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Original Article

Variational problem, Lagrangian and μ-conservation law of the generalized Rosenau-type equation

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ABSTRACT: The goal of this article is to compute conservation law, Lagrangian and μ-conservation law of the generalized Rosenau-type equation using the homotopy operator, the μ-symmetry method and the variational problem method. The generalized Rosenau-type equation includes the generalized Rosenau equation, the generalized Rosenau-RLW equation and the generalized Rosenau-KdV equation, which admits the third-order Lagrangian. The article also compares the conservation law and the μ-conservation law of these three equation.

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1. Introduction

It is known that nonlinear complex physical phenomena can be related to the mathematical model of nonlinear equations in physics. The nonlinear wave is one of the most important scientific research areas. Many scientists developed different mathematical models to explain the wave behaviour, such as the KdV equation, the RLW equation, the Rosenau equation, and many others. In the following, the article gives a short review of these important wave models. The KdV equation

$$u_t + u_x + uu_x + u_{xxx},$$

was introduced by Diederik Korteweg and Gustav de Vries [4] in 1895. There are a lot of studies on this equation and its variational form. The KdV equation, the modified Korteweg-de Vries, the generalised Korteweg-de Vries are nonlinear partial differential equations arising in the study of a number of different physical systems, e.g., water waves, plasma physics, harmonic lattices, elastic rods and nonlinear long dynamo waves observed in the Sun.

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The regularized long-wave (RLW) equation

$$u_t + u_x + uu_x - u_{xxt} = 0,$$

was first put forward as a model for small-amplitude long waves on the surface of water in a channel by Peregrine [8]. The vibrations of a one-dimensional anharmonic lattice associated with the birth of the soliton are modeled in terms of the discrete lattices. If the lattice is dense and weakly anharmonic, the KdV equation is derived. When the article studies the compact discrete systems, the KdV equation cannot model the wave to wave and wave to wall interactions for the dynamics of dense discrete systems. To overcome this difficulty of the KdV equation, Rosenau proposed the following so-called Rosenau equation [9]:

$$u_t + u_x + uu_x + u_{xxxxt} = 0.$$

This equation was derived to describe the dynamics of dense discrete systems considering higher order effects by Rosenau [10]. The generalized Rosenau equation is

$$\Delta_R : u_t + au_x + bu^n u_x + cu_{xxxxt} = 0. \tag{1}$$

where $n \geq 2$ is a positive integer and a, b, c are real valued constants.

For further considerations of nonlinear waves, the term $-u_{xxt}$ is included in the Rosenau equation. The resulting equation is usually called the Rosenau-RLW equation [7]:

$$u_t + u_x + uu_x - u_{xxt} + u_{xxxxt} = 0.$$

The above equation was further extended into the generalized Rosenau-RLW equation (the $gR - RLW$):

$$\Delta_{RW} : u_t + au_x + bu^n u_x - d_1 u_{xxt} + cu_{xxxxt} = 0, \tag{2}$$

where $n \geq 2$ is a positive integer and a, b, c and d_1 are real valued constants [11].

On the other hand, to consider another behaviour of nonlinear waves, the viscous term u_{xxx} needs to be included in the Rosenau equation (1). The resulting equation is usually called the Rosenau-KdV equation:

$$u_t + u_x + uu_x + u_{xxx} + u_{xxxxt} = 0,$$

and the above equation was further extended into the generalized Rosenau-KdV equation (the $gR - KdV$):

$$\Delta_{RK} : u_t + au_x + bu^n u_x + d_2 u_{xxx} + cu_{xxxxt} = 0, \tag{3}$$

where $n \geq 2$ is a positive integer and a, b, c and d_2 are real valued constants [2].

The outline of this article is as follows. Firstly, the article computes conservation law of the generalized Rosenau-type equation using the homotopy operator. Secondly, the article calculates variational problem and Lagrangian of the generalized Rosenau-type equation in potential form using the variational problem method. Thirdly, the article obtains μ-conservation law of the generalized Rosenau-type equation in potential form using μ-symmetry method and μ-conservation law method. Finally, μ-conservation law for the generalized Rosenau-type equation is presented and the article compares the conservation law and μ-conservation law of these equations.

2. Conservation law, variational problem, Lagrangian and the Frechet derivative

Muriel, Romero and Olver [5] have expanded the concept of variational problem and conservation law in the case of symmetries to the case of λ-symmetries of ODEs. They have presented an adapted formulation of the Nother's theorem for λ-symmetry of ODEs. Cicogna and Gaeta [1] have generalized the results obtained by Muriel, Romero and Olver in the case of λ-symmetries for ODEs to the case of μ-symmetries for PDEs, and in the case of μ-symmetry of the Lagrangian, the conservation law is referred to as μ-conservation law.

