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Original Article

Existence and convergence of fixed points for noncyclic φ-contractions

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ABSTRACT: In the paper, we introduce a new class of noncyclic φ-contractions as a generalization of the class of noncyclic contractions which was first introduced in the paper [R. Espínola, M. Gabeleh, On the structure of minimal sets of relatively nonexpansive mappings, Numerical Functional Analysis and Optimization 34 (8), 845-860, 2013] and study the existence, uniqueness and convergence of a fixed point for such class of noncyclic mapping in the framework of uniformly convex Banach spaces. We obtain existence results of the best proximity points for cyclic φ-contractions as a consequence of our main theorems.

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1. Introduction and Preliminaries

Let A and B be two nonempty subsets of a metric space (X, d). If self mapping $T\colon A\cup B \to A\cup B$ be a noncyclic map , i.e., $T(A) \subseteq A$ and $T(B) \subseteq B$; then $x^* \in A\cup B$ is a fixed point of T provided that $Tx^* = x^*$. If $T : A\cup B \to A\cup B$ be a cyclic map , i.e., $T(A) \subseteq B$ and $T(B) \subseteq A$ and $d(A, B) = \inf\{d(a, b) : \ a \in A, \ b \in B\}$; then $x^* \in A \cup B$ is called a best proximity point of T provided that $d(x^*, Tx^*) = d(A, B)$. If $d(A, B) > 0$, then a best proximity point serves as a optima for the operator equation $Tx = x$.

In 2005, Anthony Eldred, Kirk and Veeremani [5] introduced noncyclic nonexpansive mappings and studied the existence of a fixed point of such mappings. In 2006, cyclic contraction mappings on uniformly convex Banach spaces were introduced and studied by Anthony Eldred and Veeremani [6]. Since then, the problems of the existence of a best proximity point (fixed point) of cyclic (noncyclic) mappings, have been extensively studied by many authors; see for instance [1, 2, 5, 6, 7, 9, 11, 12, 13, 14] and references therein.

In 2009, Al-Thagafi and Shahzad [4] generalized cyclic contraction condition and proved existence of best proximity points.

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Definition 1.1 ([4]). *Let A and B be nonempty subsets of a metric space (X, d) and let $T : A \cup B \to A \cup B$ be a cyclic map. The map T is said to be cyclic φ-contraction if $\varphi : [0, +\infty) \to [0, +\infty)$ is a strictly increasing map and*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)),$$

for all $x \in A$ and $y \in B$.

In the recent years, many authors used varieties of weak contractive conditions to prove the existence of fixed point and best proximity point theorems.

In 2013, the class of noncyclic contractions was first introduced by Espínola and Gabeleh [8]. As a result of Theorem 2.7 of [3], for these mappings, the authors presented the following existence theorem.

Theorem 1.2. *Let A and B be nonempty convex subsets of a uniformly convex Banach space X such that A is closed and let $T : A \cup B \to A \cup B$ be a noncyclic contraction map that is, there exists $c \in [0, 1)$ such that*

$$d(Tx, Ty) \leq cd(x, y) + (1 - c)d(A, B), \tag{1}$$

for all $x \in A$ and $y \in B$. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique fixed point $x \in A$ such that $x_n \to x$.

In this paper, we introduce the concept of noncyclic φ-contractions as a generalization concept of noncyclic contractions and study the existence, uniqueness and convergence of fixed points for such mappings in the framework of uniformly convex Banach spaces. Also, iterative algorithms are furnished to determine such fixed points. We obtain existence results of the best proximity points for cyclic φ-contractions as a consequence of our main theorems. Here, we recall some definitions and facts will be used in the next section.

Definition 1.3 ([10]). *A Banach space X is said to be uniformly convex if there exists a strictly increasing function $\delta : [0, 2] \to [0, 1]$ such that the following implication holds for all $x, y, p \in X$, $R > 0$ and $r \in [0, 2R]$:*

$$\left.\begin{array}{l} \|x - p\| \leq R \\ \|y - p\| \leq R \\ \|x - y\| \geq r \end{array}\right\} \Rightarrow \left\|\frac{x + y}{2} - p\right\| \leq \big(1 - \delta(\frac{r}{R})\big)R.$$

Lemma 1.4 ([6]). *Let A be a nonempty closed and convex subset and B be nonempty closed subset of a uniformly convex Banach space. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ is a sequence in B satisfying*

(a) $\|x_n - y_n\| \to d(A, B)$;

(b) $\|z_n - y_n\| \to d(A, B)$.

