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Original Article

Almost Ricci soliton in Q^{m^*}

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ABSTRACT: In this paper, we will focus our attention on the structure of h-almost Ricci solitons on complex hyperbolic quadric. We will prove non-existence a contact real hypersurface in the complex hyperbolic quadric $Q^{m^*}, m \geq 3$, admitting the gradient almost Ricci soliton. Moreover, the gradient almost Ricci soliton function f is trivial.

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1. Introduction

In 1982, Hamilton introduced the notion of Ricci flows and Ricci solitons to find a canonical metric on a smooth manifold [10, 11]. They are natural generalizations of Einstein metrics. During the last two decades, the geometry of Ricci solitons has been the focus of attention of many mathematicians [3, 4, 5, 13]. The notion of f-almost Ricci soliton which develops naturally the notion of almost Ricci soliton has been introduced in [9]. Faraji and others obtained a complete classification of f-almost Ricci solitons with concurrent potential vector fields [7].

Gasqui and Goldschmidt presented various results concerning the geometry of the complex quadric Q_n of dimension $n \geq 3$ which are needed in the study of the infinitesimal rigidity of this space. They considered Q_n both as a complex hypersurface of the complex projective space CP^{n+1} and as a symmetric space [8]. The complex quadric Q^m is the set of oriented 2-dimensional planes in $\mathbb{R}^{m+2}$ or the set of real projective lines $\mathbb{R}P^1$ in a real projective space $\mathbb{R}P^{m+1}$ which can be regarded as a kind of real Grassmann manifold of compact type with rank 2 [14]. Shu introduced the notion of parallel Ricci tensor for real hypersurfaces in the complex quadric $Q^m = SO_{m+2}/SO_mSO_2$. According to the $\mathfrak{U}$-principal or the $\mathfrak{U}$-isotropic unit normal vector field N, he gived a complete classification of real hypersurfaces in $Q^m = SO_{m+2}/SO_mSO_2$ with parallel Ricci tensor [19]. Also, he classified real hypersurfaces with isometric Reeb flow in the complex hyperbolic quadrics $Q^{m^*} = SO^0_{2,m}/SO_mSO_2, m \geq 3$. He showed that m is even,

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say $m = 2k$, and any such hypersurface becomes an open part of a tube around a k-dimensional complex hyperbolic space $\mathbb{C}H^k$ which is embedded canonically in Q^{2k^*} as a totally geodesic complex submanifold or a horosphere whose center at infinity is $\mathfrak{U}$-isotropic singular [17].

Inspired and motivated by the above facts, In this paper, we will focus our attention on the structure of h-almost Ricci solitons on complex hyperbolic quadric. We will prove non-existence a contact real hypersurface in the complex hyperbolic quadric $Q^{m^*}, m \geq 3$, admitting the gradient almost Ricci soliton. Moreover, the gradient almost Ricci soliton function f is trivial.

2. Preliminaries and notations

In this section, we shall present some preliminaries which will be needed for the establishment of our desired results. Let M be a real hypersurface in a kahler manifold $\bar{M}$. The complex structure J on $\bar{M}$ induces locally an almost contact metric structure (ϕ, ξ, η, g) on M. In the context of contact geometry, the unit vector field ξ is often referred to as the Reeb vector field on M and its flow is known as the Reeb flow. The integral curves of ξ are geodesics in M if and only if ξ is a principal curvature vector of M everywhere. The tangent bundle TM of M splits orthogonally into $TM = \mathcal{C} \bigoplus \mathcal{F}$, where $C = ker(\eta)$ is the maximal complex subbundle of TM and $\mathcal{F} = \mathbb{R}\xi$. The structure tensor field ϕ restricted to $\mathcal{C}$ coincides with the complex structure J restricted to $\mathcal{C}$, and we have $\phi\xi = 0$ [1]. The complex quadric Q^m is a Kahler-Einstein manifold, which can be seen in several different ways, for example as a complex hypersurface of the complex projective space $\mathbb{C}P^{m+1}$, as the Grassmannian manifold of oriented 2-planes in R^{n+2} or as the homogeneous space

$$Q^m = \frac{SO_{m+2}}{SO_2 \times SO_m}.$$

The m-dimensional complex hyperbolic quadric Q^{m^*} is the non-compact dual of the m-dimensional complex quadric Q^m, i.e., the simply connected Riemannian symmetric space whose curvature tensor is the negative of the curvature tensor of Q^m.

Recall that a nonzero tangent vector $W \subset T_{[z]}Q^m$ is called singular if it is tangent to more than one maximal flat in Q^m. There are two types of singular tangent vectors for the complex quadric Q^m: 1. If there exists a conjugation $A \in \mathfrak{U}$ such that $W \in V(A)$, then W is singular. Such a singular tangent vector is called $\mathfrak{U}$-principal. 2. If there exist a conjugation $A \in \mathfrak{U}$ and orthonormal vectors $X, Y \in V(A) \subset T[z]Q^m$ such that $W/||W|| = (X + JY)/\sqrt{2}$, then W is singular. Such a singular tangent vector is called $\mathfrak{U}$-isotropic.