A variational problem consists of finding the extrema of a functional $\mathfrak{L} = \int_\Omega L(x, u^{(n)})dx$, in some class of functions $u = f(x)$ it is defined over Ω. The integrand $L(x, u^{(n)})$, called the Lagrangian of the variational problem $\mathfrak{L}$, is a smooth function of x, u and various derivative of u. The α-th Euler operator is given by $E_\alpha = \sum_J (-D)_J \dfrac{\partial}{\partial u_J^\alpha}$ for $\alpha = 1, 2, \ldots, q$.

Theorem 2.1. *If $u = f(x)$ is a smooth extremal of the variational problem $\mathfrak{L} = \int_\Omega L(x, u^{(n)})dx$, then it must be a solution of the Euler-Lagrange equations $E_\alpha(L) = 0$, for $\alpha = 1, 2, \ldots, q$.*

If $x = (x^1, x^2, \dots, x^p)$ and $\mathbf{P}(x, u^{(n)}) = (P_1(x, u^{(n)}), \dots, P_p(x, u^{(n)}))$, are p-tuple of smooth functions of x, u and the derivatives of u, it can be defined as the total divergence of $\mathbf{P}$ to be the function $\operatorname{Div}\mathbf{P} := D_1P_1 + \cdots + D_pP_p$, where each D_j is the total derivative with respect to x^j. Let $\Delta(x, u^{(n)}) = 0$, be a system of differential equation. A conservation law is a relation

$$\operatorname{Div}\mathbf{P} := \sum_{i=1}^{p} D_iP^i = 0.$$

$\operatorname{Div}\mathbf{P}$ vanishes on all solutions of the system Δ if and only if there functions $Q_v^J(x, u^{(m)})$ such that $\operatorname{Div}\mathbf{P} = \sum_{v,J} Q_v^J D_J \Delta_v$, for all (x, u). In particular, a system of the Kovalevskaya form satisfies the nondegeneracy condition. Therefore $\operatorname{Div}\mathbf{P} = \operatorname{Div}R + Q.\Delta$, where $Q = (Q_1, \dots, Q_l)$, and $Q_v = \sum_J (-D)_J Q_v^J$. Replacing $\mathbf{P}$ by $\mathbf{P} - R$, the article gets an equivalent conservation law

$$\operatorname{Div}\mathbf{P} = Q.\Delta.$$

This is called the characteristic form of a conservation law, and Q is called the characteristic of the given conservation law. Suppose $E_{\alpha^j}(\Lambda_\nu\Delta_\nu) \equiv 0, and\ j = 1, \dots, q$. Finally $\{\Lambda_\nu\}_{\nu=1}^{l}$ yields a local conservation law for the system and Λ determines a pair of nontrivial local conservation law of (ρ, ϱ), i.e.

$$D_x\rho^1 + D_t\rho^2 \equiv \Lambda\Delta.$$

To calculate (ρ^1, ρ^2), one can use strong 2-dimensional homotopy operator

$$D_x\rho^1 + D_t\rho^2 = D_x \mathsf{H}^{(x)}_{u(x,t)} f + D_t \mathsf{H}^{(t)}_{u(x,t)} f = 0.$$

Definition 2.2. *The homotopy operator is a pair vector operator of* $\left(H^{(x)}_{u(x,t)} f, H^{(t)}_{u(x,t)} f\right)$, *where*

$$H^{(x)}_{u(x,t)} f = \int_0^1 \Big(\sum_{j=1}^{q} \Upsilon^{(x)}_{u^j} f\Big)[\kappa u] \frac{d\kappa}{\kappa}, \quad H^{(t)}_{u(x,t)} f = \int_0^1 \Big(\sum_{j=1}^{q} \Upsilon^{(t)}_{u^j} f\Big)[\kappa u] \frac{d\kappa}{\kappa}.$$

The x-integrand, $\Upsilon^{(x)}_{u^j(x,t)} f$ and the t-integrand, $\Upsilon^{(t)}_{u^j(x,t)} f$ are

$$\Upsilon^{(x)}_{u^j} f = \sum_{\iota_1=1}^{N_1^j} \sum_{\iota_2=0}^{N_2^j} \Big(\sum_{r_1=0}^{\iota_1-1} \sum_{r_2=0}^{\iota_2} \mathbf{J}^{(x)} u^j_{x^{r_1}t^{r_2}} (-D_x)^{\iota_1-r_1-1} (-D_t)^{\iota_2-r_2} \Big) \frac{\partial f}{\partial u^j_{x^{\iota_1}t^{\iota_2}}},$$

$$\Upsilon^{(t)}_{u^j} f = \sum_{\iota_1=0}^{N_1^j} \sum_{\iota_2=1}^{N_2^j} \Big(\sum_{r_1=0}^{\iota_1} \sum_{r_2=0}^{\iota_2-1} \mathbf{J}^{(x)} u^j_{x^{r_1}t^{r_2}} (-D_x)^{\iota_1-r_1} (-D_t)^{\iota_2-r_2-1} \Big) \frac{\partial f}{\partial u^j_{x^{\iota_1}t^{\iota_2}}},$$

where N_1^j, N_2^j are the order of derivatives u in x and t and

$$\mathbf{J}^{(x)} = \mathbf{J}(r_1, r_2, \iota_1, \iota_2) = \frac{C(r_1 + r_2, r_1)C(\iota_1 + \iota_2 - r_1 - r_2 - 1, \iota_1 - r_1 - 1)}{C(\iota_1 + \iota_2, \iota_1)}.$$

Also, $\mathbf{J}^{(t)} = \mathbf{J}(r_2, r_1, \iota_2, \iota_1)$.