Then $\|x_n - z_n\|$ converges to zero.

Since the proof of next result was classic, we presented it separately.

Lemma 1.5. *Let A and B be nonempty convex subsets of a uniformly convex Banach space X such that $d(A, B) > 0$. Let $\{x_n\}$ and $\{z_n\}$ be sequences in A and $\{y_n\}$ is a sequence in B. Suppose that for each $\epsilon > 0$, there exists a positive integer N such that for all $m > n \geq N$, $\|x_n - y_n\| < d(A, B) + \epsilon$ and $\|z_m - y_n\| < d(A, B) + \epsilon$. Then for every $\epsilon > 0$ there exists a positive integer k such that for all $m > n \geq k$, $\|x_n - z_m\| < \epsilon$.*

Proof. We assume the contrary. Then there exists $\epsilon_0 > 0$ such that for each $k \geq 1$, there exist $m_k > n_k \geq k$ such that

$$\|x_{n_k} - z_{m_k}\| \geq \epsilon_0. \tag{2}$$

On the other hand, for each $\epsilon > 0$, we have

$$\|x_{n_k} - y_{n_k}\| < d(A, B) + \epsilon \tag{3}$$

and

$$\|z_{m_k} - y_{n_k}\| < d(A, B) + \epsilon. \tag{4}$$

for all $m_k > n_k \geq N$. Choose $0 < \gamma < 1$ such that $\frac{\epsilon_0}{\gamma} > d(A, B)$ and choose ϵ such that

$$0 < \epsilon < \min\big\{\frac{\epsilon_0}{\gamma} - d(A, B), \frac{d(A, B)\delta(\gamma)}{1 - \delta(\gamma)}\big\}.$$

It follows from (2), (3), (4) and the uniform convexity of X that

$$\left\|\frac{x_{n_k}+z_{m_k}}{2}-y_{n_k}\right\| \le \left(1-\delta(\frac{\epsilon_0}{d(A,B)+\epsilon})\right)(d(A,B)+\epsilon),$$

for all $m_k > n_k \ge N$. The choice of ϵ and the fact that δ is strictly increasing imply that

$$\begin{aligned}\left\|\frac{x_{n_k}+z_{m_k}}{2}-y_{n_k}\right\| &\le \left(1-\delta\left(\frac{\epsilon_0}{\frac{\epsilon_0}{\gamma}}\right)\right)(d(A,B)+\epsilon)\\ &\le (1-\delta(\gamma))(d(A,B)+\epsilon)\\ &= (1-\delta(\gamma))d(A,B)+(1-\delta(\gamma))\epsilon\\ &< (1-\delta(\gamma))d(A,B)+\delta(\gamma)d(A,B)\\ &= d(A,B),\end{aligned}$$

for all $m_k > n_k \ge N$. Thus $\frac{x_{n_k}+z_{m_k}}{2} \notin A$ for all $m_k > n_k \ge N$, this is a contradiction. □

2. Main results

To establish our results, we introduce the following new class of noncyclic maps.

Definition 2.1. *Let A and B be nonempty subsets of a metric space (X,d) and let $T\colon A\cup B \to A\cup B$ be a noncyclic map. Suppose that $\varphi\colon [0,+\infty) \to [0,+\infty)$ is a strictly increasing map. The map T is said to be noncyclic φ-contraction if for all $x \in A$ and $y \in B$, we have*

$$d(Tx,Ty) \le d(x,y) - \varphi(d(x,y)) + \varphi(d(A,B)). \tag{5}$$

It follows directly from the definition that if $\varphi(t) = (1-c)t$, then (5) reduces to (1). Also

$$\varphi(d(A,B)) \le \varphi(d(x,y)) \quad \textit{for all} \quad x\in A \quad \textit{and} \quad y \in B \tag{6}$$

and

$$d(Tx,Ty) \le d(x,y) \quad \textit{for all} \quad x\in A \quad \textit{and} \quad y \in B. \tag{7}$$

