Proof. First we obtain the upper bound. Suppose that D is a γ_{st}-set of G. First suppose that $v \in D$. If $u \in N(v)$ is the vertex with the maximum degree among others, then $(D \cup \{u\}) \setminus \{v\}$ is a strong dominating set of G, because each vertex is strong dominated by the same vertex as before or possibly by u. Now suppose that $v \notin D$. If $w \in N(v)$ is the vertex with the maximum degree among others, then by the same argument, $D \cup \{w\}$ is a strong dominating set of G. Therefore we have $\gamma_{st}(G/v) \leq \gamma_{st}(G) + 1$. To show that this bound is tight, consider graphs G and G/v in Figure 3.

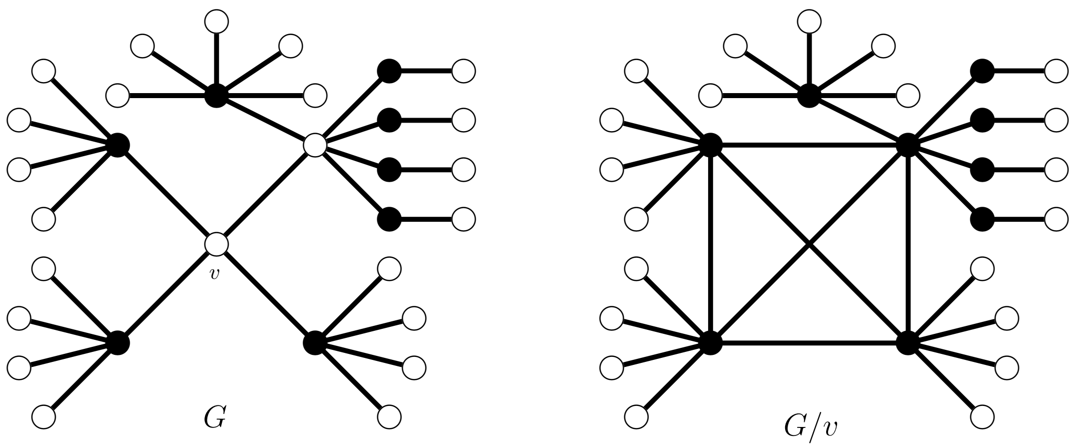

Figure 3: Graphs G and G/v with $\gamma_{st}(G) = 8$ and $\gamma_{st}(G/v) = 9$.

One can easily check that the set of black vertices is a γ_{st}-set of both graphs, and therefore $\gamma_{st}(G/v) = \gamma_{st}(G) + 1 = 8 + 1 = 9$. Now, we obtain the lower bound. To show the lower bound, first we form G/v. Suppose that S is a γ_{st}-set of G/v. We remove all the added edges and add v to form G. We consider the following cases:

(i) $N(v) \subseteq S$. Clearly $S \cup \{v\}$ is a strong dominating set of G and we have $\gamma_{st}(G) \leq \gamma_{st}(G/v) + 1$.

(ii) $N(v) \subseteq V \setminus S$. So $S \cup \{v\}$ is a strong dominating set of G, because each vertex is strong dominated by the same vertex as before (and possibly v). So $\gamma_{\rm st}(G) \leq \gamma_{\rm st}(G/v) + 1$.

(iii) For all vertices $u \in N(v)$, $\deg_G(v) \geq \deg_G(u)$. So by the same argument as Case (ii), $S \cup \{v\}$ is a strong dominating set of G, and $\gamma_{\rm st}(G) \leq \gamma_{\rm st}(G/v) + 1$.

(iv) There exists a vertex $u \in N(v) \cap S$ such that $\deg_G(u) \geq \deg_G(v)$. So $S \cup N(v)$ is a strong dominating set of G, because v is strong dominated by u and the rest of vertices are strong dominated by the same vertices as before. So $\gamma_{\rm st}(G) \leq \gamma_{\rm st}(G/v) + \deg(v) - 1$.

