



Original Article

On bijectivity and Connes bijectivity of a dual Banach algebra with respect to a w^* -closed ideal

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ABSTRACT: In this paper, we introduce a notion of Connes bijectivity for a dual Banach algebra A with respect to its w^* -closed ideal I , say I -Connes bijectivity. Some Lipschitz algebras $Lip_\alpha(X)$ and some matrix algebras are studied under this new notion. Also, with some mild assumptions, the relation between I -Connes bijectivity and left ϕ -contractibility is given, where ϕ is a w^* -continuous multiplicative linear functional on A . As an application, we characterize Connes bijectivity of some Lipschitz algebras.

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1. Introduction and Preliminaries

The concept of amenability for Banach algebras were first introduced by B. E. Johnson [5]. A Banach algebra A is amenable if and only if there exists a bounded net (m_α) in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \rightarrow 0$ and $\pi_A(m_\alpha)a \rightarrow a$ for every $a \in A$, where $\pi_A : A \otimes_p A \rightarrow A$ is denoted for the product morphism($\pi_A(a \otimes b) = ab$, for all $a, b \in A$). Indeed for a locally compact group G , $L^1(G)$ (the measure algebra $M(G)$) is amenable if and only if G is amenable (G is discrete and amenable). Helemskii in [4] and [15] studied the structure of Banach algebras thorough the homological methods of Banach algebras. He defined the concepts biflatness and bijectivity. In fact a Banach algebra A is bijective, if there exists a bounded A -bimodule morphism $\rho : A \rightarrow A \otimes_p A$ such that $\pi_A \circ \rho(a) = a$, for all $a \in A$. It is known that for a locally compact group G , the group algebra $L^1(G)$ (the measure algebra $M(G)$) is bijective if and only if G is compact (G is finite). For the history of amenability and homological properties of algebra, see [12].

There exists a class of Banach algebras which is called dual Banach algebras. This category of Banach algebras is defined by Runde [11]. Let A be a Banach algebra. Then a Banach A -bimodule E is called dual if there is a

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closed submodule E_* of E^* such that $E = (E_*)^*$. The Banach algebra A is called dual if it is dual as a Banach A -bimodule. A dual Banach A -bimodule E is normal if for each $x \in E$ the module maps $A \rightarrow E$ by $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are w^* - w^* -continuous. Let A be a Banach algebra and let E be a Banach A -bimodule. A bounded linear map $D : A \rightarrow E$ is called a bounded derivation if $D(ab) = a \cdot D(b) + D(a) \cdot b$, for every $a, b \in A$. A bounded derivation $D : A \rightarrow E$ is called inner if there exists an element x in E such that $D(a) = a \cdot x - x \cdot a$ ($a \in A$). A dual Banach algebra A is called Connes amenable if for every normal dual Banach A -bimodule E , every w^* -continuous derivation $D : A \rightarrow E$ is inner. For a given dual Banach algebra A and a Banach A -bimodule E , $\sigma wc(E)$ denote the set of all elements $x \in E$ such that the module maps $A \rightarrow E$ by $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are w^* - w -continuous. It is a closed submodule of E , see [11] and [13] for more details. Note that, since $\sigma wc(A_*) = A_*$, the adjoint of π_A maps A_* into $\sigma wc(A \otimes_p A)^*$. Therefore π_A^{**} drops to an A -bimodule morphism $\pi_{\sigma wc} : (\sigma wc(A \otimes_p A)^*)^* \rightarrow A$.

A dual Banach algebra A is called Connes-biprojective if there exists a bounded A -bimodule morphism $\rho : A \rightarrow (\sigma wc(A \otimes_p A)^*)^*$ such that $\pi_{\sigma wc} \circ \rho(a) = a$ for all $a \in A$. Shirinkalam and Pourabbas showed that a dual Banach algebra A is Connes amenable if and only if A is Connes-biprojective and it has an identity [16]. They characterized Connes-biprojectivity of the measure algebra $M(G)$ for a locally compact group G .

In this paper, we introduce the notion of I -Connes biprojective Banach algebras in the category of dual Banach algebras, where I is a w^* -closed ideal. Some matrix algebras and Lipschitz algebras are studied under this new notion. Also, with some mild assumptions, the relation between I -Connes biprojectivity and left ϕ -contractibility is given, where ϕ is a w^* -continuous multiplicative linear functional on A . As an application, we characterize Connes biprojectivity of some Lipschitz algebras.