Example 2.1. *Let $X := \mathbb{R}$ with the usual metric. For $A=[0,1]$ and $B=[-1,0]$, define noncyclic map $T\colon A\cup B \to A\cup B$ by*

$$T(x) = \begin{cases} \dfrac{x}{1+2x} & \text{if } x \in A,\\ \dfrac{y}{1-2y} & \text{if } y\in B.\end{cases}$$

If $\varphi(t) = \frac{t^2}{1+2t}$ for $t\ge 0$. Then we have

$$\begin{aligned} d(Tx,Ty) &= \left|\frac{x}{1+2x} - \frac{y}{1-2y}\right|\\ &= \frac{x}{1+2x} - \frac{y}{1-2y}\\ &= \frac{x-y-4xy}{1+2(x-y)-4xy}\\ &\le (x-y) - \frac{(x-y)^2}{1+2(x-y)}\\ &= |x-y| - \varphi(|x-y|)+\varphi(0)\\ &= d(x,y) - \varphi(d(x,y)) + \varphi(d(A,B)),\end{aligned}$$

for all $x\in A$ and $y \in B$. Hence T is a noncyclic φ-contraction map which is not noncyclic contraction.

Lemma 2.2. *Let A and B be nonempty subsets of a metric space (X,d) and let $T: A\cup B \to A\cup B$ be a cyclic φ-contraction map. Then T^2 is a noncyclic φ-contraction map.*

Proof. Applying (7) and the cyclic φ-contraction property of T, we obtain

$$d(T^2x, T^2y) \leq d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)) + \varphi(d(A, B)),$$

for all $x \in A$ and $y \in B$. □

The following example illustrates Lemma 2.2.

Example 2.2. *Let $X := \mathbb{R}$ with the usual metric. For $A = B = [0, 1]$, define noncyclic map $T : A \cup B \to A \cup B$ by $Tx = \frac{x}{1+x}$. If $\varphi(t) = \frac{t^2}{1+2t}$ for $t \geq 0$. By Example 2 of [4], T is a cyclic φ-contraction map. Also, we have*

$$\begin{aligned}
|T^2x - T^2y| &= |\frac{x}{1+2x} - \frac{y}{1+2y}| \\
&= \frac{|x-y|}{(1+2x)(1+2y)} \\
&\leq \frac{|x-y|}{1+x+y} \\
&\leq \frac{|x-y|}{1+|x-y|} \\
&= |x-y| - \frac{|x-y|^2}{1+|x-y|} \\
&= |x-y| - \varphi(|x-y|) + \varphi(0),
\end{aligned}$$

for all $x \in A$ and $y \in B$. Hence T^2 is a noncyclic φ-contraction map.

The following result will be needed to prove the main theorems of this section.

Lemma 2.3. *Let A and B be nonempty subsets of a uniformly convex Banach space X and let $T : A \cup B \to A \cup B$ be a noncyclic φ-contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ and for $y_0 \in B$, define $y_{n+1} := Ty_n$ for each $n \geq 0$. Then*

(a) $\|x_n - y_n\| \to d(A, B)$ as $n \to \infty$;

(b) if A is convex, then $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$;

(c) if B is convex, then $\|y_n - y_{n+1}\| \to 0$ as $n \to \infty$;

(d) if A and B are convex, then for each $\epsilon > 0$, there exists a positive integer N_0 such that for all $m > n \geq N_0$, $\|x_m - y_n\| < d(A, B) + \epsilon$.