Let us denote by $\mathbb{C}_1^{m+2}$ an indefinite complex Euclidean space $\mathbb{C}^{m+2}$, on which the indefinite Hermitian product

$$H(z, \omega) = -z_1\bar{\omega}_1 + z_2\bar{\omega}_2 + ... + z_{n+2}\bar{\omega}_{n+2},$$

is negative definite. The homogeneneous quadratic equation $z_1^2 + ... + z_{m+1}^2 - z_{m+2}^2 = 0$ consists of the points in $\mathbb{C}_1^{m+2}$ defines a noncompact complex hyperbolic quadric $Q^{*m} = SO_{2,m}^0/SO_2SO_m$ which can be immersed in the $(m+1)$-dimensional in complex hyperbolic space $\mathbb{C}H^{m+1} = SU_{1,m+1}/S(U_{m+1}U_1)$. The complex hypersurface Q^{m^*} in $\mathbb{C}H^{m+1}$ is known as the m-dimensional complex hyperbolic quadric. The complex structure J on $\mathbb{C}H^{m+1}$ naturally induces a complex structure on Q^{m^*} which we will denote by J as well.

The complex hyperbolic quadric Q^{m^*} admits two important geometric structures, a complex conjugation structure A and a Kahler structure J, which anti-commute with each other, that is, $AJ = -JA$. Then for $m \geq 2$ the triple (Q^{m^*}, J, g) is a Hermitian symmetric space of non-compact type and its minimal sectional curvature is equal to -4. Here we note that the unit normal vector field N is said to be $\mathfrak{U}$-principal if N is invariant under the complex conjugation A, that is, $AN = N$.

Definition 2.1. *[18] Let M be a real hypersurface in the complex hyperbolic quadric Q^{m^*} and denote by (ϕ, ξ, η, g) the induced almost contact metric structure on M and by ∇ the induced Riemannian connection on M. Note that $\xi = -JN$, where N is a (local) unit normal vector field of M. The vector field ξ is known as the Reeb vector field of M. If the integral curves of ξ are geodesics in M, the hypersurface M is called a Hopf hypersurface.*

Suh proved that the Reeb flow on a real hypersurface in $G_2^*(Cm + 2)$ is isometric if and only if M is an open part of a tube around a totally geodesic $G_2^*(Cm + 1) \in G_2^*(Cm + 2)$ or a horosphere with singular normal JN [16]. He in [17] investigated this problem for $SO_{2,m}^0/SO^2SO^m$ with isometric Reeb flow. We stated the following theorem.

Theorem 2.2. *[17] Let M be a real hypersurface of the complex hyperbolic quadric Q^{m^*}, $m \geq 3$. The Reeb flow on M is isometric if and only if m is even, say $m = 2k$, and M is locally congruent to an open part of a tube around a totally geodesic $\mathbb{C}H^k \subset Q^{2k*}$ or a horosphere whose center at infinity is $\mathfrak{U}$-isotropic singular.*

In [2] Berndt and Suh carryed out a systematic study of contact hypersurfaces in kahler manifolds. They apply their results to the complex quadric $Q^n = SO_{n+2}/SO_nSO_2$ and its noncompact dual space $Q^{n^*} = SO^o_{n,2}/SO_nSO_2$ and obtained the following result:

Theorem 2.3. *[2, 20] Let M be a pseudo-anti commuting Hopf real hypersurfaces in the complex hyperbolic quadric $Q^{m^*}, m \geq 3$. Then M is locally congruent to one of the following:*
(1) a tube around a totally geodesic $\mathbb{C}H^k \subset Q^{2k*}$, where $m = 2k$,
(2) a horosphere whose center at infinity is $\mathfrak{U}$-isotropic singular,
(3) a tube around a totally geodesic Hermitian symmetric space $Q^{(m-1)^*}$ embedded in Q^{m^*},
(4) a horosphere in Q^{m^*} whose center at infinity is the equivalence class of an $\mathfrak{U}$-principal geodesic in Q^{m^*},
(5) a tube around the m-dimensional real hyperbolic space $\mathbb{R}H^m$ which is embedded in Q^{m^*} as a real space form.

Berndt and Suh [2] have given a complete classification for contact hypersurfaces M in Q^{m^*} as follows:

Theorem 2.4. *[2] Let M be a connected orientable real hypersurface with constant mean curvature in the complex hyperbolic quadric $Q^{m^*} = SO^0_{m,2}/SO_mSO_2, m \geq 3$. Then M is a contact hypersurface if and only if M is congruent to an open part of one of the following hypersurfaces in Q^{m^*}:*
(i) a tube of radius r around the Hermitian symmetric space $Q^{(m-1)^*}$ which is embedded in Q^{m^*} as a totally geodesic complex hypersurface,
(ii) a horosphere in Q^{m^*} whose center at infinity is the equivalence class of an $\mathfrak{U}$-principal geodesic in Q^{m^*},
(iii) a tube of radius r around the m-dimensional real hyperbolic space $\mathbb{R}H^m$ which is embedded in Q^{m^*} as a real space form of Q^{m^*}.

