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Original Article

# Generalized Lorentzian Ricci solitons on 3-dimensional Lie groups associated to the Bott Connection

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**ABSTRACT:**
In this paper, we investigate which one of the non-isometric left-invariant Lorentz metrics $g$ on 3-dimensional Lie groups satisfies the generalized Ricci soliton equation $a\mathrm{Ric}^B[g] + \frac{b}{2}\mathcal{L}_X^B g + cX^\flat \otimes X^\flat = \lambda g$ associated to the Bott connection $\nabla^B$, here $X$ is a vector field and $\lambda, a, b, c$ are real constants such that $c \neq 0$. A complete classification of this structure on 3-dimensional Lorentzian Lie groups will be presented.

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## 1. Introduction

Einstein metrics play a fundamental role in many cases of equations of importance and interest in differential geometry and physics and have been extensively studied recently in both Riemannian and pseudo-Riemannian geometry. Ricci solitons as a natural generalization of Einstein metrics were first introduced by R. Hamilton. Ricci solitons are self-similar solutions to the Ricci flow equation and play an important role in understanding the singularities of the Ricci flow equation [9]. In [16], Perelman demonstrated that any closed Riemannian manifold admitting a Ricci soliton is gradient soliton. He also proved that any closed Riemannian manifold admitting a steady or expanding Ricci soliton is necessarily an Einstein manifold. Further, if $M$ is not compact then the Ricci soliton $(M, g)$ is not necessarily gradient. The Riemannian manifold $(M, g)$ is called a Ricci soliton, if there exist a smooth vector field $X$ and a real constant $\lambda$ satisfying,

$$\mathrm{Ric} + \frac{1}{2}\mathcal{L}_X g = \lambda g, \tag{1}$$

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where Ric is the Ricci tensor and $\mathcal{L}_X g$ denotes the Lie derivative of the metric $g$ with respect to the vector field $X$. Note that, in the case where $\lambda$ is a variable function the Ricci soliton is called an almost Ricci soliton. The quadruple $(M, g, X, \lambda)$ denotes the Ricci soliton and the vector field $X$ is called the potential vector field of the Ricci soliton. The Ricci soliton $(M, g, X, \lambda)$ is said to be expanding, stable or shrinking depending on $\lambda < 0$, $\lambda = 0$ or $\lambda > 0$, respectively. If the potential field $X = \nabla f$ for some real smooth function $f$ on $M$, then $(M, g)$ is called a gradient Ricci soliton and in this case equation (1) takes the following form

$$\mathrm{Ric} + \mathrm{Hess}(f) = \lambda g.$$

The Ricci soliton $(M, g, X, \lambda)$ is said to be trivial if $\mathcal{L}_X g = 0$. In this case the metric $g$ reduces to an Einstein metric.
Ricci solitons on Finsler spaces are introduced and developed by Bidabad and et al. (see [6, 26]). Recently, Ricci solitons are considered and studied extensively in pseudo-Riemannian geometry because of their application in theoretical physics (for instance see [7, 14]). In [25], the authors studied the Lorentzain Ricci solitons on nilpotent Lie groups. Moreover, Ricci solitons have been investigated on pseudo-Riemannain manifolds associated to an arbitrary affine connection. Einstein manifolds associated to affine connections were investigated in [12, 15, 18, 20, 21, 22] and affine Ricci solitons were studied in [10, 11, 13, 17, 19]. Furthermore, Wang [23] classified the affine Ricci solitons associated to canonical connections and Kobayashi-Nomizu connections on three-dimensional Lorentzian Lie groups. He also classified affine Ricci solitons on three dimensional Lorentzian Lie groups [24]. Also, the notion of generalized Ricci solitons as a generalization of Einstein manifolds were introduced by Catino et al. [8]. Thereafter, the second author in [2, 3] investigated affine generalized Ricci solitons on three dimensional Lorentzian Lie groups associated to Yano connections, canonical connections and Kobayashi-Nomizu connections. Motivation by these works, in this paper we consider three dimensional Lorentzian Lie groups and study their generalized solitons associated to the Bott connection. We consider the equation

$$a\mathrm{Ric}^B[g] + \frac{b}{2}\mathcal{L}^B_X g + cX^\flat \otimes X^\flat = \lambda g, \tag{2}$$

on three dimensional Lorentzian Lie groups and solve the corresponding system of algebraic equations. Here $\mathrm{Ric}^B$ is the Ricci tensor associated to the Bott connection, $\mathcal{L}^B_X$ is the Lie derivative in dirction $X$ with respec to to the Bott connection, and $X^\flat$ is defined by $X^\flat(Y) = g(X, Y)$ for any vector field $Y$. When $X = 0$ and $a = \lambda = 0$, the equation (2) is trivially true and we say such solutions to be trivial. Also,

- if $a = 1$ and $b = c = 0$ then the equation (2) reduces to Einstein equation associated to Bott connection,
- if $a = c = 0$ then the equation (2) is in relation to conformal-Killing vector fields associated to Bott connection,
- if $a = b = 1$ and $c = 0$ then the equation (2) reduces to Ricci soliton equation associated to Bott connection which has been studied in [23].

According to the above cases, we assume that $c \neq 0$, and we are going to characterize all 3- dimensional Lorentzian Ricci solitons associated to the Bott connection on Lie groups. Since, Lie groups are parallizable, hence the Levi-Civita connection together with the Bott connection seems to be the most natural affine connections on Lie groups. In fact, we have an affine-metric geometry on three dimensional Lorentzian Lie groups and the soliton equation describes an intrinsic relation between our metric and affine geometries.