Theorem 2.3 (Noether's Theorem). *Suppose G is a one-parameter group of symmetries of the variational problem $\mathfrak{L} = \int L(x, u^{(n)})dx$. Let $X = \xi^i(x, u)\partial_{x^i} + \varphi_\alpha(x, u)\partial_{u^\alpha}$, be the infinitesimal generator of G, and $Q_\alpha(x, u) = \varphi_\alpha - \sum_{i=1}^{p} \xi^i \dfrac{\partial u^\alpha}{\partial x^i}$, the corresponding characteristic of X. Then $Q = (Q_1, \dots, Q_q)$, is also the characteristic of a conservation law for the Euler-Lagrange equations $E(L) = 0$; in other words, there is a p-tuple $\mathbf{P}(x, u^{(n)}) = (P_1, \dots, P_p)$, such that $\operatorname{Div}\mathbf{P} = Q.E(L) = \sum_{v=1}^{q} Q_v E(L)$, is a conservation law in characteristic form for the Euler-Lagrange equations $E(L) = 0$.*

The Frechet derivative with respect to a tuple of functions $\Delta_\alpha(x, [u]) = 0$, is defined as

$$\mathbf{D}_\Delta(P) := \frac{d}{d\varepsilon} \mid_{\varepsilon=0} \Delta(x, [u + \varepsilon P(x, [u])]).$$

In components, it is $(\mathbf{D}_\Delta)_{\alpha\beta} = \sum_J \frac{\partial \Delta_\alpha}{\partial u_J^\beta} D_J$. The adjoint operator is given by $(\mathbf{D}_Q^*)_{\alpha\beta} = \sum_J (-D)_J) \frac{\partial Q^\alpha}{\partial u_J^\beta}$. For a Euler-Lagrange equations $E(L) = 0$, the associated Frechet derivative is always self-adjoint, namely $\mathbf{D}^*_{E(L)} = \mathbf{D}_{E(L)}$. Hence in some sense it implies Noether's theorem through the relation between characteristics of symmetries and conservation laws. It is also interesting to realise that self-adjointness of a Frechet derivative is sufficient but not necessary for constructing a relation between symmetries and conservation laws. For instance, skew self-adjointness is also sufficient, namely $\mathbf{D}^*_\Delta = -\mathbf{D}_\Delta$, for a system, $\Delta_\alpha(x, [u]) = 0$.

A system admits a variational formulation if and only if its Frechet derivative is self-adjoint. In fact, one can see the following theorem [6].

Theorem 2.4. *Let $\Delta = 0$ be a system of differential equation. Then Δ is the Euler-Lagrange expression for some variational problem $\mathfrak{L} = \int L dx$, i.e. $\Delta = E(L)$, if and only if the Frechet derivative $\mathbf{D}_\Delta$ is self-adjoint: $\mathbf{D}^*_\Delta = \mathbf{D}_\Delta$. In this case, a Lagrangian for Δ can be explicitly constructed using the homotopy formula*

$$L[u] = \int_0^1 u.\Delta[\lambda u] d\lambda.$$

3. Conservation law of the generalized Rosenau-type equation

All the rules in form $\Lambda = \Lambda(x, t, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt})$ of the Eq.(1) are obtained, and the solutions of the determining system are

$$\Lambda_1 = 1, \quad \Lambda_2 = u,$$

where Λ determines a pair of nontrivial local conservation law of (ρ^1, ρ^2), where

$$D_x \rho^1 + D_t \rho^2 \equiv \Lambda \Delta_R.$$

Table 1 show the local conservation law multipliers for the generalized Rosenau equation.

Table 1: Conservation laws for Eq.(1)

Λ	
$\Lambda_1 = 1$	$\Upsilon_u^{(x)} = au + bu^{n+1} + \frac{4}{5} c u_{xxxt}$ $\Upsilon_u^{(t)} = u + \frac{1}{5} c u_{xxxx}$ $\rho^1 = au + \frac{b}{n+1} u^{n+1} + \frac{4}{5} c u_{xxxt}$ $\rho^2 = u + \frac{1}{5} c u_{xxxx}$ $D_x \rho^1 + D_t \rho^2 \equiv \Lambda_1 \Delta_R$
$\Lambda_2 = U$	$\Upsilon_u^{(x)} = au^2 + bu^{n+2} + \frac{8}{5} cu u_{xxxt} - \frac{2}{5} c u_t u_{xxx} - \frac{6}{5} c u_x u_{xxt} + \frac{4}{5} c u_{xx} u_{xt}$ $\Upsilon_u^{(t)} = u^2 + \frac{2}{5} cu u_{xxxx} - \frac{2}{5} c u_x u_{xxx} + \frac{1}{5} c u_{xx}^2$ $\rho^1 = \frac{a}{2} u^2 + \frac{b}{n+2} u^{n+2} + \frac{4}{5} cu u_{xxxt} - \frac{1}{5} c u_t u_{xxx} - \frac{3}{5} c u_x u_{xxt} + \frac{2}{5} c u_{xx} u_{xt}$ $\rho^2 = \frac{1}{2} u^2 + \frac{1}{5} cu u_{xxxx} - \frac{1}{5} c u_x u_{xxx} + \frac{1}{10} c u_{xx}^2$ $D_x \rho^1 + D_t \rho^2 \equiv \Lambda_2 \Delta_R$