Proof. (a) Let $d_n := \|x_n - y_n\|$ for each $n \geq 0$. It follows from (7) that $\{d_n\}$ is decreasing and bounded. Thus $\lim_{n\to\infty} d_n = d_0$ for some $d_0 \geq d(A, B)$. If $d_{n_0} = 0$ for some $n_0 \geq 0$, there is nothing to prove. So assume that $d_n > 0$ for each $n \geq 0$. Because T is a noncyclic φ-contraction, we have

$$d_{n+1} \leq d_n - \varphi(d_n) + \varphi(d(A, B)).$$

Hence by using (6), we get

$$\varphi(d(A, B)) \leq \varphi(d_n) \leq d_n - d_{n+1} + \varphi(d(A, B)), \tag{8}$$

for each $n \geq 0$. On the other hand, since φ is strictly increasing and $d_n \geq d_0 \geq d(A, B)$ for each $n \geq 0$, it follows from (8) that

$$\varphi(d(A, B)) \leq \varphi(d_0) \leq \varphi(d_n) \leq d_n - d_{n+1} + \varphi(d(A, B)),$$

so

$$\lim_{n\to\infty} \varphi(d_n) = \varphi(d_0) = \varphi(d(A, B)).$$

As φ is strictly increasing, we have $d_0 = d(A, B)$.
(b) From (a), we get

$$\|x_n - y_n\| \to d(A, B),$$

as $n \to \infty$. By reusing (a), with starting point x_1 instead of x_0, we obtain

$$\|x_{n+1} - y_n\| \to d(A, B),$$

as $n \to \infty$. So by Lemma 1.4, we get $\|x_n - x_{n+1}\| \to 0$ as $n \to \infty$.
(c) Proof of (c) is similar to (b).
(d) Suppose the contrary. Then there exists $\epsilon_0 > 0$ such that for each $k \geq 1$, there are $m_k > n_k \geq k$ satisfying

$$\|x_{m_k} - y_{n_k}\| \geq d(A, B) + \epsilon_0 \tag{9}$$

and

$$\|x_{m_k - 1} - y_{n_k}\| < d(A, B) + \epsilon_0.$$

Using the triangle inequality, for every $k \geq 1$, we have

$$\|x_{m_k} - y_{n_k}\| \leq \|x_{m_k} - x_{m_k+1}\| + \|x_{m_k+1} - y_{n_k+1}\| + \|y_{n_k+1} - y_{n_k}\|.$$

So by (6), and the noncyclic φ-contraction property of T, we obtain

$$\begin{aligned} \|x_{m_k} - y_{n_k}\| &\leq \|x_{m_k} - x_{m_k+1}\| + \|x_{m_k} - y_{n_k}\| - \varphi(\|x_{m_k} - y_{n_k}\|) \\ &\quad + \varphi(d(A, B)) + \|y_{n_k+1} - y_{n_k}\| \\ &\leq \|x_{m_k} - x_{m_k+1}\| + \|x_{m_k} - y_{n_k}\| + \|y_{n_k+1} - y_{n_k}\|. \end{aligned}$$

Hence, for every $k \geq 1$, we get

$$\begin{aligned} 0 &\leq \|x_{m_k} - x_{m_k+1}\| - \varphi(\|x_{m_k} - y_{n_k}\|) + \varphi(d(A, B)) + \|y_{n_k+1} - y_{n_k}\| \\ &\leq \|x_{m_k} - x_{m_k+1}\| + \|y_{n_k+1} - y_{n_k}\|. \end{aligned}$$

A and B are convex. Letting $k \to \infty$ and using (b) and (c), we obtain

$$0 \leq -\lim_{k\to\infty} \varphi(\|x_{m_k} - y_{n_k}\|) + \varphi(d(A, B)) \leq 0$$

and hence

$$\lim_{k\to\infty} \varphi(\|x_{m_k} - y_{n_k}\|) = \varphi(d(A, B)). \tag{10}$$

Since φ is strictly increasing, it follows from (9) and (10) that

$$\varphi(d(A, B) + \epsilon_0) \leq \lim_{k\to\infty} \varphi(\|x_{m_k} - y_{n_k}\|) = \varphi(d(A, B)) < \varphi(d(A, B) + \epsilon_0),$$

which is absurd. □

Now we are ready to prove our main first result in this section.

