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Original Article

On biprojectivity and Connes biprojectivity of a dual Banach algebra with respect to a w^*-closed ideal

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ABSTRACT: In this paper, we introduce a notion of Connes biprojectivity for a dual Banach algebra A with respect to its w^*-closed ideal I, say I-Connes biprojectivity. Some Lipschitz algebras $Lip_\alpha(X)$ and some matrix algebras are studied under this new notion. Also, with some mild assumptions, the relation between I-Connes biprojectivity and left ϕ-contractibility is given, where ϕ is a w^*-continuous multiplicative linear functional on A. As an application, we characterize Connes biprojectivity of some Lipschitz algebras.

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1. Introduction and Preliminaries

The concept of amenability for Banach algebras were first introduced by B. E. Johnson [5]. A Banach algebra A is amenable if and only if there exists a bounded net (m_α) in $A \otimes_p A$ such that $a \cdot m_\alpha - m_\alpha \cdot a \rightarrow 0$ and $\pi_A(m_\alpha)a \rightarrow a$ for every $a \in A$, where $\pi_A : A \otimes_p A \rightarrow A$ is denoted for the product morphism($\pi_A(a \otimes b) = ab$, for all $a, b \in A$). Indeed for a locally compact group G, $L^1(G)$ (the measure algebra $M(G)$) is amenable if and only if G is amenable (G is discrete and amenable). Helemskii in [4] and [15] studied the structure of Banach algebras thorough the homological methods of Banach algebras. He defined the concepts biflatness and biprojectivity. In fact a Banach algebra A is biprojective, if there exists a bounded A-bimodule morphism $\rho : A \rightarrow A \otimes_p A$ such that $\pi_A \circ \rho(a) = a$, for all $a \in A$. It is known that for a locally compact group G, the group algebra $L^1(G)$ (the measure algebra $M(G)$) is biprojective if and only if G is compact (G is finite). For the history of amenability and homological properties of algebra, see [12].

There exists a class of Banach algebras which is called dual Banach algebras. This category of Banach algebras is defined by Runde [11]. Let A be a Banach algebra. Then a Banach A-bimodule E is called dual if there is a

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closed submodule E_* of E^* such that $E = (E_*)^*$. The Banach algebra A is called dual if it is dual as a Banach A-bimodule. A dual Banach A-bimodule E is normal if for each $x \in E$ the module maps $A \longrightarrow E$ by $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are w^*-w^*-continuous. Let A be a Banach algebra and let E be a Banach A-bimodule. A bounded linear map $D : A \to E$ is called a bounded derivation if $D(ab) = a \cdot D(b) + D(a) \cdot b$, for every $a, b \in A$. A bounded derivation $D : A \to E$ is called inner if there exists an element x in E such that $D(a) = a \cdot x - x \cdot a$ $(a \in A)$. A dual Banach algebra A is called Connes amenable if for every normal dual Banach A-bimodule E, every w^*-continuous derivation $D : A \longrightarrow E$ is inner. For a given dual Banach algebra A and a Banach A-bimodule E, $\sigma wc(E)$ denote the set of all elements $x \in E$ such that the module maps $A \to E$ by $a \mapsto a \cdot x$ and $a \mapsto x \cdot a$ are w^*-w-continuous. It is a closed submodule of E, see [11] and [13] for more details. Note that, since $\sigma wc(A_*) = A_*$, the adjoint of π_A maps A_* into $\sigma wc(A \otimes_p A)^*$. Therefore π_A^{**} drops to an A-bimodule morphism $\pi_{\sigma wc} : (\sigma wc(A \otimes_p A)^*)^* \longrightarrow A$.

A dual Banach algebra A is called Connes-biprojective if there exists a bounded A-bimodule morphism $\rho : A \longrightarrow (\sigma wc(A \otimes_p A)^*)^*$ such that $\pi_{\sigma wc} \circ \rho(a) = a$ for all $a \in A$. Shirinkalam and Pourabbas showed that a dual Banach algebra A is Connes amenable if and only if A is Connes-biprojective and it has an identity [16]. They characterized Connes-biprojectivity of the measure algebra $M(G)$ for a locally compact group G.

In this paper, we introduce the notion of I-Connes biprojective Banach algebras in the category of dual Banach algebras, where I is a w^*-closed ideal. Some matrix algebras and Lipschitz algebras are studied under this new notion. Also, with some mild assumptions, the relation between I-Connes biprojectivity and left ϕ-contractibility is given, where ϕ is a w^*-continuous multiplicative linear functional on A. As an application, we characterize Connes biprojectivity of some Lipschitz algebras.