Tables 2 and 3 show the local conservation law multipliers for the generalized Rosenau-RLW and generalized Rosenau-KdV equations.

4. Lagrangian of the generalized Rosenau-type equation

The generalized Rosenau equation do not admit a variational problem since it is of odd order, but the generalized Rosenau equation in potential form admits a variational problem. The Frechet derivative of the generalized Rosenau equation is

$$\mathbf{D}_{\Delta_R} = nbu^{n-1} u_x + D_t + (a + bu^n) D_x + c D_x^4 D_t,$$

then, it does not admit a variational problem since $\mathbf{D}^*_{\Delta_R} \neq \mathbf{D}_{\Delta_R}$. But, replacing u by v_x for the generalized Rosenau equation, the article gets the generalized Rosenau equation in potential form:

$$\Delta_{Rv} : v_{xt} + a v_{xx} + b v_x^n v_{xx} + c v_{xxxxxt} = 0\,.$$

Table 2: Conservation laws for Eq.(2)

Λ	
$\Lambda_1 = 1$	$\Upsilon_u^{(x)} = au + bu^{n+1} - \frac{2}{3}d_1 u_{xt} + \frac{4}{5}cu_{xxxt}$ $\Upsilon_u^{(t)} = u - \frac{1}{3}d_1 u_{xx} + \frac{1}{5}cu_{xxxx}$ $\rho^1 = au + \frac{b}{n+1}u^{n+1} - \frac{2}{3}d_1 u_{xt} + \frac{4}{5}cu_{xxxt}$ $\rho^2 = u - \frac{1}{3}d_1 u_{xx} + \frac{1}{5}cu_{xxxx}$
$\Lambda_2 = U$	$\Upsilon_u^{(x)} = au^2 + bu^{n+2} - \frac{4}{3}d_1 uu_{xt} + \frac{2}{3}d_1 u_x u_t + \frac{8}{5}cuu_{xxxt} - \frac{2}{5}cu_t u_{xxx} - \frac{6}{5}cu_x u_{xxt} + \frac{4}{5}cu_{xx}u_{xt}$ $\Upsilon_u^{(t)} = u^2 - \frac{2}{3}d_1 uu_{xx} + \frac{1}{3}d_1 u_x^2 + \frac{2}{5}cuu_{xxxx} - \frac{2}{5}cu_x u_{xxx} + \frac{1}{5}cu_{xx}^2$ $\rho^1 = \frac{a}{2}u^2 + \frac{b}{n+2}u^{n+2} - \frac{2}{3}d_1 uu_{xt} + \frac{1}{3}d_1 u_x u_t + \frac{4}{5}cuu_{xxxt} - \frac{1}{5}cu_t u_{xxx} - \frac{3}{5}cu_x u_{xxt} + \frac{2}{5}cu_{xx}u_{xt}$ $\rho^2 = \frac{1}{2}u^2 - \frac{1}{3}d_1 uu_{xx} + \frac{1}{6}d_1 u_x^2 + \frac{1}{5}cuu_{xxxx} - \frac{1}{5}cu_x u_{xxx} + \frac{1}{10}cu_{xx}^2$

Table 3: Conservation laws for Eq.(3)

Λ	
$\Lambda_1 = 1$	$\Upsilon_u^{(x)} = au + bu^{n+1} + d_2 u_{xx} + \frac{4}{5}cu_{xxxt}$ $\Upsilon_u^{(t)} = u + \frac{1}{5}cu_{xxxx}$ $\rho^1 = au + \frac{b}{n+1}u^{n+1} + d_2 u_{xx} + \frac{4}{5}cu_{xxxt}$ $\rho^2 = u + \frac{1}{5}cu_{xxxx}$
$\Lambda_2 = U$	$\Upsilon_u^{(x)} = au^2 + bu^{n+2} + 2d_2 uu_{xx} - d_2 u_x^2 + \frac{8}{5}cuu_{xxxt} - \frac{2}{5}cu_t u_{xxx} - \frac{6}{5}cu_x u_{xxt} + \frac{4}{5}cu_{xx}u_{xt}$ $\Upsilon_u^{(t)} = u^2 + \frac{2}{5}cuu_{xxxx} - \frac{2}{5}cu_x u_{xxx} + \frac{1}{5}cu_{xx}^2$ $\rho^1 = \frac{a}{2}u^2 + \frac{b}{n+2}u^{n+2} + d_2 uu_{xx} - \frac{1}{2}d_2 u_x^2 + \frac{4}{5}cuu_{xxxt} - \frac{1}{5}cu_t u_{xxx} - \frac{3}{5}cu_x u_{xxt} + \frac{2}{5}cu_{xx}u_{xt}$ $\rho^2 = \frac{1}{2}u^2 + \frac{1}{5}cuu_{xxxx} - \frac{1}{5}cu_x u_{xxx} + \frac{1}{10}cu_{xx}^2$