(v) There exists a vertex $u \in N(v) \cap (V \setminus S)$ such that $\deg_G(u) \geq \deg_G(v)$. If $N(v) \subseteq V \setminus S$, then it is Case (ii). So suppose that there exists a vertex w such that $w \in N(v) \cap S$. Then similar to Case (iv), $S \cup N(v)$ is a strong dominating set of G, since v is strong dominated by u, and we are done.

Hence in general, we have $\gamma_{\rm st}(G/v) \geq \gamma_{\rm st}(G) - \deg(v) + 1$. Now, we show that this bound is tight. Consider graphs G and G/v in Figure 4. The set of black vertices is a $\gamma_{\rm st}$-set of both graphs, and $\gamma_{\rm st}(G/v) = \gamma_{\rm st}(G) - \deg(v) + 1 = 20 - 4 + 1 = 17$. □

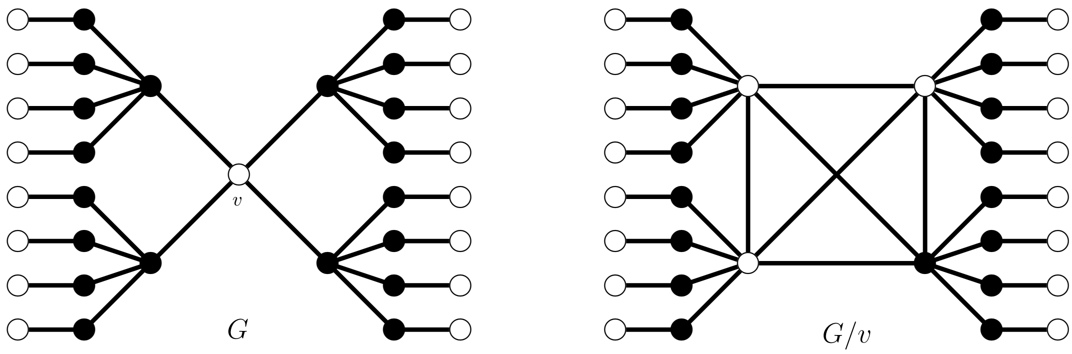

Figure 4: Graphs G and G/v with $\gamma_{\rm st}(G) = 20$ and $\gamma_{\rm st}(G/v) = 17$.

Theorem 2.3. *Let $G = (V, E)$ be a connected graph and $v \in V$ be a pendant vertex such that $uv \in E$. Then,*

$$\gamma_{\rm st}(G) - 1 \leq \gamma_{\rm st}(G/v) \leq \gamma_{\rm st}(G) + \deg(u) - 1.$$

Furthermore, these bounds are tight.

Proof. First we consider the upper bound. Suppose that D is a $\gamma_{\rm st}$-set of G. So clearly $u \in D$. We have the following cases:

(i) u do not strong dominate other vertices except v. Then D is strong dominating set of G/v and we have $\gamma_{\rm st}(G/v) \leq \gamma_{\rm st}(G)$.

(ii) u strong dominate $w \neq v$ and $\deg_G(u) \geq \deg(w) + 1$. Then $(D \cup N(u)) \setminus \{w\}$ is strong dominating set of G/v and we have $\gamma_{\rm st}(G/v) \leq \gamma_{\rm st}(G) + \deg(u) - 1$, because all vertices are strong dominating by the same vertices as before.

(iii) u strong dominate $w \neq v$ and $\deg_G(u) = \deg(w)$. Then $(D \cup N(u)) \setminus \{u\}$ is strong dominating set of G/v and we have $\gamma_{\rm st}(G/v) \leq \gamma_{\rm st}(G) + \deg(u) - 1$, because all vertices are strong dominated by the same vertices as before, and u is strong dominated by w.