Recently the notion of I-biprojectivity is given for Banach algebras. Let A be a Banach algebra and I be a closed ideal of A. Then A is called I-biprojective if there exists a bounded A-bimodule morphism $\rho : I \to A \otimes_p A$ such that $\pi_A \circ \rho(i) = i$ for all $i \in I$. For a locally compact group G, the measure algebra $M(G)$ is $L^1(G)$-biprojective if and only if G is compact [14].

Throughout this paper, $\Delta(A)$ $(\Delta_{w^*}(A))$ denotes the character space (w^*- character space) of A , that is, all non-zero ($w^* -$continuous) multiplicative linear functionals on A, respectively. Let $\phi \in \Delta(A)$. Then ϕ has a unique extension to A^{**} denoted by $\tilde{\phi}$ and defined by $\tilde{\phi}(F) = F(\phi)$, for every $F \in A^{**}$. Clearly, this extension remains to be a character on A^{**}. The projective tensor product $A \otimes_p A$ is a Banach A-bimodule by the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \qquad (b \otimes c) \cdot a = b \otimes ca, \qquad (a, b, c \in A).$$

Let X and Y be Banach A-bimodules. Then the map $T : X \to Y$ is called A-bimodule morphism if

$$T(a \cdot x) = a \cdot T(x), \qquad T(x \cdot a) = T(x) \cdot a, \qquad (a \in A, x \in X).$$

2. *I*-Connes biprojectivity

We commence this section with the definition of our new notion. We should remind that every w^*-closed ideal of a dual Banach algebra is also dual Banach algebra see [8, Lemma 2].

Definition 2.1. *Let I be a w^*-closed ideal of a dual Banach algebra A. We say that A is I-Connes biprojective, if there exists a bounded A-bimodule morphisms $\rho : I \to (\sigma wc((A \otimes_p A)^*)^*$ such that $\pi_{\sigma wc} \circ \rho(i) = i$ for all $i \in I$. We say that A is ideally Connes biprojective, if it is I-Connes biprojective for every w^*-closed ideal I of A.*

Remark 2.2. *In above definition, we can replace ρ with a bounded net of A-bimodule morphisms, say (ρ_α) from I into $\sigma wc((A \otimes_p A)^*)^*$ which $\pi_{\sigma wc} \circ \rho_\alpha(i) \xrightarrow{w^*} i$, for all $i \in I$. To see this, since (ρ_α) is a bounded net of A-bimodule morphisms, we have $(\rho_\alpha) \subseteq B(I, (\sigma wc((A \otimes_p A)^*)^*)$ (the set of bounded linear maps from I into $(\sigma wc((A \otimes_p A)^*)^*)$. On the other hand, on bounded sets the w^*-operator topology coinsides with the w^*-topology of $B(I, (\sigma wc((A \otimes_p A)^*)^*)$ where identified with $(I \otimes_p \sigma wc(A \otimes_p A)^*)^*$. It is known that the unit ball of $B(I, (\sigma wc((A \otimes_p A)^*)^*)$ is w^*-operator compact. Then (ρ_α) has a w^*-operator topology limit point say ρ. Thus*

$$\rho(i_1 i_2) = w^* - \lim \rho_\alpha(i_1 i_2) = w^* - \lim i_1 \cdot \rho_\alpha(i_2) = i_1 \cdot w^* - \lim \rho_\alpha(i_2) = i_1 \cdot \rho(i_2).$$

Similarly we have $\rho(i_1 i_2) = \rho(i_1) \cdot i_2$. Therefore ρ is a bounded A-bimodule morphism and

$$\pi_{\sigma wc} \circ \rho(i) = \pi_{\sigma wc}(w^* - \lim \rho_\alpha(i)) = w^* - \lim \pi_{\sigma wc} \rho_\alpha(i) = i.$$

So A is I-Connes biprojective.

Lemma 2.3. *Let A be a non-zero dual Banach algebra and $ab = ba = 0$ for all $a, b \in A$. Then A is not Connes biprojective.*

Proof. We assume in contradiction that A is Connes biprojective. Then there exists a bounded A-bimodule morphisms $\rho : A \to (\sigma wc(A \otimes_p A)^*)^*$ such that $\pi_{\sigma wc} \circ \rho(a) = a$ for each $a \in A$. It is known that there exists a net (u_α) in $A \otimes_p A$ such that $w^* - \lim \hat{u}_\alpha|_{\sigma wc(A \otimes_p A)^*} = \rho(a)$. Thus

$$a = \pi_{\sigma wc} \circ \rho(a) = w^* - \lim \pi_A(u_\alpha) = w^* - \lim 0 = 0.$$

It follows that $A = 0$ which is a contradiction. □

Let A be a Banach algebra and $\phi \in \Delta(A)$. Then A is called left ϕ-contractible if there exists a $m \in A$ such that $am = \phi(a)m$ and $\phi(m) = 1$ for all $a \in A$. For further information about this concept see [10].