Recently the notion of I -biprojectivity is given for Banach algebras. Let A be a Banach algebra and I be a closed ideal of A . Then A is called I -biprojective if there exists a bounded A -bimodule morphism $\rho : I \rightarrow A \otimes_p A$ such that $\pi_A \circ \rho(i) = i$ for all $i \in I$. For a locally compact group G , the measure algebra $M(G)$ is $L^1(G)$ -biprojective if and only if G is compact [14].

Throughout this paper, $\Delta(A)$ ($\Delta_{w^*}(A)$) denotes the character space (w^* - character space) of A , that is, all non-zero (w^* - continuous) multiplicative linear functionals on A , respectively. Let $\phi \in \Delta(A)$. Then ϕ has a unique extension to A^{**} denoted by $\tilde{\phi}$ and defined by $\tilde{\phi}(F) = F(\phi)$, for every $F \in A^{**}$. Clearly, this extension remains to be a character on A^{**} . The projective tensor product $A \otimes_p A$ is a Banach A -bimodule by the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca, \quad (a, b, c \in A).$$

Let X and Y be Banach A -bimodules. Then the map $T : X \rightarrow Y$ is called A -bimodule morphism if

$$T(a \cdot x) = a \cdot T(x), \quad T(x \cdot a) = T(x) \cdot a, \quad (a \in A, x \in X).$$

2. I -Connes biprojectivity

We commence this section with the definition of our new notion. We should remind that every w^* -closed ideal of a dual Banach algebra is also dual Banach algebra see [8, Lemma 2].

Definition 2.1. Let I be a w^* -closed ideal of a dual Banach algebra A . We say that A is I -Connes biprojective, if there exists a bounded A -bimodule morphisms $\rho : I \rightarrow (\sigma wc((A \otimes_p A)^*))^*$ such that $\pi_{\sigma wc} \circ \rho(i) = i$ for all $i \in I$. We say that A is ideally Connes biprojective, if it is I -Connes biprojective for every w^* -closed ideal I of A .

Remark 2.2. In above definition, we can replace ρ with a bounded net of A -bimodule morphisms, say (ρ_α) from I into $\sigma wc((A \otimes_p A)^*)^*$ which $\pi_{\sigma wc} \circ \rho_\alpha(i) \xrightarrow{w^*} i$, for all $i \in I$. To see this, since (ρ_α) is a bounded net of A -bimodule morphisms, we have $(\rho_\alpha) \subseteq B(I, (\sigma wc((A \otimes_p A)^*))^*)$ (the set of bounded linear maps from I into $(\sigma wc((A \otimes_p A)^*))^*$). On the other hand, on bounded sets the w^* -operator topology coincides with the w^* -topology of $B(I, (\sigma wc((A \otimes_p A)^*))^*)$ where identified with $(I \otimes_p \sigma wc(A \otimes_p A)^*)^*$. It is known that the unit ball of $B(I, (\sigma wc((A \otimes_p A)^*))^*)$ is w^* -operator compact. Then (ρ_α) has a w^* -operator topology limit point say ρ . Thus

$$\rho(i_1 i_2) = w^* - \lim \rho_\alpha(i_1 i_2) = w^* - \lim i_1 \cdot \rho_\alpha(i_2) = i_1 \cdot w^* - \lim \rho_\alpha(i_2) = i_1 \cdot \rho(i_2).$$

Similarly we have $\rho(i_1 i_2) = \rho(i_1) \cdot i_2$. Therefore ρ is a bounded A -bimodule morphism and

$$\pi_{\sigma wc} \circ \rho(i) = \pi_{\sigma wc}(w^* - \lim \rho_\alpha(i)) = w^* - \lim \pi_{\sigma wc} \rho_\alpha(i) = i.$$

So A is I -Connes biprojective.

Lemma 2.3. Let A be a non-zero dual Banach algebra and $ab = ba = 0$ for all $a, b \in A$. Then A is not Connes biprojective.