By using theorem 2.4, we have:

Lemma 2.5. *[21] Let M be a contact real hypersurface in the complex hyperbolic quadric Q^{m^*}. Then the Reeb function α and the non-vanishing principal curvature μ are respectively given by*

$$\alpha = \sqrt{2}\coth(\sqrt{2}r), \qquad \text{and} \qquad \mu = \sqrt{2}\tanh(\sqrt{2}r),$$

$$\alpha = \sqrt{2}, \qquad \text{and} \qquad \mu = 2,$$

and

$$\alpha = \sqrt{2}\tanh(\sqrt{2}r), \qquad \text{and} \qquad \mu = \sqrt{2}\coth\sqrt{2}r.$$

Suh obtained a classification for pseudo-Einstein Hopf real hypersurfaces in the complex hyperbolic Q^{m^*} as follows:

Theorem 2.6. *[20] There does not exist a Hopf–Ricci soliton (M, g, ξ, ρ) in the complex hyperbolic quadric Q^{m^*}, $m \geq 3$.*

Lemma 2.7. *[20] Let M be a Hopf hypersurface in the complex hyperbolic quadric Q^{m^*}, $m \geq 3$. Then we obtain*

$$(2S\phi S - \alpha(\phi S + S\phi) + 2\phi)X = 2\rho(X)(B\xi + \delta\xi) + 2g(X, B\xi + \delta\xi)\phi B\xi. \tag{1}$$

In the following, we present some concepts of Ricci solitons.

Definition 2.8. *[7] The Ricci flow is the equation*

$$\frac{\partial}{\partial t}g_{ij} = -2R_{ij},$$

evolving a Riemannian metric by its Ricci curvature [10]. It now occupies a central position as one of the key tools of geometry. A Riemannian manifold (M^m, g) is said to be a Ricci soliton if there exists a smooth vector field X on M^m such that

$$\mathcal{L}_X g + 2Ric = 2\lambda g, \tag{2}$$

where λ is a real constant, Ric and $\mathcal{L}_X$ stand for the Ricci tensor and Lie derivative operator, respectively.

We denote a Ricci soliton by (M^m, g, X, λ). The smooth vector field X mentioned above, is called a potential field for the Ricci soliton. A Ricci soliton (M^m, g, X, λ) is said to be steady, shrinking or expanding if $\lambda = 0$, $\lambda > 0$ or $\lambda < 0$, respectively. Also, a Ricci soliton (M^m, g, X, λ) is said to be gradient soliton if there exists a smooth function l on M such that $X = \nabla l$. In this case, l is called a potential function for the Ricci soliton and the equation (2) can be rewritten as follows

$$Ric + \nabla^2 l = \lambda g,$$

where $\nabla^2 l$ is the Hessian of l [6, 11]. Given a Ricci soliton, let Y_t be the time dependent vector field

$$Y_t = -\frac{1}{2\gamma(t)} X,$$

where γ is a smooth function respect to t and let ϕ_t be the flow generated by Y_t. If we set

$$g(t) = -2\gamma(t)\phi_t^* g,$$

then $g(t)$ satisfies the Ricci flow equation

$$\frac{\partial g(t)}{\partial t} = -2Ric(g(t)).$$

A Ricci soliton is a self-similar solution to the Ricci flow equation since it is obtained as a rescaling limit of a singularity [12, 15].

Definition 2.9. *A Riemannian manifold (M^m, g) is said to be an f-almost Ricci soliton if there exists a smooth vector field X on M^m and a smooth function $f : M^m \to \mathbb{R}$, such that*

$$f\mathcal{L}_X g + 2Ric = 2\lambda g, \tag{3}$$

where λ is a smooth function on M, Ric and $\mathcal{L}_X$ stand for the Ricci tensor and Lie derivative, respectively. In the case λ is constant we simply say that it is an f-Ricci soliton.

We will denote the f-almost Ricci soliton by (M^m, g, X, f, λ). All concepts related to Ricci soliton can be defined for f-almost Ricci soliton, accordingly. An f-almost Ricci soliton is said to be shirinking, steady or expanding if λ is positive, zero or negative, respectively. Also, if $X = \nabla l$ for a smooth function l, then we say $(M^m, g, \nabla l, f, \lambda)$ is a gradient f-almost Ricci soliton with potential function l. In such cases the equation (3) can be rewritten as follows

$$Ric + f\nabla^2 l = \lambda g,$$

where $\nabla^2 l$ denotes the Hessian of l. Note that when the potential function l be a real constant then, the underlying Ricci soltion is simply Einstein metric [9].

3. Main Results

In this section, we announce our main results and theorems.

Lemma 3.1. *Let (M^m, g, X, f, λ) be a Holf-f-almost Ricci soliton real hypersurface with the potential Reeb field ξ in the complex hyperbolic Q^{m^*}.*
I)If N is $\mathfrak{U}$-principal, then

$$\lambda = -2(m-1) + h\alpha - \alpha^2.$$

II) If N is $\mathfrak{U}$-isotropic, then

$$\lambda = -2(m-2) + h\alpha - \alpha^2.$$

Proof. Let $A \in \mathfrak{U}$ such that $AN = N$. Then we have $A\xi = -\xi$ and

$$Y\alpha = (\xi\alpha)\eta(Y),$$

for any vector field Y on M. Since $grad^M \alpha = (\xi\alpha)\xi$, we obtain

$$(Hess^M \alpha)(X, Y) = g(\nabla_X grad^M \alpha, Y) = X(\xi\alpha)\eta(Y) + (\xi\alpha)g(\phi SX, Y).$$