## 2. Ricci tensor associated to Bott connection

In the rest of this paper, $\{G_i\}_{i=1}^7$ denote the connected, simply connected 3-dimensional Lie groups equipped left-invariant Lorentzian metrics and $\{\mathfrak{g}_i\}_{i=1}^7$ as their Lie algebras (see [4]). Let $\nabla$ be the Levi-Civita connection of $G_i$. Recall the definition of the Bott connection $\nabla^B$ on a parallelizable pseudo-Riemannian manifold $(M, g)$ with the Levi-Civita connection $\nabla$, whose tangent bundle $TM = \mathrm{span}\{e_1, e_2, e_3\}$. Take the distribution $D = \mathrm{span}\{e_1, e_2\}$ and $D^\perp = \mathrm{span}\{e_3\}$, then the Bott connection $\nabla^B$ is defined as follows: (see [1, 5])

$$\nabla^B_X Y = \begin{cases} \pi_D(\nabla_X Y), & X, Y \in \Gamma^\infty(D), \\ \pi_D([X, Y]), & X \in \Gamma^\infty(D^\perp), Y \in \Gamma^\infty(D), \\ \pi_{D^\perp}([X, Y]), & X \in \Gamma^\infty(D), Y \in \Gamma^\infty(D^\perp), \\ \pi_{D^\perp}(\nabla_X Y), & X, Y \in \Gamma^\infty(D^\perp), \end{cases}$$

where $\pi_D$ (resp. $\pi_{D^\perp}$) is the projection on $D$ (resp. $D^\perp$). The Riemannian curvature tensor of $\nabla^B$ which we denote it by $R^B$ is given by

$$R^B(X,Y)(Z) = \nabla^B_X \nabla^B_Y Z - \nabla^B_Y \nabla^B_X Z - \nabla^B_{[X,Y]} Z.$$

Now, by means of the metric tensor $g$ on $G_i$, we can define the Ricci curvature tensor of $(G_i, g)$ associated to the Bott connection $\nabla^B$ as

$$\mathrm{Ric}^B(X,Y) = \frac{B(X,Y) + B(Y,X)}{2},$$

where

$$B(X,Y) = g(R^B(X,e_3)Y, e_3) - g(R^B(X,e_2)Y, e_2) - g(R^B(X,e_1)Y, e_1).$$

Also, we define the Lie derivative of the metric $g$ associated to Bott connection as follows

$$\left(\mathcal{L}^B_V g\right)(X,Y) := g(\nabla^B_X V, Y) + g(X, \nabla^B_Y V).$$

## 3. Generalized Lorentz Ricci solitons with respect to Bott connection

In this section, we classify three dimensional Lorentz Lie groups associated to Bott connection.

### *3.1. Generalized Lorentz Ricci soliton on $G_1$*

By [4], we have the following Lie algebra of $G_1$ satisfies

$$\begin{aligned}
[e_1, e_2] &= \alpha e_1 - \beta e_3,\\
[e_1, e_3] &= -\alpha e_1 - \beta e_2,\\
[e_2, e_3] &= \beta e_1 + \alpha e_2 + \alpha e_3,
\end{aligned}$$

where $\alpha \neq 0$ and $\{e_i\}_{i=1}^3$ pseudo-orthonormal basis, with $e_3$ is timelike. The Bott connection $\nabla^B$ of $G_1$ is given by

$$\nabla^B_{e_i} e_j = \begin{pmatrix} -\alpha e_2 & \alpha e_1 & 0 \\ 0 & 0 & \alpha e_3 \\ \alpha e_1 + \beta e_2 & -\beta e_1 - \alpha e_2 & 0 \end{pmatrix},$$

and the Ricci curvature of the Bott connection $\nabla^B$ of $(G_1, g)$ is determined by

$$\mathrm{Ric}^B(e_i, e_j) = \begin{pmatrix} -(\alpha^2+\beta^2) & \alpha\beta & -\frac{\alpha\beta}{2} \\ \alpha\beta & -(\alpha^2+\beta^2) & \frac{\alpha^2}{2} \\ -\frac{\alpha\beta}{2} & \frac{\alpha^2}{2} & 0 \end{pmatrix}.$$

Let $X = X^i e_i$ be a left-invariant vector field. By definition of $\mathcal{L}^B_X g$, we have

$$(\mathcal{L}^B_X g)(e_i, e_j) = \begin{pmatrix} 2\alpha X^2 & -\alpha X^1 & \alpha X^1 - \beta X^2 \\ -\alpha X^1 & 0 & \beta X^1 - \alpha(X^3 + X^2) \\ \alpha X^1 - \beta X^2 & \beta X^1 - \alpha(X^3 + X^2) & 0 \end{pmatrix}.$$

According to th definition of $X^\flat$, we get

$$X^\flat(e_1) = X^1, \quad X^\flat(e_2) = X^2, \quad X^\flat(e_3) = -X^3.$$

The corresponding generalized Lorentz Ricci soliton equation (2), concludes the following system of algebraic equations

$$\begin{cases} -a(\alpha^2+\beta^2)+b\alpha X^2+c(X^1)^2=\lambda, \\ 2a\alpha\beta-b\alpha X^1+2cX^1X^2=0, \\ -a(\alpha^2+\beta^2)+c(X^2)^2=\lambda, \\ a\alpha^2-b\alpha(X^3+X^2)+b\beta X^1-2cX^2X^3=0, \\ b\alpha X^1-b\beta X^2-a\alpha\beta-2cX^1X^3=0, \\ c(X^3)^2=-\lambda. \end{cases} \tag{3}$$

The second equation indicates $X^1(2cX^2-b\alpha)=-2a\alpha\beta$, so $2cX^2-b\alpha=0$ or $X^1=-\dfrac{2a\alpha\beta}{2cX^2-b\alpha}$.

**Case 1**: Let us assume $2cX^2-b\alpha=0$ (equivalently, $X^2=\dfrac{b\alpha}{2c}$) and $\beta=0$, then the third equation of (3) implies that

$$\lambda=\frac{b^2\alpha^2}{4c}-a\alpha^2.$$

Using this equality in the first equation, we obtain $c(X^1)^2=-\dfrac{b^2\alpha^2}{4c}$, hence $X^1=0$. Since $\alpha\neq 0$, so $b=0$. In this case, $X^2$ vanishes and $\lambda=-a\alpha^2$. But, the fourth equation can be rewritten as $a\alpha^2=0$ which means $a=0$. So, we must have $X^1=X^2=X^3=a=\lambda=0$.