Proof. We assume in contradiction that A is Connes biprojective. Then there exists a bounded A -bimodule morphisms $\rho : A \rightarrow (\sigma wc(A \otimes_p A))^*$ such that $\pi_{\sigma wc} \circ \rho(a) = a$ for each $a \in A$. It is known that there exists a net (u_α) in $A \otimes_p A$ such that $w^* - \lim \hat{u}_\alpha|_{\sigma wc(A \otimes_p A)^*} = \rho(a)$. Thus

$$a = \pi_{\sigma wc} \circ \rho(a) = w^* - \lim \pi_A(u_\alpha) = w^* - \lim 0 = 0.$$

It follows that $A = 0$ which is a contradiction. □

Let A be a Banach algebra and $\phi \in \Delta(A)$. Then A is called left ϕ -contractible if there exists a $m \in A$ such that $am = \phi(a)m$ and $\phi(m) = 1$ for all $a \in A$. For further information about this concept see [10].

Proposition 2.4. *Let A be a commutative dual Banach algebra and $\phi \in \Delta_{w^*}(A)$. Suppose that I is a w^* -closed ideal of A . If A is I -Connes biprojective, then A is left ϕ -contractible, provided that $\phi|_I \neq 0$.*

Proof. Let i_0 be an element of I such that $\phi(i_0) = 1$. Since A is I -Connes biprojective, there exists a bounded A -bimodule morphism $\rho : I \rightarrow (\sigma wc((A \otimes_p A)^*))^*$ such that $\pi_{\sigma wc} \circ \rho(i) = i$ for all $i \in I$. Put $m = \rho(i_0)$. One can see that $a \cdot m = m \cdot a$ and $\pi_{\sigma wc}(m)i = i$ for all $i \in I$ and $a \in A$. Define $T : A \otimes_p A \rightarrow A$ by $T(a \otimes b) = \phi(b)a$ for each $a, b \in A$. It is easy to see that

$$aT(x) = T(a \cdot x), \quad T(x \cdot a) = \phi(a)T(x) \quad \phi \circ T(x) = \phi \circ \pi_A(x) \quad (a \in A, x \in X).$$

Since T^{**} is a w^* -continuous map, for each $x \in (A \otimes_p A)^{**}$ and $a \in A$, we have

$$aT^{**}(x) = T^{**}(a \cdot x), \quad T^{**}(x \cdot a) = \phi(a)T^{**}(x) \quad \tilde{\phi} \circ T^{**}(x) = \tilde{\phi} \circ \pi_A^{**}(x).$$

On the other hand, for all $a \in A, f \in A^*$ and $x \in A \otimes_p A$ consider

$$\langle x, a \cdot T^*(f) \rangle = \langle x \cdot a, T^*(f) \rangle = \langle T(x \cdot a), f \rangle = \phi(a)\langle T(x), f \rangle = \phi(a)\langle x, T^*(f) \rangle$$

also

$$\langle x, T^*(f) \cdot a \rangle = \langle a \cdot x, T^*(f) \rangle = \langle T(a \cdot x), f \rangle = \langle a \cdot T(x), f \rangle = \langle T(x), f \cdot a \rangle,$$

which follow that

$$a \cdot T^*(f) = \phi(a)T^*(f), \quad T^*(f) \cdot a = T^*(f \cdot a).$$

These last facts with the w^* -continuity of ϕ gives that $T^*(\sigma wc(A^*)) \subseteq \sigma wc(A \otimes_p A)^*$. Let $q : A^{**} \rightarrow \sigma wc((A)^*)^*$ be the quotient map. For each $a \in A$ and $f \in \sigma wc(A^*)$, we have

$$\begin{aligned} \langle f \cdot a, q \circ T^{**}(m) \rangle &= \langle f \cdot a, T^{**}(m)|_{\sigma wc(A)^*} \rangle = \langle f \cdot a, T^{**}(m) \rangle \\ &= \langle f, a \cdot T^{**}(m) \rangle \\ &= \langle f, T^{**}(a \cdot m) \rangle \\ &= \langle T^*(f), a \cdot m \rangle \\ &= \langle T^*(f), m \cdot a \rangle \\ &= \langle f, T^{**}(m \cdot a) \rangle \\ &= \langle f, \phi(a)T^{**}(m) \rangle \\ &= \phi(a)\langle f, T^{**}(m) \rangle \\ &= \phi(a)\langle f, T^{**}(m)|_{\sigma wc(A)^*} \rangle \\ &= \phi(a)\langle f, q \circ T^{**}(m) \rangle. \end{aligned}$$