Also, because $Hess^M\alpha$ is a symmetric bilinear form, the following equation obtain

$$(\xi\alpha)g((S\phi+\phi S)X,Y)=0,$$

for all vector fields X,Y on M. Now let us consider an open subset $\mathcal{U}=\{p\in M|(\xi\alpha)_p\neq 0\}$. Then $(S\phi+\phi S)=0$ on $\mathcal{U}$. Now we continue our discussion on this open subset $\mathcal{U}$. From equation (1), $AN=N$, $A\xi=-\xi$ and the condition $(S\phi+\phi S)=0$ imply

$$S^2\phi X-\phi X=0,$$

replacing X by ϕX, we have

$$S^2X=X+(\alpha^2-1)\eta(X)\xi. \tag{4}$$

By using $X\alpha=(\xi\alpha)\eta(X)$ and differentating (4), we give

$$\begin{aligned}(\nabla_XS)SY-S(\nabla_XS)Y&=2\alpha(X\alpha)\eta(Y)\xi+(\alpha^2-1)[g(\nabla_X\xi,Y)\xi+\eta(Y)\nabla_X\xi]\\&=2\alpha(\xi\alpha)\eta(X)\eta(Y)\xi+(\alpha^2-1)[g(\phi SX,Y)\xi+\eta(Y)\phi SX].\end{aligned} \tag{5}$$

If the unite normal N is $\mathfrak{U}$-principal and from (5), we obtain

$$Ric(X)=-(2m-1)X+2\eta(X)\xi+AX+hSX-S^2X.$$

Since (M,g,ξ,f,λ) is Hopf-f-almost Ricci soliton, and $A\xi=-\xi$ for the $\mathfrak{U}$-principal unite normal then we can write

$$\begin{aligned}\lambda&=\frac{f}{2}(\mathcal{L}_\xi g)(\xi,\xi)+Ric(\xi,\xi)\\&=g(Ric(\xi),\xi)\\&=-2(m-1)+h\alpha-\alpha^2.\end{aligned}$$

If the unite normal N is $\mathfrak{U}$-isotropic and from (5), we have

$$Ric(X)=-(2m-1)X+3\eta(X)\xi-g(AX,N)AN-g(AX,\xi)A\xi+hSX-S^2X.$$

On the other hand, (M,g,ξ,f,λ) is Hopf-f-almost Ricci soliton and $\mathfrak{U}$-isotropic, we get

$$\begin{aligned}\lambda&=\frac{f}{2}(\mathcal{L}_\xi g)(\xi,\xi)+Ric(\xi,\xi)\\&=g(Ric(\xi),\xi)\\&=-2(m-2)+h\alpha-\alpha^2.\end{aligned}$$

We obtain the desired result.

□

Theorem 3.2. *There does not exist a Hopf-f-almost Ricci soliton (M,g,f,ξ,ρ) in the complex hyperboic quadric $Q^{m^*}, m\geq 3$.*

Proof. Let (M,g,f,ξ,λ) be a Hopf-f-almost Ricci soliton in the complex hyperbolic quadric Q^{m^*}.
The first case: $\mathfrak{U}$-principal unit normal vector field N.
By Lemma 3.1 for the $\mathfrak{U}$-principal unit normal N, we obtain

$$[-1-(h\alpha-\alpha^2)]X+2\eta(X)\xi+AX+hSX-S^2X+\frac{1}{2}(\phi S-S\phi)X=0. \tag{6}$$

Since, the Hopf-f-almost Ricci soliton (M,g,f,ξ,λ) satisfies the condition of pseudo-anti commuting $Ric.\phi+\phi.Ric=\kappa\phi, \kappa=2\lambda$, then by (i) in Theorem 2.4 for $\mathfrak{U}$-principal unit normal N, a hypersurface M is locally congruent to a tube over a totally geodesic and totally complex submanifold $Q^{(m-1)^*}$ in Q^{m^*}, horosphere, and totally geodesic totally real submanifold $\mathbb{R}H^m$ in Q^{m^*}. Furthermore, λ, α and μ are respectively given by

$$\lambda=\frac{1}{\sqrt{2}}\tanh(\sqrt{2}r),\qquad \alpha=\sqrt{2}\coth(\sqrt{2}r),\qquad \textit{and}\qquad \mu=\sqrt{2}\tanh(\sqrt{2}r),$$

$$\lambda = \frac{1}{\sqrt{2}}, \qquad \alpha = \sqrt{2}, \qquad and \qquad \mu = \sqrt{2},$$

and

$$\lambda = \frac{1}{\sqrt{2}}\coth(\sqrt{2}r), \qquad \alpha = \sqrt{2}\tanh(\sqrt{2}r), \qquad and \qquad \mu = \sqrt{2}\coth(\sqrt{2}r).$$