Now, let $X^2=\dfrac{b\alpha}{2c}$ and $\beta\neq 0$, then the second equation shows $a=0$. Comparing the third and the last equations leads $\lambda=0$. Therefore $X^1=X^2=X^3=0$.

**Case 2**: Let's now consider the case in which $X^1=-\dfrac{2a\alpha\beta}{2cX^2-b\alpha}$. The first and the third equations lead us to

$$(X^1)^2-(X^2)^2=-\frac{b\alpha X^2}{c},$$

substituting $X^1=-\dfrac{2a\alpha\beta}{2cX^2-b\alpha}$ in the above equation, we obtain

$$4\alpha^2a^2\beta^2c-c(X^2)^2(2cX^2-b\alpha)^2+b\alpha X^2(2cX^2-b\alpha)^2=0,$$

and by solving this equation, we have

$$X^2=\frac{1}{4}\frac{2\alpha bc\pm\sqrt{2c^2b^2\alpha^2+2c^2\alpha\sqrt{b^2\alpha^2+64c^2a^2\beta^2}}}{c^2}.$$

Using the third equation, we compute

$$\lambda=\Big(\frac{1}{4}\frac{2\alpha bc\pm\sqrt{2c^2b^2\alpha^2+2c^2\alpha\sqrt{b^2\alpha^2+64c^2a^2\beta^2}}}{c^2}\Big)^2-a(\alpha^2+\beta^2).$$

In the ray of the last equation, we arrive at

$$X^3=\pm\sqrt{\frac{a(\alpha^2+\beta^2)}{c}-\frac{1}{c}\Big(\frac{1}{4}\frac{2\alpha bc\pm\sqrt{2c^2b^2\alpha^2+2c^2\alpha\sqrt{b^2\alpha^2+64c^2a^2\beta^2}}}{c^2}\Big)^2}.$$

Now, the fourth and the fifth equations in (3) provide conditions that our parameters $a, b, c, \alpha$, and $\beta$ have to satisfy them. Thus, we have the following theorem:

**Theorem 3.1.** $(G_1, g)$ *admits non-trivial, non-steady generalized Lorentzian soliton with respect to the Bott connection.*

### 3.2. Generalized Lorentz Ricci soliton on $G_2$

By [4], we have the following Lie algebra of $G_2$ satisfies

$$\begin{aligned}
&[e_1, e_2] = \gamma e_2 - \beta e_3,\\
&[e_1, e_3] = -\beta e_2 - \gamma e_3,\\
&[e_2, e_3] = \alpha e_1,
\end{aligned}$$

where $\gamma \neq 0$ and $\{e_i\}_{i=1}^3$ pseudo-orthonormal basis, with $e_3$ is timelike. The Bott connection $\nabla^B$ of $G_2$ is given by

$$\nabla^B_{e_i} e_j = \begin{pmatrix} 0 & 0 & -\gamma e_3 \\ -\gamma e_2 & \gamma e_1 & 0 \\ \beta e_2 & -\alpha e_1 & 0 \end{pmatrix},$$

and the Ricci curvature of the Bott connection $\nabla^B$ of $(G_2, g)$ is obtained by

$$\mathrm{Ric}^B(e_i, e_j) = \begin{pmatrix} -(\beta^2+\gamma^2) & 0 & 0 \\ 0 & -(\gamma^2+\alpha\beta) & -\frac{\alpha\gamma}{2} \\ 0 & -\frac{\alpha\gamma}{2} & 0 \end{pmatrix}.$$

Let $X = X^i e_i$ be a left-invariant vector field, then we get

$$(\mathcal{L}^B_X g)(e_i, e_j) = \begin{pmatrix} 0 & \gamma X^2 & \gamma X^3 - \alpha X^2 \\ \gamma X^2 & -2\gamma X^1 & \beta X^1 \\ \gamma X^3 - \alpha X^2 & \beta X^1 & 0 \end{pmatrix}.$$

By definition of $X^\flat$, we have

$$X^\flat(e_1) = X^1, \quad X^\flat(e_2) = X^2, \quad X^\flat(e_3) = -X^3.$$

The equation (2), implies that the following system of algebraic equations

$$\begin{cases} -a(\beta^2+\gamma^2) + c(X^1)^2 = \lambda, \\ b\gamma X^2 + 2cX^1X^2 = 0, \\ b\gamma X^3 - b\alpha X^2 - 2cX^1X^3 = 0, \\ -a(\gamma^2+\alpha\beta) - b\gamma X^1 + c(X^2)^2 = \lambda, \\ -a\alpha\gamma + b\beta X^1 - 2cX^2X^3 = 0, \\ c(X^3)^2 = -\lambda. \end{cases} \tag{4}$$

The second equation yields $X^2 = 0$ or $X^1 = -\frac{b\gamma}{2c}$.

**Case 1**: Let us assume $X^2 = 0$, then the above system of equations reduces to

$$\begin{cases} -a(\beta^2+\gamma^2) + c(X^1)^2 = \lambda, \\ b\gamma X^3 - 2cX^1X^3 = 0, \\ -a(\gamma^2+\alpha\beta) - b\gamma X^1 = \lambda, \\ -a\alpha\gamma + b\beta X^1 = 0, \\ c(X^3)^2 = -\lambda. \end{cases} \tag{5}$$

In this case, if $X^3 = 0$, then $\lambda = 0$ and we have

$$\begin{cases} -a(\beta^2+\gamma^2) + c(X^1)^2 = 0, \\ -a(\gamma^2+\alpha\beta) - b\gamma X^1 = 0, \\ -a\alpha\gamma + b\beta X^1 = 0. \end{cases}$$

Hence, we deduce that $X^1 = \pm\sqrt{\dfrac{a(\beta^2+\gamma^2)}{c}}$ and the equations $-a(\gamma^2+\alpha\beta) - b\gamma X^1 = 0$ and $-a\alpha\gamma + b\beta X^1 = 0$ provide conditions that our parameters must satisfy.