It follows that $a \cdot q \circ T^{**}(m) = \phi(a)q \circ T^{**}(m)$ for all $a \in A$. Moreover we know that ϕ is a w^* -multiplicative linear functional. It follows that $\phi \in A_*$. On the other hand $A_* \subseteq \sigma wc(A)^*$. Then $\phi \in \sigma wc(A)^*$. Hence

$$\langle \phi, q \circ T^{**}(m) \rangle = \langle \phi, T^{**}(m)|_{\sigma wc(A)^*} \rangle = \langle \phi, T^{**}(m) \rangle = \langle \phi, \pi_A^{**}(m) \rangle = 1.$$

Thus it gives that A is left ϕ -contractible. □

Theorem 2.5. *Let A be a dual Banach algebra and $\phi \in \Delta_{w^*}(A)$. Suppose that I is a w^* -closed ideal which $\phi|_I \neq 0$ and $I \ker \phi|_I = \ker \phi|_I$. If A is I -Connes biprojective, then A is left ϕ -contractible.*

Proof. Let A be I -Connes biprojective. Then there exists a bounded A -bimodule morphism $\rho : I \rightarrow (\sigma wc(A \otimes_p A)^*)^*$ such that $\pi_{\sigma wc} \circ \rho(i) = i$ for all $i \in I$. Let i_0 be an element of I such that $\phi(i_0) = 1$. It is known that $\ker \phi$ is a closed ideal of A . Thus $\frac{A}{\ker \phi}$ is a Banach A -bimodule, naturally. We denote the identity map on A by id_A . Also $q : A \rightarrow \frac{A}{\ker \phi}$ is denoted for the quotient map. Define $id_A \otimes q : A \otimes_p A \rightarrow A \otimes_p \frac{A}{\ker \phi}$ by $id_A \otimes q(a \otimes b) = a \otimes (b + \ker \phi)$ for all $a, b \in A$. Clearly $id_A \otimes q$ is a bounded A -bimodule morphism. It implies that

$$(id_A \otimes q)^*(\sigma wc(A \otimes_p \frac{A}{\ker \phi})^*) \subseteq \sigma wc(A \otimes_p A)^*.$$

Using this fact, set

$$\theta : ((id_A \otimes q)^*|_{\sigma wc(A \otimes_p \frac{A}{\ker \phi})^*})^* : (\sigma wc(A \otimes_p A)^*)^* \rightarrow (\sigma wc(A \otimes_p \frac{A}{\ker \phi})^*)^*.$$

Clearly we observe that θ is a w^* -continuous A -bimodule morphism. Put

$$\eta = \theta \circ \rho : I \rightarrow (\sigma wc(A \otimes_p \frac{A}{\ker \phi})^*)^*.$$

We can see that η is a bounded A -bimodule morphism. Since $\overline{I \ker \phi|_I} = I$, we may assume that for each $l \in \ker \phi|_I$ there is $l_1 \in \ker \phi|_I$ and $i_1 \in I$ such that $l = i_1 l_1$. On the other hand we know that there exists a quotient map q from $(A \otimes_p A)^{**} \rightarrow (\sigma wc(A \otimes_p A)^*)^*$ and compose q with the embedding map from $A \otimes_p A$ into $(A \otimes_p A)^{**}$ gives a continuous A -bimodule map $\tau : A \otimes_p A \rightarrow (\sigma wc(A \otimes_p A)^*)^*$ which has a w^* -dense range. We denote \bar{u} for $\tau(u) = \hat{u}|_{\sigma wc(A \otimes_p A)^*}$, where $u \in A \otimes_p A$ and \hat{u} is the image of embedding map at u in $(A \otimes_p A)^{**}$. So for $\rho(i_1) \in (\sigma wc(A \otimes_p A)^*)^*$ there exists a net (u_α) in $A \otimes_p A$ which $w^* - \lim \bar{u}_\alpha = \rho(i_1)$. Applying the w^* -continuity of θ implies that

$$\begin{aligned} \eta(l) &= \theta \circ \rho(i_1 l_1) = \theta(\rho(i_1) \cdot l_1) \\ &= \theta((w^* - \lim \bar{u}_\alpha) \cdot l_1) \\ &= w^* - \lim \theta(\bar{u}_\alpha \cdot l_1) \\ &= w^* - \lim ((id_A \otimes q)^*|_{\sigma wc(A \otimes_p \frac{A}{\ker \phi})^*})^*(u_\alpha \cdot l_1) = 0, \end{aligned}$$