Proposition 2.4. *Let A be a commutative dual Banach algebra and $\phi \in \Delta_{w^*}(A)$. Suppose that I is a w^*-closed ideal of A. If A is I-Connes biprojective, then A is left ϕ-contractible, provided that $\phi|_I \neq 0$.*

Proof. Let i_0 be an element of I such that $\phi(i_0) = 1$. Since A is I-Connes biprojective, there exists a bounded A-bimodule morphism $\rho : I \to (\sigma wc((A \otimes_p A)^*)^*$ such that $\pi_{\sigma wc} \circ \rho(i) = i$ for all $i \in I$. Put $m = \rho(i_0)$. One can see that $a \cdot m = m \cdot a$ and $\pi_{\sigma wc}(m)i = i$ for all $i \in I$ and $a \in A$. Define $T : A \otimes_p A \to A$ by $T(a \otimes b) = \phi(b)a$ for each $a, b \in A$. It is easy to see that

$$aT(x) = T(a \cdot x), \quad T(x \cdot a) = \phi(a)T(x) \quad \phi \circ T(x) = \phi \circ \pi_A(x) \qquad (a \in A, x \in X).$$

Since T^{**} is a w^*-continuous map, for each $x \in (A \otimes_p A)^{**}$ and $a \in A$, we have

$$aT^{**}(x) = T^{**}(a \cdot x), \quad T^{**}(x \cdot a) = \phi(a)T^{**}(x) \quad \tilde{\phi} \circ T^{**}(x) = \tilde{\phi} \circ \pi_A^{**}(x).$$

On the other hand, for all $a \in A$, $f \in A^*$ and $x \in A \otimes_p A$ consider

$$\langle x, a \cdot T^*(f)\rangle = \langle x \cdot a, T^*(f)\rangle = \langle T(x \cdot a), f\rangle = \phi(a)\langle T(x), f\rangle = \phi(a)\langle x, T^*(f)\rangle$$

also

$$\langle x, T^*(f) \cdot a\rangle = \langle a \cdot x, T^*(f)\rangle = \langle T(a \cdot x), f\rangle = \langle a \cdot T(x), f\rangle = \langle T(x), f \cdot a\rangle,$$

which follow that

$$a \cdot T^*(f) = \phi(a)T^*(f), \quad T^*(f) \cdot a = T^*(f \cdot a).$$

These last facts with the w^*-continuity of ϕ gives that $T^*(\sigma wc(A^*)) \subseteq \sigma wc(A \otimes_p A)^*$. Let $q : A^{**} \to \sigma wc((A)^*)^*$ be the quotient map. For each $a \in A$ and $f \in \sigma wc(A^*)$, we have

$$\begin{aligned}
\langle f \cdot a, q \circ T^{**}(m)\rangle = \langle f \cdot a, T^{**}(m)|_{\sigma wc(A)^*}\rangle &= \langle f \cdot a, T^{**}(m)\rangle \\
&= \langle f, a \cdot T^{**}(m)\rangle \\
&= \langle f, T^{**}(a \cdot m)\rangle \\
&= \langle T^*(f), a \cdot m\rangle \\
&= \langle T^*(f), m \cdot a\rangle \\
&= \langle f, T^{**}(m \cdot a)\rangle \\
&= \langle f, \phi(a)T^{**}(m)\rangle \\
&= \phi(a)\langle f, T^{**}(m)\rangle \\
&= \phi(a)\langle f, T^{**}(m)|_{\sigma wc(A)^*}\rangle \\
&= \phi(a)\langle f, q \circ T^{**}(m)\rangle.
\end{aligned}$$

It follows that $a \cdot q \circ T^{**}(m) = \phi(a)q \circ T^{**}(m)$ for all $a \in A$. Moreover we know that ϕ is a w^*-multiplicative linear functional. It follows that $\phi \in A_*$. On the other hand $A_* \subseteq \sigma wc(A)^*$. Then $\phi \in \sigma wc(A)^*$. Hence

$$\langle \phi, q \circ T^{**}(m)\rangle = \langle \phi, T^{**}(m)|_{\sigma wc(A)^*}\rangle = \langle \phi, T^{**}(m)\rangle = \langle \phi, \pi_A^{**}(m)\rangle = 1.$$