Now, we consider the case in which $X^3 \neq 0$, then the second equation in the system (5) leads to $X^1 = \dfrac{b\gamma}{2c}$ and

$$\begin{aligned}
\lambda &= \frac{b^2\gamma^2}{2c} - a(\beta^2+\gamma^2),\\
X^3 &= \pm\sqrt{\frac{ac(\beta^2+\gamma^2) - b^2\gamma^2}{2c^2}},\\
&ac\beta(\alpha-\beta) + b^2\gamma^2 = 0,\\
&b^2\beta\gamma - 2ac\alpha\gamma = 0.
\end{aligned}$$

**Case 2**: If $X^1 = -\dfrac{b\gamma}{2c}$, then $\lambda = \dfrac{b^2\gamma^2}{4c} - a(\beta^2+\gamma^2)$. Now the system (4) can be rewritten as

$$\begin{cases}
2b\gamma X^3 - b\alpha X^2 = 0,\\
-a(\gamma^2+\alpha\beta) + \dfrac{b^2\gamma^2}{2c} + c(X^2)^2 = \dfrac{b^2\gamma^2}{4c} - a(\beta^2+\gamma^2),\\
a\alpha\gamma + \dfrac{b^2\beta\gamma}{2c} + 2cX^2X^3 = 0,\\
c(X^3)^2 = a(\beta^2+\gamma^2) - \dfrac{b^2\gamma^2}{4c}.
\end{cases} \tag{6}$$

Using the second and the fourth equations, we obtain $X^2$ and $X^3$. In fact,

$$X^2 = \pm\frac{1}{2|c|}\sqrt{4ac(\alpha\beta-\beta^2) - b^2\gamma^2},$$

and

$$X^3 = \pm\frac{1}{2|c|}\sqrt{4ac(\beta^2+\gamma^2) - b^2\gamma^2}.$$

Then the first and the third equations in (6) provide conditions that the parameters have to satisfy them. Therefore, we have the following theorem:

**Theorem 3.2.** *$(G_2, g, X, \lambda)$ is a generalized Lorentzian soliton associated to the Bott connection if and only if one of the following cases hold:*

i) $X^1 = \pm\sqrt{\dfrac{a(\beta^2+\gamma^2)}{c}}$, $X^2 = X^3 = \lambda = 0$, *such that* $a(\gamma^2+\alpha\beta) + b\gamma X^1 = 0$ *and* $-a\alpha\gamma b\beta X^1 - a\alpha\gamma = 0$,

ii) $X^1 = \dfrac{b\gamma}{2c}$, $X^2 = 0$, $X^3 = \pm\sqrt{\dfrac{ac(\beta^2+\gamma^2) - b^2\gamma^2}{2c^2}}$, $\lambda = \dfrac{b^2\gamma^2}{2c} - a(\beta^2+\gamma^2)$,
*such that* $ac\beta(\alpha-\beta) + b^2\gamma^2 = 0$ *and* $b^2\beta\gamma - 2ac\alpha\gamma = 0$,

iii) $X^1 = -\dfrac{b\gamma}{2c}$, $X^2 = \pm\dfrac{1}{2|c|}\sqrt{4ac(\alpha\beta-\beta^2) - b^2\gamma^2}$, $X^3 = \pm\dfrac{1}{2|c|}\sqrt{4ac(\beta^2+\gamma^2) - b^2\gamma^2}$, $\lambda = \dfrac{b^2\gamma^2}{4c} - a(\beta^2+\gamma^2)$,
*such that* $2b\gamma X^3 - b\alpha X^2 = 0$ *and* $a\alpha\gamma + \dfrac{b^2\beta\gamma}{2c} + 2cX^2X^3 = 0$.

### *3.3. Generalized Lorentz Ricci soliton on $G_3$*

By [4], we have the following Lie algebra of $G_3$ satisfies

$$\begin{aligned}
[e_1, e_2] &= -\gamma e_3,\\
[e_1, e_3] &= -\beta e_2,\\
[e_2, e_3] &= \alpha e_1,
\end{aligned}$$

where $\{e_i\}_{i=1}^3$ pseudo-orthonormal basis, with $e_3$ is timelike. The Bott connection $\nabla^B$ of $G_3$ is given by

$$\nabla^B_{e_i} e_j = \begin{pmatrix} 0 & 0 & -\gamma e_3 \\ 0 & 0 & 0 \\ \beta e_2 & -\alpha e_1 & 0 \end{pmatrix},$$

and the Ricci curvature of the Bott connection $\nabla^B$ of $(G_3, g)$ is presented by

$$\mathrm{Ric}^B(e_i, e_j) = \begin{pmatrix} -\beta\gamma & 0 & 0 \\ 0 & -\gamma\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $X = X^i e_i$ be a left-invariant vector field. We get

$$(\mathcal{L}^B_X g)(e_i, e_j) = \begin{pmatrix} 0 & 0 & -\alpha X^2 \\ 0 & 0 & \beta X^1 \\ -\alpha X^2 & \beta X^1 & 0 \end{pmatrix},$$

and

$$X^\flat(e_1) = X^1, \quad X^\flat(e_2) = X^2, \quad X^\flat(e_3) = -X^3.$$

On Lie group $G_3$, generalized Lorentz Ricci soliton equation (2) yields

$$\begin{cases} c(X^1)^2 - a\beta\gamma = \lambda, \\ cX^1X^2 = 0, \\ 2cX^1X^3 + b\alpha X^2 = 0, \\ a\gamma\alpha - c(X^2)^2 = -\lambda, \\ b\beta X^1 - 2cX^2X^3 = 0, \\ c(X^3)^2 = -\lambda. \end{cases}$$