The Frechet derivative of the Δ_{Rv} is

$$\mathbf{D}_{\Delta_{Rv}} = D_x D_t + nbv_x^{n-1}v_{xx}D_x + (a + bv_x^n)D_x^2 + cD_x^5 D_t\,,$$

and it is self-adjoint: $\mathbf{D}^*_{\Delta_{Rv}} = \mathbf{D}_{\Delta_{Rv}}$. According to Theorem 2.4, the Δ_{Rv} has a Lagrangian of the following form

$$L[v] = \int_0^1 v.\Delta_{Rv}[\lambda v]d\lambda = -\frac{1}{2}\Big(v_x v_t + av_x^2 + \frac{2}{(n+1)(n+2)}bv_x^{n+2} + cv_{xxx}v_{xxt}\Big) + \mathrm{Div}\mathbf{P}.$$

Corollary 4.1 (Lagrangian of the Δ_{Rv}). *The 3-th order Lagrangian of the Δ_{Rv}, up to* Div*-equivalence is*

$$\mathcal{L}_{\Delta_{Rv}}[v] = -\frac{1}{2}\Big(v_x v_t + av_x^2 + \frac{2}{(n+1)(n+2)}bv_x^{n+2} + cv_{xxx}v_{xxt}\Big).$$

Tables 4 and 5 show the following results for the Lagrangian are obtained from the generalized Rosenau-RLW equation in potential form (the Δ_{RWv}) and the generalized Rosenau-KdV equation in potential form (the Δ_{RKv}).

Table 4: Lagrangian for the Δ_{RWv}

The Frechet derivative	$\mathbf{D}_{gR-RLW_v} = D_x D_t + nbv_x^{n-1}v_{xx}D_x + (a + bv_x^n)D_x^2 - d_1 D_x^3 D_t + cD_x^5 D_t$
Lagrangian	$\mathcal{L}_{gR-RLW_v}[v] = -\frac{1}{2}\Big(v_x v_t + av_x^2 + \frac{2}{(n+1)(n+2)}bv_x^{n+2} + d_1 v_{xx}v_{xt} + cv_{xxx}v_{xxt}\Big)$

5. μ-conservation law and the 3−th order Lagrangian

Let $\Delta(x, u^{(n)}) = 0$ be a scalar PDEs for $u = u(x^1, \ldots, x^p)$ and $\mu = \lambda_i dx^i$, be horizontal one-form on first order jet space $(J^{(1)}M, \pi, M)$ with condition $D_i\lambda_j - D_j\lambda_i = 0$, where $\lambda_i\colon J^{(1)}M \longrightarrow \mathbb{R}$ [3]. Suppose $X = \xi^i\partial_{x^i} + \varphi\partial_u$,

Table 5: Lagrangian for the Δ_{RKv}

The Frechet derivative	$\mathbf{D}_{gR-RLW_v} = D_xD_t + nbv_x^{n-1}v_{xx}D_x + (a+bv_x^n)D_x^2 + d_2D_x^4 + cD_x^5D_t$
Lagrangian	$\mathcal{L}_{gR-KdV_v}[v] = -\frac{1}{2}\Big(v_xv_t + av_x^2 + \frac{2}{(n+1)(n+2)}bv_x^{n+2} - d_2v_{xx}^2 + cv_{xxx}v_{xxt}\Big)$

is a vector field on M. The μ-prolongation of X on n-th order jet space J^nM is $Y = X + \sum_{J=1}^k \Psi_J\,\partial_{u_J}$, and its coefficient satisfies the μ-prolongation formula

$$\Psi_{J,i} = (D_i+\lambda_i)\Psi_J - u_{J,m}(D_i+\lambda_i)\xi^m\,, \tag{4}$$

where $\Psi_0 = \varphi$. Let $Y : \mathcal{S} \longrightarrow T\mathcal{S}$, and $\mathcal{S} \subset J^{(k)}M$ be the solution manifold for Δ, then X is a μ-symmetry for Δ.

A conservation law is a relation $\operatorname{Div}\mathbf{P} := \sum_{i=1}^p D_iP^i = 0$, where $\mathbf{P} = (P^1,\dots,P^p)$ is a $p-$dimensional vector. A μ-conservation law is a relation as

$$(D_i+\lambda_i)P^i = 0,$$

where P^i is a vector and the $M-$vector P^i is called a μ-conserved vector.