Then in this case of N is $\mathfrak{U}$-principal, we consider $X \in T_\upsilon, \upsilon = 0$. Now we investigate the following three condition.
Condition 1: $X \in V(A) \bigcap T_z M, \quad z \in M$.
Then $AX = X$. Moreover, $S\phi X = \frac{2}{\alpha}\phi X$ for $X \in T_\upsilon$. By applying equation (6), we get a contradiction.
Condition 2: $X \in JV(A) \bigcap T_z M, \; z \in M$.
In this subcase $AX = -X, \quad SX = 0$, and $S\phi X = \frac{2}{\alpha}\phi X$. Together with equation (6), we obtain a contraction.
Condition 3: $X \in (V(A) \bigoplus J(V(X)) \bigcap T_z M, \; z \in M$.
At first, we consider $X = \frac{1}{\sqrt{2}}(Y + Z)$, where $Y \in V(A)$ and $Z \in JV(A)$ such that $Y \perp \phi Z$. Then $AX = \frac{1}{\sqrt{2}}(Y - Z)$ and $X \in T_\upsilon$, $\phi X \in T_\mu$, where $\upsilon = 0$ and $\mu = \frac{2}{\alpha}$. From equation (6), we get

$$[-1 - (h\alpha - \alpha^2)](Y + Z) + (Y - Z) - \frac{1}{\alpha}(\phi Y + \phi Z) = 0.$$

By using $g(\phi Z, Y) = 0$ and taking the inner product Y and Z, respectively, we have $h\alpha - \alpha^2 = -2$ and $h\alpha - \alpha^2 = 0$. We get a contradiction.
If $X = \frac{1}{\sqrt{2}}(Y + \phi Z)$, where $Y \in V(A)$ and $\phi Y \in JV(A)$, Then $AX = \frac{1}{2}(Y - \phi Y)$ and $S\phi X = \frac{2}{\alpha}(\phi Y - Y)$. Then putting these in equation (6) gives

$$[-1 - (h\alpha - \alpha^2)](Y + \phi Y) + (Y - \phi Y) - \frac{1}{\alpha}(\phi Y - Y) = 0.$$

By taking the inner product Y and ϕY respectively, we have $-\alpha^2 + h\alpha = \frac{1}{\alpha}$ and $-\alpha^2 + h\alpha = -2 - \frac{1}{\alpha}$. It follows from two equations that $\alpha = -1$. On the other hand, the Reeb function mentioned above $\alpha = \sqrt{2}\coth(\sqrt{2}r)$, $\alpha = \sqrt{2}$, $\alpha = \sqrt{2}\tanh(\sqrt{2}r)$ is all positive. we reach a contradiction.
According to the description provided, there do not exist any f-almost Ricci soliton real hypersurfaces in the complex hyperbolic Q^{m^*} with $\mathfrak{U}$-principal unite normal vector field.

The second case: $\mathfrak{U}$-isotropic unite normal vector field N.
By Theorem (2.3), Ricci soliton hypersurfacees in the complex quadric Q^{m^*} satisfy the condition of pseudo-anti commuting Ricci tensor, Then M is locally congruent to a tube over a totally complex hyperbolic space $\mathbb{C}H^k$ in Q^{2k^*} and the shape operator S of the pseudo-anti commuting Hopf hypersurface in Q^{m^*} can be obtained as follows

$$S = \begin{bmatrix} 2\coth 2r & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \coth r & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \coth r & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & \tanh r & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \tanh r \end{bmatrix}.$$

Since N is $\mathfrak{U}$-isotropic, we know that

$$\frac{f}{2}(\mathcal{L}_\xi g)(X, Y) + Ric(X, Y) = \lambda g(X, Y).$$

By Lemma 3.1, λ is given by

$$\lambda = -2(m - 1) + h\alpha - \alpha^2.$$

Then it becomes the following

$$\frac{1}{2}(\phi S - S\phi)X + (-3 - h\alpha + \alpha^2)X + 3\eta(X)\xi - g(AX, N)AN - g(A\xi, X)A\xi + hSX - S^2X = 0.$$

By putting $SX = \coth rX, S\phi X = \coth r\lambda X$, we have

$$-3 + h\coth r - \coth^2 r = h(\coth r + \tanh r) - (\coth r + \tanh r)^2,$$

for any X orthogonal to the vector fields ξ, $A\xi$ and AN.
Then this yields

$$\tanh^2 r - h\tanh r - 1 = 0,$$

where the trace h is given by $h = \alpha + 2(k-1)(\tanh r + \coth r) = (2k-1)(\tanh r + \coth r)$. Then we have

$$\tanh^2 r - 1 = h\tan r = (2k-1)(\coth r + \tanh r)\tanh r = 2k - 1 + (2k-1)\tan^2 r.$$

This implies $\tanh^2 r = -\frac{k}{k-1}$, which obtain a contradiction. So in second case we proved that there does not exist a Hopf-f-almost Ricci soliton (M, g, f, ξ, ρ) in the complex hyperboic quadric Q^{m^*}.
Then we give a complete proof of theorem. □

Theorem 3.3. *There dose not exist a real hypersurface with isometric Reeb flow in the complex hyperbolic quadric $Q^{m^*}, m \geq 3$, admitting gradient almost Ricci soliton.*