The second equation leads to $X^1 = 0$ or $X^2 = 0$.
**Case 1**: Let $X^1 = 0$, then we have

$$\begin{cases} -a\beta\gamma = \lambda, \\ b\alpha X^2 = 0, \\ a\gamma\alpha - c(X^2)^2 = -\lambda, \\ -2cX^2X^3 = 0, \\ c(X^3)^2 = -\lambda. \end{cases}$$

If $X^2 = 0$, then $a\gamma(\beta - \alpha) = 0$ and $X^3 = \pm\sqrt{\frac{a\beta\gamma}{c}}$. Else if $X^2 \neq 0$, then $X^3 = \lambda = 0$, $X^2 = \pm\sqrt{\frac{a\gamma(\alpha-\beta)}{c}}$, and $b\alpha = 0$.
**Case 2**: Suppose that $X^2 = 0$. Then, we have

$$\begin{cases} c(X^1)^2 - a\beta\gamma = \lambda, \\ 2cX^1X^3 = 0, \\ a\gamma\alpha = -\lambda, \\ b\beta X^1 = 0, \\ c(X^3)^2 = -\lambda. \end{cases}$$

The second equation proposes that $X^1 = 0$ or $X^3 = 0$. The case $X^2 = X^1 = 0$ has mentioned before, so we assume $X^1 \neq 0$, then $X^3 = \lambda = 0$, $b = \alpha = 0$, and $X^1 = \pm\sqrt{\frac{a\beta\gamma}{c}}$.
Therefore, we have the following theorem:

**Theorem 3.3.** $(G_3, g_2, X, \lambda)$ *is a generalized Lorentzian soliton associated to the Bott connection if and only if one of the following cases hold:*

i) $X^1 = X^2 = 0,\ X^3 = \pm\sqrt{\dfrac{a\beta\gamma}{c}},\ \lambda = -a\beta\gamma,$

ii) $X^1 = 0, X^2 = \pm\sqrt{\dfrac{a\gamma(\alpha-\beta)}{c}},\ X^3 = \lambda = 0,$ *such that* $b\alpha = 0,$

iii) $X^1 = \pm\sqrt{\dfrac{a\beta\gamma}{c}},\ X^2 = X^3 = \lambda = b = \alpha = 0.$

### *3.4. Generalized Lorentz Ricci soliton on $G_4$*

By [4], we have the following Lie algebra of $G_4$ satisfies

$$\begin{aligned} &[e_1, e_2] = -e_2 + (2\eta - \beta)e_3, \qquad \eta = \pm 1,\\ &[e_1, e_3] = e_3 - \beta e_2,\\ &[e_2, e_3] = \alpha e_1, \end{aligned}$$

where $\{e_i\}_{i=1}^3$ pseudo-orthonormal basis, with $e_3$ is timelike. The Bott connection $\nabla^B$ of $G_4$ is determined by

$$\nabla^B_{e_i} e_j = \begin{pmatrix} 0 & 0 & e_3 \\ e_2 & -e_1 & 0 \\ \beta e_2 & -\alpha e_1 & 0 \end{pmatrix},$$

and the Ricci curvature of the Bott connection $\nabla^B$ of $(G_4, g)$ is obtained by

$$\mathrm{Ric}^B(e_i, e_j) = \begin{pmatrix} -(\beta-\eta)^2 & 0 & 0 \\ 0 & 2\alpha\eta - \alpha\beta - 1 & \dfrac{\alpha}{2} \\ 0 & \dfrac{\alpha}{2} & 0 \end{pmatrix}.$$

If $X = X^i e_i$ is a left-invariant vector field then

$$(\mathcal{L}^B_X g)(e_i, e_j) = \begin{pmatrix} 0 & -X^2 & -\alpha X^2 - X^3 \\ -X^2 & 2X^1 & \beta X^1 \\ -\alpha X^2 - X^3 & \beta X^1 & 0 \end{pmatrix},$$

and

$$X^\flat(e_1) = X^1, \quad X^\flat(e_2) = X^2, \quad X^\flat(e_3) = -X^3.$$

By virtue of the generalized Lorentz Ricci soliton equation (2), we have

$$\begin{cases} -a(\beta-\eta)^2 + c(X^1)^2 = \lambda, \\ 2cX^1X^2 - bX^2 = 0, \\ b(\alpha X^2 + X^3) + 2cX^1X^3 = 0, \\ a(2\alpha\eta - \alpha\beta - 1) + bX^1 + c(X^2)^2 = \lambda, \\ a\alpha + b\beta X^1 - 2cX^2X^3 = 0, \\ c(X^3)^2 = -\lambda. \end{cases}$$

According to the second equation one can deduces that $X^2 = 0$ or $X^1 = \dfrac{b}{2c}$.

**Case 1**: When $X^2 = 0$, we have the following system of algebraic equations

$$\begin{cases} -a(\beta-\eta)^2 + c(X^1)^2 = \lambda, \\ bX^3 + 2cX^1X^3 = 0, \\ a(2\alpha\eta - \alpha\beta - 1) + bX^1 = \lambda, \\ a\alpha + b\beta X^1 = 0, \\ c(X^3)^2 = -\lambda. \end{cases}$$

If $X^3 = 0$ then $\lambda = 0$ and

$$\begin{cases} -a(\beta-\eta)^2 + c(X^1)^2 = 0, \\ a(2\alpha\eta - \alpha\beta - 1) + bX^1 = 0, \\ a\alpha + b\beta X^1 = 0. \end{cases}$$