Theorem 5.1. *Consider the $n-th$ order Lagrangian $\mathcal{L} = \mathcal{L}(x,u^{(n)})$, and vector field X, then X is a μ-symmetry for $\mathcal{L}$, i.e. $Y[\mathcal{L}] = 0$ if and only if there exists $M-$vector P^i satisfying the μ-conservation law $(D_i+\lambda_i)P^i = 0$ [1].*

Let $\mathcal{L}$ be a second order Lagrangian and the vector field $X = \varphi\,(\partial/\partial u)$ be a μ-symmetry for $\mathcal{L}$, then the $M-$vector

$$P^i := \varphi\frac{\partial\mathcal{L}}{\partial u_i} + ((D_j+\lambda_j)\varphi)\frac{\partial\mathcal{L}}{\partial u_{ij}} - \varphi D_j\frac{\partial\mathcal{L}}{\partial u_{ij}},$$

is a μ-conserved vector.

Theorem 5.2. *Consider the $3-th$ order Lagrangian $\mathcal{L} = \mathcal{L}(x,t,u_x,\dots,u_{ttt})$, and vector field X, then $X = \varphi\,(\partial/\partial u)$ is a μ-symmetry for $\mathcal{L}$, i.e. $Y[\mathcal{L}] = 0$ if and only if the $M-$vector*

$$P^i := \varphi\frac{\partial\mathcal{L}}{\partial u_i} + [(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{ij}} - \varphi D_j\frac{\partial\mathcal{L}}{\partial u_{ij}} - (D_k+\lambda_k)\Big([(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{jki}} - \varphi D_j\frac{\partial\mathcal{L}}{\partial u_{jki}}\Big), \tag{5}$$

satisfying the μ-conservation law $(D_i+\lambda_i)P^i = 0$.

Proof. Let $X = \varphi\,(\partial/\partial u)$ be a μ-symmetry for $\mathcal{L}$, its three μ-prolongation is

$$Y = \varphi\frac{\partial}{\partial u} + [(D_x+\lambda_1)\varphi]\frac{\partial}{\partial u_x} + [(D_t+\lambda_2)\varphi]\frac{\partial}{\partial u_t} + \cdots + [(D_t+\lambda_2)^3\varphi]\frac{\partial}{\partial u_{ttt}}.$$

Applying this to the Lagrangian $\mathcal{L}$, one can see

$$Y[\mathcal{L}] = \varphi\frac{\partial\mathcal{L}}{\partial u} + [(D_x+\lambda_1)\varphi]\frac{\partial\mathcal{L}}{\partial u_x} + [(D_t+\lambda_2)\varphi]\frac{\partial\mathcal{L}}{\partial u_t} + \cdots + [(D_t+\lambda_2)^3\varphi]\frac{\partial\mathcal{L}}{\partial u_{ttt}}\,,$$

and integrating by parts, one gets

$$\begin{aligned} Y[\mathcal{L}] = \varphi\Big(&\frac{\partial\mathcal{L}}{\partial u} - D_x\varphi\frac{\partial\mathcal{L}}{\partial u_x} - D_t\varphi\frac{\partial\mathcal{L}}{\partial u_t} + D_x^2\varphi\frac{\partial\mathcal{L}}{\partial u_{xx}} + \cdots - D_t^3\varphi\frac{\partial\mathcal{L}}{\partial u_{ttt}}\Big) \\ &+ (D_x+\lambda_1)\Big[\varphi\frac{\partial\mathcal{L}}{\partial u_x} + [(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{xj}} - \varphi D_j\frac{\partial\mathcal{L}}{\partial u_{xj}} - (D_k+\lambda_k)\Big([(D_j+\lambda_j)\varphi] \\ &\cdot\frac{\partial\mathcal{L}}{\partial u_{jkx}} - \varphi D_j\frac{\partial\mathcal{L}}{\partial u_{jkx}}\Big)\Big] + (D_t+\lambda_2)\Big[\varphi\frac{\partial\mathcal{L}}{\partial u_t} + [(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{tj}} - \varphi D_j\frac{\partial\mathcal{L}}{\partial u_{tj}} \\ &- (D_k+\lambda_k)\Big([(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{jkt}} - \varphi D_j\frac{\partial\mathcal{L}}{\partial u_{jkt}}\Big)\Big]. \end{aligned}$$

To put

$$P^i := \varphi\frac{\partial\mathcal{L}}{\partial u_i} + [(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{ij}} - \varphi D_j\frac{\partial\mathcal{L}}{\partial u_{ij}} - (D_k+\lambda_k)\Big([(D_j+\lambda_j)\varphi]\frac{\partial\mathcal{L}}{\partial u_{jki}} - \varphi D_j\frac{\partial\mathcal{L}}{\partial u_{jki}}\Big).$$