Thus it gives that A is left ϕ-contractible. □

Theorem 2.5. *Let A be a dual Banach algebra and $\phi \in \Delta_{w^*}(A)$. Suppose that I is a w^*-closed ideal which $\phi|_I \neq 0$ and $\overline{I \ker \phi|_I} = \ker \phi|_I$. If A is I-Connes biprojective, then A is left ϕ-contractible.*

Proof. Let A be I-Connes biprojective. Then there exists a bounded A-bimodule morphism $\rho : I \to (\sigma wc(A \otimes_p A)^*)^*$such that $\pi_{\sigma wc} \circ \rho(i) = i$ for all $i \in I$. Let i_0 be an element of I such that $\phi(i_0) = 1$. It is known that $\ker \phi$ is a closed ideal of A. Thus $\frac{A}{\ker \phi}$ is a Banach A-bimodule, naturally. We denote the identity map on A by id_A. Also $q : A \to \frac{A}{\ker \phi}$ is denoted for the quotient map. Define $id_A \otimes q : A \otimes_p A \to A \otimes_p \frac{A}{\ker \phi}$ by $id_A \otimes q(a \otimes b) = a \otimes (b + \ker \phi)$ for all $a, b \in A$. Clearly $id_A \otimes q$ is a bounded A-bimodule morphism. It implies that

$$(id_A \otimes q)^*(\sigma wc(A \otimes_p \frac{A}{\ker \phi})^*) \subseteq \sigma wc(A \otimes_p A)^*.$$

Using this fact, set

$$\theta : ((id_A \otimes q)^*|_{\sigma wc(A \otimes_p \frac{A}{\ker \phi})^*})^* : (\sigma wc(A \otimes_p A)^*)^* \to (\sigma wc(A \otimes_p \frac{A}{\ker \phi})^*)^*.$$

Clearly we observe that θ is a w^*-continuous A-bimodule morphism. Put

$$\eta = \theta \circ \rho : I \to (\sigma wc(A \otimes_p \frac{A}{\ker \phi})^*)^*.$$

We can see that η is a bounded A-bimodule morphism. Since $\overline{I \ker \phi|_I} = I$, we may assume that for each $l \in \ker \phi|_I$ there is $l_1 \in \ker \phi|_I$ and $i_1 \in I$ such that $l = i_1 l_1$. On the other hand we know that there exists a quotient map q from $(A \otimes_p A)^{**} \to (\sigma wc(A \otimes_p A)^*)^*$ and compose q with the embedding map from $A \otimes_p A$ into $(A \otimes_p A)^{**}$ gives a continuous A-bimodule map $\tau : A \otimes_p A \to (\sigma wc(A \otimes_p A)^*)^*$ which has a w^*-dense range. We denote $\overline{u}$ for $\tau(u) = \hat{u}|_{\sigma wc(A \otimes_p A)^*}$, where $u \in A \otimes_p A$ and $\hat{u}$ is the image of embedding map at u in $(A \otimes_p A)^{**}$. So for $\rho(i_1) \in (\sigma wc(A \otimes_p A)^*)^*$ there exists a net (u_α) in $A \otimes_p A$ which $w^* - \lim \overline{u}_\alpha = \rho(i_1)$. Applying the w^*-continuity of θ implies that

$$\begin{aligned}
\eta(l) = \theta \circ \rho(i_1 l_1) &= \theta(\rho(i_1) \cdot l_1) \\
&= \theta((w^* - \lim \overline{u}_\alpha) \cdot l_1) \\
&= w^* - \lim \theta(\overline{u}_\alpha \cdot l_1) \\
&= w^* - \lim((id_A \otimes q)^*|_{\sigma wc(A \otimes_p \frac{A}{\ker \phi})^*})^*(u_\alpha \cdot l_1) = 0,
\end{aligned}$$

the last equality holds because $q(l_1) = 0$. So $\eta(l) = 0$. So η induces a map from $\frac{A}{\ker \phi}$ into $(\sigma wc(A \otimes_p A)^*)^*$ which is a bounded A-bimodule morphism. Since $\phi \in \Delta_{w^*}(A)$, we denote $\overline{\phi} : \frac{A}{\ker \phi} \to \mathbb{C}$ for a character which is given by $\overline{\phi}(a + \ker \phi) = \phi(a)$ for all $a \in A$. Clearly $\overline{\phi}$ is a character. Put $id_A \otimes \overline{\phi} : A \otimes_p \frac{A}{\ker \phi} \to A$ which is defined by $id_A \otimes \overline{\phi}(a \otimes b + \ker \phi) = \phi(b)a$ for every $a, b \in A$. One can readily see that for each $f \in A^*$ and $a \in A$

$$(id_A \otimes \overline{\phi})^*(f) \cdot a = (id_A \otimes \overline{\phi})^*(f \cdot a), \quad a \cdot ((id_A \otimes \overline{\phi})^*(f)) = \phi(a)(id_A \otimes \overline{\phi})^*(f).$$