Proof. Let M is the complex hyperbolic quadric Q^{m^*} with isometric Reeb flow that it admits gradient almost Ricci soliton (M, Df, ψ, g), where Df denotes the gradient of the smooth function f on M. Then

$$\nabla_X Df + Ric(X) = \psi X, \tag{7}$$

where ψ is a smooth function on M. From the $\mathfrak{U}$-isotropic unit normal, it follows that $g(A\xi, \xi) = 0$, $g(AN, N) = 0$ and $g(A\xi, N) = 0$. So we have

$$Ric(X) = -(2m-1)X + 3\eta(X)\xi - g(AX, N)AN - g(AX, \xi)A\xi + hSX - S^2X.$$

Put $X = \xi$. Since M is Hopf and the properties of $\mathfrak{U}$-isotropic, we have the following

$$Ric(\xi) = k\xi,$$

where the constant k is given by

$$k = -2(m-2) + h\alpha - \alpha^2.$$

Since we have assumed that M has isometric Reeb flow, by taking the covariant derivative we obtain the following equations

$$(\nabla_\xi Ric)\xi = k\phi SX,$$

and

$$\begin{aligned}(\nabla_\xi Ric)X = &-g(X, \nabla_\xi(AN))AN - g(X, AN)\nabla_\xi(AN) - g(X, \nabla_\xi(A\xi))A\xi \\ &- g(X, A\xi)\nabla_\xi(A\xi) + h(\nabla_\xi S)X - (\nabla_\xi S^2)X.\end{aligned}$$

From (7) and together with the above two formulas, we obtain

$$\begin{aligned}R(\xi, Y)Df &= \nabla_\xi\nabla_Y Df - \nabla_Y\nabla_\xi Df - \nabla_{[\xi,Y]}Df \\ &= (\nabla_Y Ric)\xi - (\nabla_\xi Ric)Y + (\xi(\psi)Y - Y(\psi)\xi) \\ &= k\phi SY + g(Y, \nabla_\xi(AN))AN + g(Y, AN)\nabla_\xi(AN) \\ &+ g(Y, \nabla_\xi(A\xi))A\xi + g(Y, A\xi)\nabla_\xi(A\xi) \\ &- h(\nabla_\xi S)Y + (\nabla_\xi S^2)Y.\end{aligned} \tag{8}$$

By the equation of Gauss, since M is $\mathfrak{U}$-isotropic and the vector fields $A\xi$ and AN are tangent vector fields on M, it follows that

$$\begin{aligned}\nabla_X(A\xi) &= \bar{\nabla}_X(A\xi) - \sigma(X, A\xi) \\ &= [(\bar{\nabla}_X A)\xi + A\bar{\nabla}_X\xi] - \sigma(X, A\xi) \\ &= q(X)JA\xi + A\phi SX + g(SX, \xi)AN - g(SX, A\xi)N,\end{aligned} \tag{9}$$

and

$$\begin{aligned}\nabla_X(AN) &= \bar{\nabla}_X(AN) - \sigma(X, AN) \\ &= [(\bar{\nabla}_X A)N + A\bar{\nabla}_X N] - \sigma(X, AN) \\ &= q(X)JAN - ASX - g(SX, A\xi)N.\end{aligned} \tag{10}$$

If we put $X = \xi$ into the equations (9) and (10), we obtain

$$\nabla_\xi(A\xi) = -[q(\xi) - \alpha]AN, \qquad and \qquad \nabla_\xi(AN) = [q(\xi) - \alpha]A\xi.$$

Then (8) can be written as follows:

$$\begin{aligned}R(\xi, Y)Df = {} & k\phi SY - h(\nabla_\xi S)Y + (\nabla_\xi S^2)Y \\ & + (q(\xi) - \alpha)[g(Y, A\xi)AN + g(Y, AN)A\xi \\ & - g(Y, AN)A\xi - g(Y, A\xi)AN].\end{aligned} \tag{11}$$

Moreover, from the curvature tensor of M in Q^{m^*}, we get

$$\begin{aligned}R(\xi, Y)Df = {} & -g(Y, Df)\xi + g(\xi, Df)Y - g(AY, Df)A\xi \\ & + g(A\xi, Df)AY - g(JAY, Df)JA\xi + g(JA\xi, Df)JAY \\ & + g(SY, Df)S\xi - g(S\xi, Df)SY.\end{aligned} \tag{12}$$

From this equation, we can take $Y \in \mathfrak{Q}$ which is orthogonal to ξ, $A\xi$ and AN such that $SY = \coth rY$. Then $Y \in T_\upsilon \subset V(A)$, $\upsilon = \coth r$ and $\phi Y \in T_\upsilon \subset V(A)$, Because of the commuting property $S\phi = \phi S$ in theorem (2.2). That is $SY = \coth rY, AY = Y, A\phi Y = \phi Y, JAY = \phi AY$ and $JA\xi = -AN$.
Using these properties into (11) and (12), we have

$$\begin{aligned}k\phi SY - h(\nabla_\xi S)Y + (\nabla_\xi S^2)Y = {} & -g(Y, Df)\xi + g(\xi, Df)Y - g(Y, Df)A\xi \\ & + g(A\xi, Df)Y + g(\phi Y, Df)AN - g(AN, Df)\phi Y \\ & + \alpha\upsilon g(Y, Df)\xi - \alpha g(\xi, Df)SY,\end{aligned} \tag{13}$$

where $\alpha = 2\coth 2r = \coth r + \tanh r$ and $\upsilon = \coth r$.
By taking the inner product of (13) with the Reeb vector field ξ, we obtain $g(Y, Df) = 0$ for any $Y \in T_\upsilon$.
Let us take $Y \in \mathfrak{Q}$ is orthogonal to ξ, $A\xi$ and AN such that $SY = \tanh rY$. Because of the commuting property $S\phi = \phi S$ in theorem (2.2), we have $Y \in T_\mu \subset JV(A)$, $\mu = \tanh r$ and $\phi Y \in T_\mu \subset JV(A)$. That is,

$$SY = \tanh rY, \quad AY = -Y, \quad A\phi Y = -\phi Y, \quad JAY = \phi AY = -\phi Y, \quad and \quad JA\xi = -AN.$$