Hence $\gamma_{\rm st}(G/v) \leq \gamma_{\rm st}(G) + \deg(u) - 1$. Now we show that this bound is tight. Consider graph G in Figure 5. One can easily check that the set of black vertices is a $\gamma_{\rm st}$-set of G, say S, and we have $\gamma_{\rm st}(G) = 17$, and $(S \cup N(u)) \setminus \{u\}$ is a $\gamma_{\rm st}$-set of G/v, and we have $\gamma_{\rm st}(G/v) = 20$. Now, we consider the lower bound. First, we form G/v and find a $\gamma_{\rm st}$-set of G/v, say D. Then one can easily check that $D \cup \{u\}$ is a strong dominating set of G, and we have $\gamma_{\rm st}(G) \leq \gamma_{\rm st}(G/v) + 1$. If we consider G as path graph of order $3k + 1$, where $k \in \mathbb{N}$, then we see that $\gamma_{\rm st}(G/v) = \gamma_{\rm st}(G) - 1$, and the tightness holds.

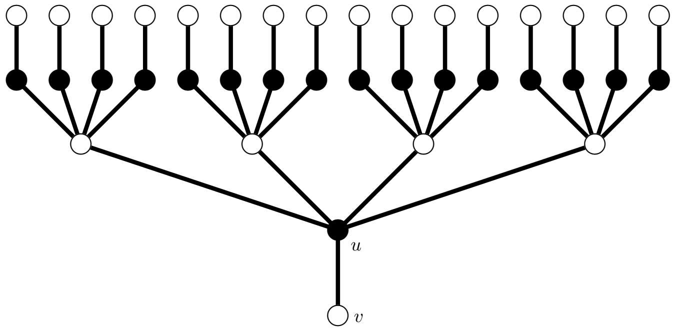

Figure 5: Graph G with $\gamma_{\mathrm{st}}(G) = 17$ and $\gamma_{\mathrm{st}}(G/v) = 20$.

□

We have the following result as an immediate outcome of Theorem 2.3.

Corollary 2.4. *If $G = (V, E)$ is a connected graph and $e = uv \in E$ such that v is the pendant vertex, then*

$$\gamma_{\mathrm{st}}(G) - 1 \leq \gamma_{\mathrm{st}}(G/e) \leq \gamma_{\mathrm{st}}(G) + \deg(u) - 1.$$

Here we consider a modified graph which is obtained by another operation on a vertex. We denote by $G \odot v$ the graph obtained from G by the removal of all edges between any pair of neighbors of v, note v is not removed from the graph. This operation removes triangles from the graph and for the first time has considered for computation of domination polynomial of a graph, which is the generating function for the number of dominating sets of graphs ([2]).

Theorem 2.5. *Let $G = (V, E)$ be a connected graph and $v \in V$. Then*

$$\gamma_{\mathrm{st}}(G \odot v) \leq \gamma_{\mathrm{st}}(G) + 1 - 2\deg_G(v) + \sum_{u \sim v} \deg_{G\odot v}(u).$$

Furthermore, this bound is tight.

Proof. If v is a pendant vertex, then we have nothing to prove, because $\gamma_{\mathrm{st}}(G \odot v) = \gamma_{\mathrm{st}}(G)$. So in the following, suppose that v is not a pendant vertex, and D is a γ_{st}-set of G. If $x \in N(v) \cap D$, and does not strong dominate other vertices, then we simply keep it for strong dominating set of $G \odot v$, and add v to D. If $x \in N(v) \cap (V \setminus D)$, then we add x and add v to D. So, in every cases, all $x \in N(v)$ are strong dominate some other vertices, and these vertices are not in $N(v)$. Suppose that the vertex x strong dominate the vertex y and $y \notin N(v)$. If after forming $G \odot v$, $\deg_{G\odot v}(x) \geq \deg(y)$, then we just add v to D. But sometimes, we need to add all neighbours of x to D and remove x from D. So in this case, The set

$$D' = (D \setminus N(v)) \cup \{v\} \bigcup_{u \sim v} (N_{G\odot v}(u) \setminus \{v\}),$$

is a strong dominating set of $G \odot v$ with the biggest size other than what ever we mentioned before, and we have

$$\gamma_{\mathrm{st}}(G \odot v) \leq \gamma_{\mathrm{st}}(G) + 1 - \deg_G(v) + \sum_{u \sim v} \left(\deg_{G\odot v}(u) - 1\right),$$

and we are done. Now, we show that this bound is tight. Consider graph G and $G \odot v$ in Figure 6. One can easily check that the set of black vertices is a γ_{st}-set of both graphs, and we have $\gamma_{\mathrm{st}}(G) = 65$ and $\gamma_{\mathrm{st}}(G \odot v) = 76$. Therefore the equality holds. □