Theorem 2.4. *Let A and B be nonempty subsets of a uniformly convex Banach space X such that A is closed. Let $T : A \cup B \to A \cup B$ be a noncyclic φ-contraction map. For $x_0 \in A$ define $x_{n+1} := Tx_n$ for each $n \geq 0$. If $d(A, B) = 0$, then T has a unique fixed point $x \in A \cap B$ and $x_n \to x$ as $n \to \infty$.*

Proof. By lemma 2.3, with starting point x_1 instead of x_0, we get

$$\|x_{n+1} - y_n\| \to 0, \tag{11}$$

as $n \to \infty$ and with starting point y_1 instead of y_0, we get

$$\|x_n - y_{n+1}\| \to 0, \tag{12}$$

as $n \to \infty$. It follows from (11), (12), the triangle inequality and Lemma 2.3(a) that

$$\|x_n - x_{n+1}\| \to 0 \quad \textit{and} \quad \|y_n - y_{n+1}\| \to 0. \tag{13}$$

Let $\epsilon > 0$ be given. From (13), similarly to Lemma 2.3(d), we can deduce that there exists N_1 such that

$$\|x_m - y_n\| < \epsilon, \tag{14}$$

for all $m > n \geq N_1$. By Lemma 2.3(a), there exists N_2 such that

$$\|x_n - y_n\| < \epsilon, \tag{15}$$

for all $n \geq N_2$. Let $N := \max\{N_1, N_2\}$. It follows from (14), (15) and the triangle inequality that

$$\|x_m - x_n\| \leq \|x_m - y_n\| + \|x_n - y_n\| < 2\epsilon,$$

for all $m > n \geq N$. Thus $\{x_n\}$ is a Cauchy sequence in A. Now, the completeness of X and the closedness of A imply that $x_n \to x$ as $n \to \infty$. Because $\|x_n - y_n\| \to 0$ as $n \to \infty$, hence $y_n \to x$ as $n \to \infty$. Also from (7), $\|Tx - y_n\| \leq \|x - y_{n-1}\|$, hence $y_n \to Tx$ as $n \to \infty$. So x is a fixed point of T and hence $x \in A \cap B$. If $z \in A \cap B$ is another fixed point of T, from Lemma 2.3(a), we obtain

$$\|z - x\| = \lim_{n\to\infty} \|T^n z - y_n\| = 0,$$

and so $z = x$. □

Lemma 2.5. *Let A and B be nonempty convex subsets of a uniformly convex Banach space X such that A is closed and let $T : A \cup B \to A \cup B$ be a noncyclic φ-contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then $\{x_n\}$ is a Cauchy sequence.*

Proof. If $d(A, B) = 0$, the result follows from Theorem 2.4. So we assume that $d(A, B) > 0$. For $y_0 \in B$, define $y_{n+1} := Ty_n$ for each $n \geq 0$. By Lemma 2.3(a), for each $\epsilon > 0$ there exists N_1 such that

$$\|x_n - y_n\| \leq d(A, B) + \epsilon,$$

for all $n \geq N_1$. Also, by Lemma 2.3(d) there exists N_2 such that

$$\|x_m - y_n\| \leq d(A, B) + \epsilon,$$

for all $m > n \geq N_2$. Let $N := \max\{N_1, N_2\}$. It follows from Lemma 1.5, for every $\epsilon > 0$ there exists a positive integer k such that for all $m > n \geq k$, $\|x_n - x_m\| < \epsilon$. So $\{x_n\}$ is a Cauchy sequence. □

Now we are ready to prove our main second result in this section.

Theorem 2.6. *Let A and B be nonempty convex subsets of a uniformly convex Banach space X such that A is closed and let $T : A \cup B \to A \cup B$ be a noncyclic φ-contraction map. For $x_0 \in A$, define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique fixed point $x \in A$ such that $x_n \to x$.*