In this case from (11) and (12) it follows that

$$\begin{aligned}k\phi SY - h(\nabla_\xi S)Y + (\nabla_\xi S^2)Y = & - g(Y, Df)\xi + g(\xi, Df)Y - g(Y, Df)A\xi \\ & + g(A\xi, Df)Y + g(\phi Y, Df)AN - g(AN, Df)\phi Y \\ & + \alpha\mu g(Y, Df)\xi - \alpha g(\xi, Df)SY.\end{aligned} \tag{14}$$

Where $\alpha = 2\coth 2r$. Also, by taking the inner product (14) with the Reeb vector field ξ, we obtain

$$0 = (-1 + \alpha\mu)g(Y, Df) = \tanh^2 rg(Y, Df),$$

for any $Y \in T_\mu$. Then we have $g(Y, Df) = 0$ for $Y \in T_\mu$.
Since $g(Y, Df) = 0$ for any $Y \in T_\upsilon$, then we get

$$Df = g(Df, \xi)\xi + g(Df, AN)AN + g(Df, A\xi)A\xi. \tag{15}$$

On the other hand, by taking the inner product of (13) with the Reeb vector field $Y \in T_\upsilon$ and $\phi Y \in T_\upsilon$, respectively, we have

$$g(Df, A\xi) = -1(1 - \alpha\upsilon)g(\xi, Df) = \coth^2 rg(\xi, Df)$$

and

$$g(AN, Df) = -k\coth r.$$

From equation (15) and these two formulas, we get

$$Df = g(\xi, Df)[\xi + \coth^2 r A\xi] - k \coth r AN. \tag{16}$$

Moreover, by taking the inner product of (14) with $Y \in T_\mu, \mu = \tanh r$, and $\phi Y \in T_v$, respectively, we obtain

$$g(Df, A\xi) = (1 - \alpha\mu)g(\xi, Df) = -\tanh^2 r g(\xi, Df) \tag{17}$$

and

$$g(AN, Df) = k\mu = k \tanh r. \tag{18}$$

Note that equations (15), (17), and (18) imply

$$Df = g(\xi, Df)[\xi - \tanh^2 r A\xi] + k \tanh r AN. \tag{19}$$

By substituting the equations (19) into (16), we obtain

$$(\coth^2 r + \tanh^2 r)A\xi - k(\coth r + \tanh r)AN = 0.$$

Since the vector fields $A\xi$ and AN are independent then $\coth r = \tanh r = 0$. We obtain a contradiction. Then we conclude the proof of the theorem. □

We want to give a property for gradient almost Ricci soliton on a real hypersurface M in the complex hyperbolic quadric Q^{m^*}.

Theorem 3.4. *There dose not exist a contact real hypersurface in the complex hyperbolic quadric $Q^{m^*}, m \geq 3$, admitting the gradient almost Ricci soliton. Moreover, the gradient vector field Df is identically vanishing.*

Proof. Let (M, g, Df, ψ) be a almost Ricci solition on a Riemannian manifold for any tangent vector field X on M,

$$\nabla_X Df + Ric(X) = \psi X. \tag{20}$$

By differentiating (20), we have

$$\begin{aligned} R(X,Y)Df &= \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df \\ &= -(\nabla_X Ric)Y - Ric(\nabla_X Y) + X(\psi)Y + \psi \nabla_X Y \\ &\quad + (\nabla_Y Ric)X + Ric(\nabla_Y X) - Y(\psi)X - \psi \nabla_Y X \\ &\quad + Ric([X,Y]) - \psi[X,Y] \\ &= (\nabla_Y Ric)X - (\nabla_X Ric)Y + [X(\psi)Y - Y(\psi)X]. \end{aligned} \tag{21}$$

Let M is a contact real hypersurface in Q^m. So it is Hopf and $\mathfrak{U}$-principal and we get

$$Ric(X) = -(2m-1)X + 2\eta(X)\xi + AX + hSX - S^2X,$$

for any tangent vector field X on M.
Put $X = \xi$, then M being Hopf and from $A\xi = -\xi$ we obtain

$$Ric(\xi) = k\xi,$$

where $k = -2(m-1) + h\alpha - \alpha^2$ is constant, and the mean curvature $h = TrS$ constant for a contact hypersurface M in Q^m.
By taking covariant derivative to the Ricci operator, we obtain

$$(\nabla_X Ric)\xi = (Xk)\xi + k\nabla_X \xi = k\phi SX, \tag{22}$$

and

$$\begin{aligned} (\nabla_\xi Ric)X &= \nabla_\xi(RicX) - Ric(\nabla_\xi X) \\ &= -(\nabla_\xi A)X + h(\nabla_\xi S)X - (\nabla_\xi S^2)X \\ &= h(\nabla_\xi S)X - (\nabla_\xi S^2)X, \end{aligned} \tag{23}$$