the last equality holds because $q(l_1) = 0$. So $\eta(l) = 0$. So η induces a map from $\frac{A}{\ker \phi}$ into $(\sigma wc(A \otimes_p A)^*)^*$ which is a bounded A -bimodule morphism. Since $\phi \in \Delta_{w^*}(A)$, we denote $\bar{\phi} : \frac{A}{\ker \phi} \rightarrow \mathbb{C}$ for a character which is given by $\bar{\phi}(a + \ker \phi) = \phi(a)$ for all $a \in A$. Clearly $\bar{\phi}$ is a character. Put $id_A \otimes \bar{\phi} : A \otimes_p \frac{A}{\ker \phi} \rightarrow A$ which is defined by $id_A \otimes \bar{\phi}(a \otimes b + \ker \phi) = \phi(b)a$ for every $a, b \in A$. One can readily see that for each $f \in A^*$ and $a \in A$

$$(id_A \otimes \bar{\phi})^*(f) \cdot a = (id_A \otimes \bar{\phi})^*(f \cdot a), \quad a \cdot ((id_A \otimes \bar{\phi})^*(f)) = \phi(a)(id_A \otimes \bar{\phi})^*(f).$$

Using w^* -continuity of ϕ and $\sigma wc(A_*) = A_*$ implies that

$$(id_A \otimes \bar{\phi})(A_*) = (id_A \otimes \bar{\phi})(\sigma wc(A_*)) \subseteq \sigma wc(A \otimes_p \frac{A}{\ker \phi})^*.$$

It follows that

$$\psi = ((id_A \otimes \bar{\phi})|_{A_*})^* : (\sigma wc(A \otimes_p \frac{A}{\ker \phi})^*)^* \rightarrow A$$

is a w^* -continuous left A -module morphism. Set $y = \psi \circ \eta$. Hence y is a bounded left A -module morphism from $\frac{A}{\ker \phi}$ into A . Note that y is a non-zero map. To see this, we show that $\phi \circ \psi = \phi \circ \pi_{\sigma wc}$. Clearly for each $a, b \in A$, we have

$$\begin{aligned} \phi \circ (id_A \otimes \bar{\phi}) \circ (id_A \otimes q)(a \otimes b) &= \phi \circ (id_A \otimes \bar{\phi})(a \otimes (b + \ker \phi)) = \phi(a\phi(b)) \\ &= \phi(a)\phi(b) \\ &= \phi \circ \pi_A. \end{aligned}$$

On the other hand for each $v \in A \otimes_p \frac{A}{\ker \phi}$ we have $\psi(\hat{v}|_{(A \otimes_p \frac{A}{\ker \phi})^*}) = (id_A \otimes \bar{\phi})(v)$. Also, for each $u \in A \otimes_p A$, we have $\pi_{\sigma wc}(\bar{u}) = \pi_A(u)$. Let $m \in (\sigma wc(A \otimes_p A)^*)^*$. Then there exists a net (u_α) in $A \otimes_p A$ such that $m = w^* - \lim \bar{u}_\alpha$.

As we know that ϕ, θ, ψ and $\pi_{\sigma wc}$ are w^* -continuous maps. So

$$\begin{aligned} \phi \circ \psi \circ \theta(m) &= \phi \circ \psi \circ \theta(w^* - \lim \bar{u}_\alpha) = w^* - \lim \phi \circ \psi \circ \theta(\bar{u}_\alpha) \\ &= w^* - \lim \phi \circ (id_A \otimes \bar{\phi}) \circ (id_A \otimes q)(u_\alpha) \\ &= w^* - \lim \phi \circ \pi_A(u_\alpha) \\ &= w^* - \lim \phi \circ \pi_{\sigma wc}(\bar{u}_\alpha) \\ &= \phi \circ \pi_{\sigma wc}(m). \end{aligned}$$

Thus for $i_0 \in I \subseteq A$, we have

$$\begin{aligned} \phi \circ y(i_0 + \ker \phi) &= \phi \circ \psi \circ \eta(i_0 + \ker \phi) = \phi \circ \psi \circ \theta \circ \rho(i_0) \\ &= \phi \circ \pi_{\sigma wc} \circ \rho(i_0) = \phi(i_0) = 1. \end{aligned}$$

It implies that y is nonzero map as desired. Also for each $a \in A$, we have

$$\begin{aligned} ay(i_0 + \ker \phi) &= y(ai_0 + \ker \phi) = y(\phi(a)i_0 + \ker \phi) \\ &= \phi(a)y(i_0 + \ker \phi). \end{aligned}$$

Hence A is left ϕ -contractible. □

We give a dual Banach algebra A with a w^* -closed ideal I which neither A nor I is Connes biprojective. But A is I -Connes biprojective.