Now, the first equation of the last above system gives $X^1 = \pm|\beta - \eta|\sqrt{\dfrac{a}{c}}$ and the other equations provide some conditions on the parameters.
If $X^3 \neq 0$ then $X^1 = -\dfrac{b}{2c}$. Hence, we obtain

$$\lambda = \frac{b^2}{4c} - a(\beta-\eta)^2,$$

and $X^3 = \pm\dfrac{1}{2|c|}\sqrt{4ac(\beta-\eta)^2 - b^2}$. In this case, our parameters must satisfy the following equations

$$a(2\alpha\eta - \alpha\beta - 1) + bX^1 = \lambda, \quad a\alpha + b\beta X^1 = 0.$$

**Case 2**: Considering the case $X^1 = \dfrac{b}{2c}$, we obtain

$$\begin{cases} -a(\beta-\eta)^2 + \dfrac{b^2}{4c} = \lambda, \\ b\alpha X^2 + 2bX^3 = 0, \\ a(2\alpha\eta - \alpha\beta - 1) + \dfrac{b^2}{2c} + c(X^2)^2 = \lambda, \\ a\alpha + \dfrac{\beta b^2}{2c} - 2cX^2X^3 = 0, \\ c(X^3)^2 = -\lambda. \end{cases}$$

Now, from the last above equation, we obtain

$$X^3 = \pm\frac{1}{2|c|}\sqrt{4ac(\beta-\eta)^2 - b^2},$$

and, the third equation implies

$$X^2 = \pm\frac{1}{2|c|}\sqrt{4ac\big(1 + \alpha\beta - 2\alpha\eta - (\beta-\eta)^2\big) - b^2}.$$

Also, the second and the fourth equations are our technical and give the conditions that our parameters have to obey them.

**Theorem 3.4.** *$(G_4, g, X, \lambda)$ is a generalized Lorentzian soliton associated to the Bott connection if and only if one of the following cases hold:*

i) $X^1 = \pm|\beta-\eta|\sqrt{\dfrac{a}{c}}, X^2 = X^3 = \lambda = 0,$ *such that* $a(2\alpha\eta - \alpha\beta - 1) + bX^1 = 0,$ *and* $a\alpha + b\beta X^1 = 0,$

ii) $X^1 = -\dfrac{b}{2c}, \quad X^2 = 0, X^3 = \pm\dfrac{1}{2|c|}\sqrt{4ac(\beta-\eta)^2 - b^2}, \quad \lambda = \dfrac{b^2}{4c} - a(\beta-\eta)^2,$

*such that* $bX^1 + a(2\alpha\eta - \alpha\beta - 1) = \lambda,$ *and* $a\alpha + b\beta X^1 = 0,$

iii) $X^1 = \dfrac{b}{2c}, \quad X^2 = \pm\dfrac{1}{2|c|}\sqrt{4ac\big(1 + \alpha\beta - 2\alpha\eta - (\beta-\eta)^2\big) - b^2}, \quad X^3 = \pm\dfrac{1}{2|c|}\sqrt{4ac(\beta-\eta)^2 - b^2},$

$\lambda = -\dfrac{1}{4}(4ac(\beta-\eta)^2 - b^2),$ *such that* $b\alpha X^2 + 2bX^3 = 0,$ *and* $a\alpha + \dfrac{\beta b^2}{2c} - 2cX^2X^3 = 0.$

*3.5. Generalized Lorentz Ricci soliton on $G_5$*

From [4], we have the following Lie algebra of $G_5$ satisfies

$$\begin{aligned}
&[e_1, e_2] = 0,\\
&[e_1, e_3] = \alpha e_1 + \beta e_2,\\
&[e_2, e_3] = \gamma e_1 + \delta e_2, \qquad \alpha + \delta \neq 0, \qquad \alpha\gamma + \beta\delta = 0,
\end{aligned}$$

where $\{e_i\}_{i=1}^{3}$ pseudo-orthonormal basis, with $e_3$ is timelike. The Bott connection $\nabla^B$ of $G_5$ is obtained by

$$\nabla^B_{e_i} e_j = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\alpha e_1 - \beta e_2 & -\gamma e_1 - \delta e_2 & 0 \end{pmatrix},$$

and the Riemannian curvature of $\nabla^B$ is identically zero, so $G_5$ is Ricci flat with respect to Bott connection. If $X = X^i e_i$ is a left-invariant vector field then

$$(\mathcal{L}^B_X g)(e_i, e_j) = \begin{pmatrix} 0 & -X^2 & -\alpha X^1 - \gamma X^2 \\ -X^2 & 2X^1 & \beta X^1 - \delta X^2 \\ -\alpha X^1 - \gamma X^2 & \beta X^1 - \delta X^2 & 0 \end{pmatrix},$$

and by definition of $X^\flat$, we have

$$X^\flat(e_1) = X^1, \quad X^\flat(e_2) = X^2, \quad X^\flat(e_3) = -X^3.$$

Using the generalized Lorentz Ricci soliton equation (2), we obtain

$$\begin{cases} c(X^1)^2 = \lambda, \\ bX^1 + c(X^2)^2 = \lambda, \\ c(X^3)^2 = -\lambda, \\ 2cX^1X^2 - bX^2 = 0, \\ b(\alpha X^1 + \gamma X^2) + 2cX^1X^3 = 0, \\ b(\beta X^1 - \delta X^2) - 2cX^2X^3 = 0. \end{cases} \tag{7}$$

The first and the third equations of the system (7) imply that $c((X^1)^2 + (X^2)^2) = 0$. Since $c \neq 0$ we conclude $X^1 = X^3 = \lambda = 0$. The second equation of the system (7) yields $X^2 = 0$.