Then there is:

$$Y[\mathcal{L}] = \varphi E(\mathcal{L}) + (D_i+\lambda_i)P^i\,,$$

where E is the Euler-Lagrange operator. The Euler-Lagrange equations $E(\mathcal{L}) = 0$ it vanishes the three term on solutions to the equations, hence this reduces to

$$Y[\mathcal{L}] = (D_i+\lambda_i)P^i\,.$$

This shows that $Y[\mathcal{L}] = 0$ implies $(D_i+\lambda_i)P^i = 0$. The $M-$vector of P^i implies $Y[\mathcal{L}] = 0$. $\square$

6. μ-conservation laws of the generalized Rosenau-type equation in potential form

The author considers the 3−th order Lagrangian $\mathcal{L}_{\Delta_{Rv}}[v]$ for the generalized Rosenau equation in potential form, then

$$\Delta_{Rv} = E(\mathcal{L}_{\Delta_{Rv}}).$$

Let $X = \varphi\partial_v$ be a vector field for $\mathcal{L}_{\Delta_{Rv}}[v]$ and $\mu = \lambda_1 dx + \lambda_2 dt$ be a horizontal one-form so that $D_t\lambda_1 = D_x\lambda_2$ when $\Delta_{Rv} = 0$. According to (4), μ-prolongation of order 3 of X is

$$Y = \varphi\partial_v + \Psi^x\partial_{v_x} + \Psi^t\partial_{v_t} + \Psi^{xx}\partial_{v_{xx}} + \cdots + \Psi^{ttt}\partial_{v_{ttt}},$$

where coefficients Y are as the following:

$$\begin{aligned}
\Psi^x &= (D_x+\lambda_1)\varphi, &\quad \Psi^t &= (D_t+\lambda_2)\varphi, &\quad \Psi^{xx} &= (D_x+\lambda_1)\Psi^x,\\
\Psi^{xt} &= (D_t+\lambda_2)\Psi^x, & \Psi^{tt} &= (D_t+\lambda_2)\Psi^t\,, & \Psi^{xxx} &= (D_x+\lambda_1)\Psi^{xx},\\
\Psi^{xxt} &= (D_t+\lambda_2)\Psi^{xx}, & \Psi^{xtt} &= (D_t+\lambda_2)\Psi^{xt}\,, & \Psi^{ttt} &= (D_t+\lambda_2)\Psi^{tt}.
\end{aligned}$$

Therefore, the μ-prolongation Y acts on the $\mathcal{L}_{\Delta_{Rv}}[v]$, and replacing v_t by $\Big(av_x^2 + (2/((n+1)(n+2)))bv_x^{n+2} + cv_{xxx}v_{xxt}\Big)/(1/2)v_x$, one can find the system of equations

$$-(3/2)c\varphi_{vv} = 0\,, \qquad -(1/2)c\lambda_2\varphi_v = 0, \quad -(1/2)c\varphi_{vt} = 0, \quad \ldots\,. \tag{6}$$

Let $F(x,t)$ be an arbitrary positive function satisfying $\mathcal{L}_{\Delta_{Rv}}[v] = 0$, and $\varphi = F(x,t)$, then

$$\lambda_1 = -\frac{F_x(x,t)}{F(x,t)}\,, \qquad \lambda_2 = -\frac{F_t(x,t)}{F(x,t)},$$

are special solutions of the system (6), where $D_t\lambda_1 = D_x\lambda_2$. Therefore, $X = F(x,t)\partial_v$ is a μ-symmetry for $\mathcal{L}_{\Delta_{Rv}}[v]$ and according to Theorem 5.1, there exists M−vector P^i satisfying the μ-conservation law $(D_i+\lambda_i)P^i = 0$. Then, using (5), the M−vector P^i are as the followings

$$\begin{aligned}
P^1 &= -\frac{1}{2}F(x,t)\Big(v_t + 2av_x + \frac{2}{n+1}bv_x^{n+1} + cv_{xxxxt}\Big),\\
P^2 &= -\frac{1}{2}F(x,t)\Big(v_x + cv_{xxxxx}\Big)\,,
\end{aligned} \tag{7}$$

and $(D_i+\lambda_i)P^i = 0$, is a μ-conservation law for 3-th order Lagrangian $\mathcal{L}_{\Delta_{Rv}}[v]$.

Corollary 6.1. *(μ-conservation law of the Δ_{Rv})*
The μ-symmetry of $\mathcal{L}_{gR_v}[v]$ is $X = F(x,t)\partial_v$ and μ-conservation law for the generalized Rosenau equation in potential form is $(D_i+\lambda_i)P^i = 0$, where P^1 and P^2 are the M−vector P^i of (7).

Corollary 6.2. *(μ-conservation law of the Δ_{Rv} and the Noether's Theorem)*
μ-conservation law of the generalized Rosenau equation in potential form, satisfying to the Noether's Theorem for μ-symmetry, i.e.

$$\begin{aligned}
(D_i+\lambda_i)P^i &= (D_x+\lambda_1)P^1 + (D_t+\lambda_2)P^2\\
&= F(x,t)(v_{xt} + av_{xx} + bv_x^n v_{xx} + cv_{xxxxxt})\\
&= QE(\mathcal{L}_{\Delta_{Rv}})\,.
\end{aligned}$$

Tables 6 and 7 show μ-symmetry, μ-conservation law and the Noether's Theorem for the generalized Rosenau-RLW and generalized Rosenau-KdV equations in potential forms.