Example 2.1. Let $A = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \mid a, b, c \in \mathbb{C} \right\}$ and $I = \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in \mathbb{C} \right\}$. With matrix operations and the ℓ^1 -norm, A becomes a dual Banach algebra and I becomes a w^* -closed ideal of A . We assume in contradiction that A is Connes biprojective. Since A is unital by [16, Theorem 2.2] A is Connes amenable. Define $\phi : A \rightarrow \mathbb{C}$ by $\phi\left(\begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix}\right) = c$. Clearly ϕ is a character on A . Put $J = \left\{ \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \mid b, c \in \mathbb{C} \right\}$. It is easy to verify that J is a w^* -closed ideal of A which $\phi|_J \neq 0$. It is easy to see that Connes amenability of A implies that A is left ϕ -amenable (or A is left ϕ -contractible), see [7]. So by similar method as in [6, Lemma 3.1] we have J is left ϕ -contractible. That is there is an element $m = \begin{pmatrix} 0 & b_0 \\ 0 & c_0 \end{pmatrix}$ in J such that $jm = \phi(j)m$ and $\phi(m) = 1$ for each $j \in J$, where $b_0, c_0 \in \mathbb{C}$. Suppose that $j = \begin{pmatrix} 0 & j_1 \\ 0 & j_2 \end{pmatrix}$ is an arbitrary element of J , where j_1 and j_2 in \mathbb{C} . Thus

$$jm = \begin{pmatrix} 0 & j_1 \\ 0 & j_2 \end{pmatrix} \begin{pmatrix} 0 & b_0 \\ 0 & c_0 \end{pmatrix} = \begin{pmatrix} 0 & j_1 c_0 \\ 0 & j_2 c_0 \end{pmatrix} = \phi(j)m = j_2 \begin{pmatrix} 0 & b_0 \\ 0 & c_0 \end{pmatrix} = \begin{pmatrix} 0 & j_2 b_0 \\ 0 & j_2 c_0 \end{pmatrix}$$

and $\phi\left(\begin{pmatrix} 0 & b_0 \\ 0 & c_0 \end{pmatrix}\right) = c_0 = 1$. It follows that for each j_1 and j_2 in \mathbb{C} we have $j_2 b_0 = j_1$. Put $j_2 = 0$ and $j_1 = 1$. Then contradiction reveals.

Since for each element i_1 and i_2 in I , we have $i_1 i_2 = 0$, Lemma 2.3 follows that I is not Connes biprojective. To show that A is I -Connes biprojective define $\rho : I \rightarrow A \otimes_p A \subseteq (\sigma wc(A \otimes_p A))^*$ by

$$\rho\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \quad (b \in \mathbb{C})$$

Clearly ρ is a bounded A -bimodule morphism and $\pi_{\sigma wc} \circ \rho(i) = i$ for all $i \in I$.

3. Applications for Lipschitz algebras

Let X be a compact metric space and $\alpha > 0$. The space of complex valued function on X is denoted by $Lip_\alpha(X)$ which

$$p_\alpha(f) = \sup\left\{ \frac{|f(x) - f(y)|}{d(x, y)^\alpha} : x, y \in X, x \neq y \right\}$$

is finite. Also

$$lip_\alpha(X) = \{f \in Lip_\alpha(X) : \frac{|f(x) - f(y)|}{d(x, y)^\alpha} \rightarrow 0 \text{ as } d(x, y) \rightarrow 0\}.$$

Define

$$\|f\|_\alpha = \|f\|_\infty + p_\alpha(f).$$

With $\|\cdot\|_\alpha$ and the pointwise operations $Lip_\alpha(X)$ and $lip_\alpha(X)$ become Banach algebras. It is known that for $0 < \alpha < 1$, $lip_\alpha(X)^{**}$ is isometrically isomorphism with $Lip_\alpha(X)$. Also $Lip_\alpha(X)$ and $lip_\alpha(X)$ are Arens regular Banach algebras for more details, see [2]. Recently Minapour and Zivari-Kazmpour showed that $Lip_\alpha(X)$ is a dual Banach algebra, [9].