**Theorem 3.5.** $(G_5, g)$ *does not admit any no-trivial generalized Lorentzian Ricci soliton associated to the Bott connection.*

*3.6. Lorentz Ricci soliton on $G_6$*

By [4], we have the following Lie algebra of $G_6$ satisfies

$$\begin{aligned}
&[e_1, e_2] = \alpha e_2 + \beta e_3,\\
&[e_1, e_3] = \gamma e_2 + \delta e_3,\\
&[e_2, e_3] = 0, \quad \alpha + \delta \neq 0, \quad \alpha\gamma - \beta\delta = 0,
\end{aligned}$$

where $\{e_i\}_{i=1}^{3}$ pseudo-orthonormal basis, with $e_3$ is timelike. The Bott connection $\nabla^B$ of $G_6$ is given by

$$\nabla^B_{e_i} e_j = \begin{pmatrix} 0 & 0 & \delta e_3 \\ -\alpha e_2 & \alpha e_1 & 0 \\ -\gamma e_2 & 0 & 0 \end{pmatrix},$$

and the Ricci curvature of the Bott connection $\nabla^B$ of $(G_6, g)$ is determined by

$$\mathrm{Ric}^B(e_i, e_j) = \begin{pmatrix} -(\alpha^2+\beta\gamma) & 0 & 0 \\ 0 & -\alpha^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For any left-invariant vector field $X = X^i e_i$, we have

$$(\mathcal{L}^B_X g)(e_i, e_j) = \begin{pmatrix} 0 & \alpha X^2 & -\delta X^3 \\ \alpha X^2 & -2\alpha X^1 & -\gamma X^1 \\ -\delta X^3 & -\gamma X^1 & 0 \end{pmatrix},$$

and

$$X^\flat(e_1) = X^1, \quad X^\flat(e_2) = X^2, \quad X^\flat(e_3) = -X^3.$$

Plugging the above equations in (2), we obtain

$$\begin{cases} c(X^1)^2 - a(\alpha^2+\beta\gamma) = \lambda, \\ b\alpha X^2 + 2cX^1X^2 = 0, \\ b\delta X^3 + 2cX^1X^3 = 0, \\ c(X^2)^2 - a\alpha^2 - b\alpha X^1 = \lambda, \\ b\gamma X^1 + 2cX^1X^3 = 0, \\ c(X^3)^2 = -\lambda. \end{cases}$$

The fifth equation indicates $X^1 = 0$ or $X^3 = -\dfrac{b\gamma}{2c}$. Hence, we consider two cases.

**Case 1**: When $X^1 = 0$, we have

$$\begin{cases} -a(\alpha^2+\beta\gamma) = \lambda, \\ b\alpha X^2 = 0, \\ b\delta X^3 = 0, \\ c(X^2)^2 - a\alpha^2 = \lambda, \\ c(X^3)^2 = -\lambda. \end{cases}$$

Thus, $X^2 = \pm\sqrt{-\dfrac{a\beta\gamma}{c}}$ and $X^3 = \pm\sqrt{\dfrac{a(\alpha^2+\beta\gamma)}{c}}$. In this case, our parameters have to satisfy $b\alpha X^2 = b\delta X^3 = 0$.

**Case 2**: Suppose that $X^3 = -\dfrac{b\gamma}{2c}$, then we have $\lambda = -\dfrac{b^2\gamma^2}{4c}$ and the following system of equations hold

$$\begin{cases} c(X^1)^2 - a(\alpha^2+\beta\gamma) = -\dfrac{b^2\gamma^2}{4c}, \\ b\alpha X^2 + 2cX^1X^2 = 0, \\ \dfrac{b^2\delta\gamma}{2c} + b\gamma X^1 = 0, \\ c(X^2)^2 - a\alpha^2 - b\alpha X^1 = -\dfrac{b^2\gamma^2}{4c}. \end{cases}$$

Therefore, from the first equation we have $X^1 = \pm\dfrac{1}{2|c|}\sqrt{4ac(\alpha^2+\beta\gamma) - b^2\gamma^2}$. Now, the last equation in the above system gives $X^2$ and the second and the third equations are our conditions on the structural parameters.

**Theorem 3.6.** *$(G_6, g, X, \lambda)$ is a generalized Lorentzian soliton associated to the Bott connection if and only if one of the following cases hold:*

i) $X^1 = 0, X^2 = \pm\sqrt{\frac{a\beta\gamma}{c}}, \quad X^3 = \pm\sqrt{\frac{a(\alpha^2+\beta\gamma)}{c}}, \quad \lambda = -a(\alpha^2+\beta\gamma),$ *such that* $b\alpha X^2 = b\gamma X^3 = 0,$

ii) $X^1 = \pm\frac{1}{2|c|}\sqrt{4ac(\alpha^2+\beta\gamma)-b^2\gamma^2}, \quad X^2 = \pm\sqrt{\frac{a\alpha^2}{c} \pm \frac{b\alpha}{2c|c|}\sqrt{4ac(\alpha^2+\beta\gamma)-b^2\gamma^2} - \frac{b^2\gamma^2}{4c^2}}, \quad X^3 = -\frac{b\gamma}{2c},$
$\lambda = -\frac{b^2\gamma^2}{4c}$ *such that* $\frac{b^2\delta\gamma}{2c} + b\gamma X^1 = 0.$

### *3.7. Generalized Lorentz Ricci soliton on $G_7$*

By [4], we have the following Lie algebra of $G_7$ satisfies

$$\begin{aligned}
&[e_1, e_2] = -\alpha e_1 - \beta e_2 - \beta e_3,\\
&[e_1, e_3] = \alpha e_1 + \beta e_2 + \beta e_3,\\
&[e_2, e_3] = \gamma e_1 + \delta e_2 + \delta e_3, \quad \alpha + \delta \neq 0, \quad \alpha\gamma = 0,
\end{aligned}$$

where $\{e_i\}_{i=1}^3$ pseudo-orthonormal basis, with $e_3$ is timelike. The Bott connection $\nabla^B$ of $G_7$ is given by

$$\nabla^B_{e_i} e_j = \begin{pmatrix} \alpha e_2 & -\alpha e_1 & \beta e_3 \\ \beta e_2 & -\beta e_1 & \delta e_3 \\ -\alpha e_1 - \beta e_2 & -\gamma e_1 - \delta e_2 & 0 \end{pmatrix},$$

and the Ricci curvature of the Bott connection $\nabla^B$ of $(G_7, g)$ is obtained by

$$\mathrm{Ric}^B(e_i, e_j) = \begin{pmatrix} -\alpha^2 & \frac{\beta(\delta-\alpha)}{2} & \delta(\alpha+\delta) \\ \frac{\beta(\delta-\alpha)}{2} & -(\alpha^2+\beta^2+\beta\gamma) & \delta^2 + \frac{\beta\gamma+\alpha\delta}{2} \\ \delta(\alpha+\delta) & \delta^2 + \frac{\beta\gamma+\alpha\delta}{2}\delta(\alpha+\delta) & 0 \end{pmatrix}.$$