7. μ-conservation laws of the generalized Rosenau-type equation

The author considers the generalized Rosenau equation in potential form:

$$D_x(v_t + av_x + (b/(n+1))v_x^{n+1} + cv_{xxxxt}) = 0,$$

and that is equivalent to

$$v_t + av_x + (b/(n+1))v_x^{n+1} + cv_{xxxxt} = f(t),$$

Table 6: μ-conservation law for the Δ_{RWv}

μ-symmetry of $\mathcal{L}_{gR-RLW_v}[v]$	$X = F(x,t)\partial_v$
μ-conservation law	$P^1 = -\frac{1}{2}F(x,t)\Big(v_t + 2av_x + \frac{2}{n+1}bv_x^{n+1} - 2d_1v_{xxt} + cv_{xxxxt}\Big)$ $P^2 = -\frac{1}{2}F(x,t)\Big(v_x + cv_{xxxxx}\Big)$
The Noether's Theorem	$(D_i + \lambda_i)P^i = QE(\mathcal{L}_{\Delta_{RKv}})$

Table 7: μ-conservation law for the Δ_{RKv}

μ-symmetry of $\mathcal{L}_{gR-KdV_v}[v]$	$X = F(x,t)\partial_v$
μ-conservation law	$P^1 = -\frac{1}{2}F(x,t)\Big(v_t + 2av_x + \frac{2}{n+1}bv_x^{n+1} + 2d_2v_{xxx} + cv_{xxxxt}\Big)$ $P^2 = -\frac{1}{2}F(x,t)\Big(v_x + cv_{xxxxx}\Big)$
The Noether's Theorem	$(D_i + \lambda_i)P^i = QE(\mathcal{L}_{\Delta_{RKv}})$

where $f(t)$ is an arbitrary function. One can substitutes $f(t) - av_x - (b/(n+1))v_x^{n+1} - cv_{xxxxt}$ for v_t and substitutes u for v_x in the $M-$vector P^i of (7), then, one obtain $M-$vectors P^1 and P^2:

$$\begin{aligned} P^1 &= -\frac{1}{2}F(x,t)\Big(f(t) + au + \frac{b}{n+1}u^{n+1}\Big)\,, \\ P^2 &= -\frac{1}{2}F(x,t)\Big(u + cu_{xxxx}\Big)\,. \end{aligned} \tag{8}$$

Corollary 7.1 (μ-conservation law of the Eq.(1)). *μ-conservation law for the generalized Rosenau equation is $(D_i + \lambda_i)P^i = 0$, where P^1 and P^2 are the $M-$vector P^i of (8).*

Corollary 7.2 (the Eq.(1) and characteristic form). *The generalized Rosenau equation satisfying to the characteristic form, i.e.*

$$\begin{aligned} (D_i + \lambda_i)P^i &= (D_x + \lambda_1)P^1 + (D_t + \lambda_2)P^2 \\ &= F(x,t)(u_t + au_x + bu^n u_x + cu_{xxxxt}) \\ &= Q.\Delta_R\,. \end{aligned}$$

Therefore, table 8 shows μ-conservation law for the generalized Rosenau equation.

Table 8: μ-conservation law for the Eq.(1)

μ-conservation law	$P^1 = -\frac{1}{2}F(x,t)\Big(f(t) + au + \frac{b}{n+1}u^{n+1}\Big)$ $P^2 = -\frac{1}{2}F(x,t)\Big(u + cu_{xxxx}\Big)$
Characteristic form	$(D_i + \lambda_i)P^i = Q.\Delta_R$

Tables 9 and 10 show μ-conservation law for the generalized Rosenau-RLW and generalized Rosenau-KdV equations.

Table 9: μ-conservation law for the Eq.(2)

μ-conservation law	$P^1 = -\frac{1}{2}F(x,t)\Big(f(t) + au + \frac{b}{n+1}u^{n+1} - d_1u_{xt}\Big)$ $P^2 = -\frac{1}{2}F(x,t)\Big(u + cu_{xxxx}\Big)$
Characteristic form	$(D_i + \lambda_i)P^i = Q.\Delta_{RW}$

Table 10: μ-conservation law for the Eq.(3)

μ-conservation law	$P^1 = -\frac{1}{2}F(x,t)\Big(f(t) + au + \frac{b}{n+1}u^{n+1} + d_2 u_{xx}\Big)$ $P^2 = -\frac{1}{2}F(x,t)\Big(u + cu_{xxxx}\Big)$
Characteristic form	$(D_i + \lambda_i)P^i = Q.\Delta_{RK}$

Conclusion

Tables 1, 2, 3 and tables 8, 9, 10 also compare the conservation law and the μ-conservation law of the generalized Rosenau equation, the generalized Rosenau-RLW equation and the generalized Rosenau-KdV equation.

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