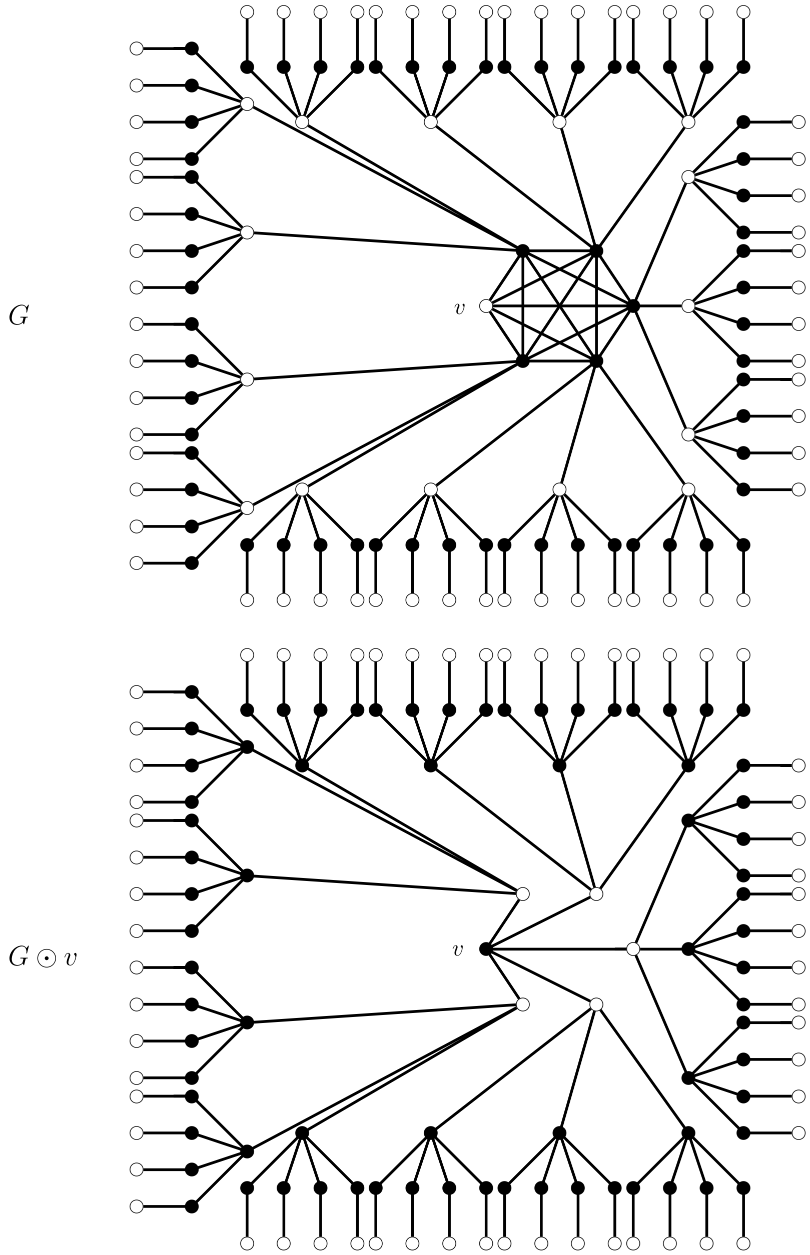

Figure 6: Graphs G and $G \odot v$ with $\gamma_{\mathrm{st}}(G) = 65$ and $\gamma_{\mathrm{st}}(G \odot v) = 76$.

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