Theorem 3.1. *Let X be a compact metric space and $0 < \alpha < 1$. Then $Lip_\alpha(X)$ is Connes biprojective if and only if X is finite.*

Proof. Suppose that $Lip_\alpha(X)$ is Connes biprojective. Since $Lip_\alpha(X)$ posses an identity, by [16, Theorem 2.2] $Lip_\alpha(X)$ is Connes amenable. Thus $Lip_\alpha(X) \cong (lip_\alpha(X))^{**}$ is Connes amenable. It is easy to see that $lip_\alpha(X)$ is a closed ideal of $Lip_\alpha(X)$. Applying [12, Theorem 4.4.8] follows that $lip_\alpha(X)$ is amenable. By the main result of [3] X is finite.

Converse is clear. □

A Banach algebra A is called biflat if there exists a bounded A -bimodule morphism $\rho : A \rightarrow (A \otimes_p A)^{**}$ such that $\pi_A^{**} \circ \rho(a) = a$ for each $a \in A$ [12].

Theorem 3.2. *Let X be a compact metric space and $0 < \alpha < 1$. Then $Lip_\alpha(X)$ is $lip_\alpha(X)$ -biprojective if and only if X is finite.*

Proof. Suppose that $Lip_\alpha(X)$ is $lip_\alpha(X)$ -biprojective. Then by [14, Lemma 3.5] $lip_\alpha(X)$ is biflat. Clearly $lip_\alpha(X)$ posses an identity. Thus $lip_\alpha(X)$ is amenable. Applying [3] X is finite.

For converse, let X be finite. Then by [1, Corollary 2.2] $Lip_\alpha(X)$ separates the point of X . Applying [1, Theorem 3.2] follows that $Lip_\alpha(X)$ is biprojective. Then there exists a bounded $Lip_\alpha(X)$ -bimodule morphism ρ from $Lip_\alpha(X)$ into $Lip_\alpha(X) \otimes_p Lip_\alpha(X)$ such that $\pi_{Lip_\alpha(X)} \circ \rho(a) = a$ for all $a \in Lip_\alpha(X)$. Restrict ρ on $lip_\alpha(X)$ finishes the proof. □

Let X be a metric space. A subalgebra A of $C_b(X)$ (Banach algebra of bounded and continuous functions) is called strongly separating the points of X , if for each $x, y \in X$ with $x \neq y$, there exists $f \in A$ such that $f(x) \neq f(y)$.

Proposition 3.3. *Let G be a metric space which is a compact group and $\alpha > 0$. Suppose that $Lip_\alpha(G)$ is strongly separating the points of G . Let I be a non-zero closed ideal of $Lip_\alpha(G)$. Then $Lip_\alpha(G)$ is I -biprojective if and only if G finite.*

Proof. Since I is a non-zero closed ideal of $Lip_\alpha(G)$, semisimplicity of $Lip_\alpha(G)$ gives that there exists a non-zero multiplicative linear functional ϕ_g on $Lip_\alpha(G)$ such that $\phi_g|_I \neq 0$. By some modifications of the arguments as in Proposition 2.4, I -biprojectivity of $Lip_\alpha(G)$ implies that $Lip_\alpha(G)$ is left ϕ_g -contractible. So there exists $m \in Lip_\alpha(G)$ such that $fm = \phi_g(f)m$ and $\phi_g(m) = m(g) = 1$ for all $f \in Lip_\alpha(G)$. Let $y \neq g$ be an arbitrary element of G . Since $Lip_\alpha(G)$ is strongly separating the element of G , by [1, Proposition 2.1] there exists a $f_0 \in Lip_\alpha(G)$ such that $f_0(g) = 1$ and $f_0(y) = 0$. we know that $m = \phi_g(f_0)m = f_0m$. Thus $m(y) = f_0m(y) = f_0(y)m(y) = 0$ and $m(g) = f_0m(g) = f_0(g)m(g) = 1$. So $m = \chi_{\{g\}} \in Lip_\alpha(G)$, the characteristic function at g . Since m is a continuous function on G , it gives that G is discrete (and compact). Then G is finite.

The converse is similar to the only if part of previous Theorem. □

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