where we have used $\nabla_\xi A = 0$, because $(\nabla_\xi A)A + A(\nabla_\xi A) = 2(\nabla_\xi A)A = 0$ from $A^2 = I$ and $A \in End(TQ^m)$ for an $\mathfrak{U}$-principal unit normal N. By putting $X = \xi$ and from equtions (21), (22), and (23), we have

$$\begin{aligned} R(\xi, Y)Df &= (\nabla_Y Ric)\xi - (\nabla_\xi Ric)Y \\ &= k\phi SY - h(\nabla_\xi S)Y + (\nabla_\xi S^2)Y. \end{aligned} \tag{24}$$

Then with an a straightforward calculation the diagonalization of the shape operator S of the contact real hypersurface in complex hyperbolic quadric Q^{m^*} is obtained

$$S = \begin{bmatrix} \alpha & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \frac{2}{\alpha} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & \frac{2}{\alpha} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

By Lemma 2.5, for the case(i) the principal curvatures are given by $\alpha = \sqrt{2}\coth\sqrt{2}r, \upsilon = \frac{2}{\alpha} = \sqrt{2}\tanh\sqrt{2}r$ and $\mu = 0$, for the case(ii) the principal curvatures are given by $\alpha = \sqrt{2}, \upsilon = \sqrt{2}$ and $\mu = 0$ and for the case(iii) the principal curvatures are given by $\alpha = \sqrt{2}\coth\sqrt{2}r, \upsilon = \frac{2}{\alpha} = \sqrt{2}\tanh\sqrt{2}r$ and $\mu = 0$ in Theorem 2.4 with multiplicities 1, $2m-1$ and $2m-1$ respectively. All of these principal curvatures satisfy $\alpha\upsilon = 2$.
Also, the curvature tensor $R(X,Y)Z$ of M induced from $\bar{R}(X,Y)Z$ of the complex quadric Q^{m^*}

$$\begin{aligned} R(\xi, Y)Df &= -g(Y, Df)\xi + g(\xi, Df)Y - g(AY, Df)A\xi + g(A\xi, Df)AY \\ &\quad - g(JAY, Df)\phi A\xi + g(\phi A\xi, Df)JAY \\ &\quad + g(SY, Df)S\xi - g(S\xi, Df)SY \\ &= \alpha g(SY, Df)\xi - \alpha\eta(Df)SY \end{aligned} \tag{25}$$

for any $Y \in T_\upsilon \subset V(A), \upsilon = \sqrt{2}\tanh\sqrt{2}r, \upsilon = \sqrt{2}$ or $\upsilon = \sqrt{2}\coth\sqrt{2}r$ such that $SY = \upsilon Y, AY = Y$ and $A\xi = -\xi$ for a contact real hypersurface M in the complex hyperbolic quadric Q^{m^*}.
From equations (24) and (25), we obtain

$$k\phi SY - h(\nabla_\xi S)Y + (\nabla_\xi S^2)Y = \alpha g(SY, Df)\xi - \alpha\eta(Df)SY.$$

By taking the inner product with the Reeb vector field ξ, we get

$$\alpha g(SY, Df) - \alpha^2\eta(Df)\eta(Y) = 0.$$

Also, for any $Y \in T_\upsilon \subset V(A)$ in (25) it follows that

$$0 = \alpha g(SY, Df) = \alpha\upsilon g(Y, Df) = 2g(Y, Df). \tag{26}$$

Then, Df is orthogonal to the eigenspace T_λ for principal curvatures, $\upsilon = \sqrt{2}\tanh\sqrt{2}r, \upsilon = \sqrt{2}$ or $\upsilon = \sqrt{2}\coth\sqrt{2}r$, respectively.
Also, for $Y \in T_\mu \subset JV(A), \mu = 0$ it follows that $SY = \mu Y = 0, A\xi = -\xi$ and $AY = -Y$. Using these properties in (24) and (25) implies the following

$$k\phi SY - h(\nabla_\xi S)Y + (\nabla_\xi S^2)Y = 2g(Y, Df)\xi - 2g(\xi, Df)Y.$$

By taking the Reeb vector field ξ, we obtain

$$g(Y, Df) = 0 \qquad for \quad any \qquad Y \in T_\mu. \tag{27}$$

On the other hand, if $Y \in T_\mu$, and use $SY = 0$, we get

$$\begin{aligned} -2g(\xi, Df) &= kg(\phi SY, Y) - hg((\nabla_\xi S)Y, Y) + g((\nabla_\xi S^2)Y, Y) \\ &= -hg(\nabla_\xi(SY) - S\nabla_\xi Y, Y) + g(\nabla_\xi(S^2Y) - S^2\nabla_\xi Y, Y) \\ &= 0. \end{aligned} \tag{28}$$

From (26), (27), and (28) we have $Df = 0$ and M is Einstein. On the other hand, Theorem 2.6 gives that there does not an Einstein real hypersurface in the complex hyperbolic quadric Q^{m^*}. Then, we give a complete proof of theorem. □

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