Let $X = X^i e_i$ be a left-invariant vector field. By definition of Lie derivative associated to the Bott connection, we have

$$(\mathcal{L}^B_X g)(e_i, e_j) = \begin{pmatrix} -2\alpha X^2 & \alpha X^1 - \beta X^2 & -\beta X^3 - \alpha X^1 - \gamma X^2 \\ \alpha X^1 - \beta X^2 & 2\beta X^1 & -\beta X^1 - \delta X^2 - \delta X^3 \\ -\beta X^3 - \alpha X^1 - \gamma X^2 & -\beta X^1 - \delta X^2 - \delta X^3 & 0 \end{pmatrix}.$$

By definition of $X^\flat$, we get

$$X^\flat(e_1) = X^1, \quad X^\flat(e_2) = X^2, \quad X^\flat(e_3) = -X^3.$$

The generalized Lorentz Ricci soliton equation (2), implies that the following system of algebraic equations

$$\begin{cases}
c(X^1)^2 - a\alpha^2 - b\alpha X^2 = \lambda,\\
a\beta(\delta-\alpha) + b(\alpha X^1 - \beta X^2) + 2cX^1X^2 = 0,\\
2a\delta(\alpha+\delta) - b(\beta X^3 + \alpha X^1 + \gamma X^2) - 2cX^1X^3 = 0,\\
b\beta X^1 - a(\alpha^2+\beta^2+\beta\gamma) + c(X^2)^2 = \lambda,\\
a(2\delta^2+\beta\gamma+\alpha\delta) - b(\beta X^1 + \delta X^2 + \delta X^3) - 2cX^2X^3 = 0,\\
c(X^3)^2 = -\lambda.
\end{cases} \tag{8}$$

As we mentioned before, $\alpha\gamma = 0$, so $\alpha = 0$ or $\gamma = 0$. We consider the case $\alpha = 0$ and obtain

$$\begin{cases} c(X^1)^2 = \lambda, \\ a\beta\delta - b\beta X^2 + 2cX^1X^2 = 0, \\ 2a\delta^2 - b(\beta X^3 + \gamma X^2) - 2cX^1X^3 = 0, \\ b\beta X^1 - a(\beta^2 + \beta\gamma) + c(X^2)^2 = \lambda, \\ a(2\delta^2 + \beta\gamma) - b(\beta X^1 + \delta X^2 + \delta X^3) - 2cX^2X^3 = 0, \\ c(X^3)^2 = -\lambda. \end{cases} \tag{9}$$

The second and the last equations of the system (9) imply that $(X^1)^2 + (X^3)^2 = 0$. Then $X^1 = X^3 = \lambda = 0$ and the system (9) becomes

$$\begin{cases} a\beta\delta - b\beta X^2 = 0, \\ 2a\delta^2 - b\gamma X^2 = 0, \\ -a(\beta^2 + \beta\gamma) + c(X^2)^2 = 0, \\ a(2\delta^2 + \beta\gamma) - b\delta X^2 = 0. \end{cases}$$

If $b\beta = 0$ then $a = X^2 = 0$. If $b\beta \neq 0$ then $X^2 = \frac{a\delta}{b}$, $a(2\delta - \gamma) = 0$, $a\delta c - ab\beta(\beta + \gamma) = 0$, and $a\delta + 2\beta a = 0$. Now assume that $\alpha \neq 0$ and $\gamma = 0$. In this case, the system (8) reduces to

$$\begin{cases} c(X^1)^2 - a\alpha^2 - b\alpha X^2 = \lambda, \\ a\beta(\delta - \alpha) + b(\alpha X^1 - \beta X^2) + 2cX^1X^2 = 0, \\ 2a\delta(\alpha + \delta) - b(\beta X^3 + \alpha X^1) - 2cX^1X^3 = 0, \\ b\beta X^1 - a(\alpha^2 + \beta^2) + c(X^2)^2 = \lambda, \\ a(2\delta^2 + \alpha\delta) - b(\beta X^1 + \delta X^2 + \delta X^3) - 2cX^2X^3 = 0, \\ c(X^3)^2 = -\lambda. \end{cases}$$

If $b = 0$ then $X^1 = \pm\sqrt{\frac{\lambda + a\alpha^2}{c}}$, $X^2 = \pm\sqrt{\frac{\lambda + a(\alpha^2+\beta^2)}{c}}$, and $X^3 = \pm\sqrt{-\frac{\lambda}{c}}$ such that

$$\begin{aligned} a\beta(\delta - \alpha) \pm 2c\sqrt{\frac{(\lambda + a\alpha^2)(\lambda + a(\alpha^2 + \beta^2))}{c}} &= 0, \\ a\delta(\alpha + \delta) = \pm 2c\sqrt{\frac{-\lambda(\lambda + a\alpha^2)}{c}}&, \\ a\delta(2\delta + \alpha) = 2c\sqrt{\frac{-\lambda(\lambda + a(\alpha^2 + \beta^2))}{c}}&. \end{aligned}$$

**Theorem 3.7.** $(G_7, g)$ *admits non-trivial, non-steady generalized Lorentzian soliton with respect to the Bott connection.*

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