Proof. Fix $y_0 \in B$, define $y_{n+1} := Ty_n$ for each $n \geq 0$. If $d(A, B) = 0$, the result follows from Theorem 2.4. So we assume that $d(A, B) > 0$. By Lemma 2.5, $\{x_n\}$ is a Cauchy sequence and hence $x_n \to x \in A$ as $n \to \infty$. From (7) and the triangle inequality, we have

$$\|Tx - y_n\| \leq \|x - y_{n-1}\| \leq \|x - x_{n-1}\| + \|x_{n-1} - y_{n-1}\|$$

and

$$\|x - y_n\| \leq \|x - x_n\| + \|x_n - y_n\|.$$

So, from Lemma 2.3(a), we get $\|Tx - y_n\| \to d(A, B)$ and $\|x - y_n\| \to d(A, B)$ as $n \to \infty$. Lemma 1.4, implies that $Tx = x$. To show that uniqueness of x, suppose that there exists another fixed point $z \in A$ of T. For $y_0 \in B$, define $y_{n+1} := Ty_n$ for each $n \geq 0$. By Lemma 2.3(d), there exists N_1 such that

$$\|x - y_{n_k}\| \leq d(A, B) + \epsilon,$$

for all $n_k \geq N_1$. Also, there exists N_2 such that

$$\|z - y_{n_k}\| \leq d(A, B) + \epsilon,$$

for all $n_k \geq N_2$. Let $N := \max\{N_1, N_2\}$. Applying Lemma 1.5, we obtain $z = x$. □

Corollary 2.7. *Let A and B be nonempty closed and convex subsets of a uniformly convex Banach space X and let $T : A \cup B \to A \cup B$ be a noncyclic φ-contraction map. Then, T has a unique optimal pair of fixed points $(x^*, y^*) \in A \times B$ that is*

$$Tx^* = x^*, \quad Ty^* = y^* \quad \textit{and} \quad d(x^*, y^*) = d(A, B).$$

Example 2.3. *All conditions of Theorem 2.6 are satisfied in Example 2.1, and $x = 0$ is the unique best proximity point of T in A.*

In the following, we obtain Theorems 6 and 8 of [4] as special cases of Theorems 2.4 and 2.6, respectively.

Corollary 2.8 ([4, Theorem 6]). *Let A and B be nonempty subsets of a uniformly convex Banach space X such that A is closed. Let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction map. For $x_0 \in A$ define $x_{n+1} := Tx_n$ for each $n \geq 0$. If $d(A, B) = 0$, then T has a unique fixed point $x \in A \cap B$ and $x_n \to x$ as $n \to \infty$.*

Proof. From Lemma 2.2, T^2 is a noncyclic φ-contraction map. So by Theorem 2.4, T^2 has a unique fixed point $x \in A \cap B$ such that $x_n \to x$ as $n \to \infty$. Applying the cyclic φ-contraction property of T, we obtain

$$\|x - Tx\| = \|T^2x - Tx\| \leq \|x - Tx\| - \varphi(\|x - Tx\|) + \varphi(0),$$

hence $\varphi(\|x - Tx\|) \leq \varphi(0)$. Since φ is strictly increasing, it follows $\|x - Tx\| = 0$, so $x = Tx$ and $x \in A \cap B$. □

Before stating the next result, let me remind you of one point. At the end of the proof of Lemma 2 of [4], the authors mentioned that $\|x_{2n+3} - x_{2n+1}\| \to 0$ is obtained in a similar way, and since x_{2n+3} and x_{2n+1} are in B, B must be convex. For this reason, in [4] theorems 5, 7 and 8, B must also be convex.

Corollary 2.9 ([4, Theorem 8]). *Let A and B be nonempty convex subsets of a uniformly convex Banach space X such that A is closed. Let $T : A \cup B \to A \cup B$ be a cyclic φ-contraction map. For $x_0 \in A$ define $x_{n+1} := Tx_n$ for each $n \geq 0$. Then there exists a unique $x \in A$ such that $x_{2n} \to x$, $T^2x = x$ and $\|x - Tx\| = d(A, B)$.*

Proof. From Lemma 2.2, T^2 is a noncyclic φ-contraction map. So by Theorem 2.6, T^2 has a unique fixed point $x \in A$ such that $x_{2n} \to x$. Applying the cyclic φ-contraction property of T, we obtain

$$\|x - Tx\| = \|T^2x - Tx\| \leq \|x - Tx\| - \varphi(\|x - Tx\|) + \varphi(d(A, B)),$$

hence $\varphi(\|x - Tx\|) \leq \varphi(d(A, B))$. Since φ is strictly increasing, it follows $\|x - Tx\| = d(A, B)$. □

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