Using w^*-continuity of ϕ and $\sigma wc(A_*) = A_*$ implies that

$$(id_A \otimes \overline{\phi})(A_*) = (id_A \otimes \overline{\phi})(\sigma wc(A_*)) \subseteq \sigma wc(A \otimes_p \frac{A}{\ker \phi})^*.$$

It follows that

$$\psi = ((id_A \otimes \overline{\phi})|_{A_*})^* : (\sigma wc(A \otimes_p \frac{A}{\ker \phi})^*)^* \to A$$

is a w^*-continuous left A-module morphism. Set $y = \psi \circ \eta$. Hence y is a bounded left A-module morphism from $\frac{A}{\ker \phi}$ into A. Note that y is a non-zero map. To see this, we show that $\phi \circ \psi = \phi \circ \pi_{\sigma wc}$. Clearly for each $a, b \in A$, we have

$$\begin{aligned}
\phi \circ (id_A \otimes \overline{\phi}) \circ (id_A \otimes q)(a \otimes b) = \phi \circ (id_A \otimes \overline{\phi})(a \otimes (b + \ker \phi)) &= \phi(a\phi(b)) \\
&= \phi(a)\phi(b) \\
&= \phi \circ \pi_A.
\end{aligned}$$

On the other hand for each $v \in A \otimes_p \frac{A}{\ker \phi}$ we have $\psi(\hat{v}|_{(A \otimes_p \frac{A}{\ker \phi})^*}) = (id_A \otimes \overline{\phi})(v)$. Also, for each $u \in A \otimes_p A$, we have $\pi_{\sigma wc}(\overline{u}) = \pi_A(u)$. Let $m \in (\sigma wc(A \otimes_p A)^*)^*$. Then there exists a net (u_α) in $A \otimes_p A$ such that $m = w^* - \lim \overline{u}_\alpha$.

As we know that ϕ, θ, ψ and $\pi_{\sigma wc}$ are w^*-continuous maps. So

$$\begin{aligned}\phi\circ\psi\circ\theta(m)=\phi\circ\psi\circ\theta(w^*-\lim\overline{u}_\alpha)&=w^*-\lim\phi\circ\psi\circ\theta(\overline{u}_\alpha)\\&=w^*-\lim\phi\circ(id_A\otimes\overline{\phi})\circ(id_A\otimes q)(u_\alpha)\\&=w^*-\lim\phi\circ\pi_A(u_\alpha)\\&=w^*-\lim\phi\circ\pi_{\sigma wc}(\overline{u}_\alpha)\\&=\phi\circ\pi_{\sigma wc}(m).\end{aligned}$$

Thus for $i_0\in I\subseteq A$, we have

$$\begin{aligned}\phi\circ y(i_0+\ker\phi)=\phi\circ\psi\circ\eta(i_0+\ker\phi)&=\phi\circ\psi\circ\theta\circ\rho(i_0)\\&=\phi\circ\pi_{\sigma wc}\circ\rho(i_0)=\phi(i_0)=1.\end{aligned}$$

It implies that y is nonzero map as desired. Also for each $a\in A$, we have

$$\begin{aligned}ay(i_0+\ker\phi)=y(ai_0+\ker\phi)&=y(\phi(a)i_0+\ker\phi)\\&=\phi(a)y(i_0+\ker\phi).\end{aligned}$$

Hence A is left ϕ-contractible. □

We give a dual Banach algebra A with a w^*-closed ideal I which neither A nor I is Connes biprojective. But A is I-Connes biprojective.

Example 2.1. *Let* $A=\{\begin{pmatrix}a&b\\0&c\end{pmatrix}|a,b,c\in\mathbb{C}\}$ *and* $I=\{\begin{pmatrix}0&b\\0&0\end{pmatrix}|b\in\mathbb{C}\}$. *With matrix operations and the* ℓ^1*-norm, A becomes a dual Banach algebra and I becomes a* w^**-closed ideal of A. We assume in contradiction that A is Connes biprojective. Since A is unital by [16, Theorem 2.2] A is Connes amenable. Define* $\phi:A\to\mathbb{C}$ *by* $\phi(\begin{pmatrix}0&b\\0&c\end{pmatrix})=c$. *Clearly ϕ is a character on A. Put* $J=\{\begin{pmatrix}0&b\\0&c\end{pmatrix}|b,c\in\mathbb{C}\}$. *It is easy to verify that J is a* w^**-closed ideal of A which* $\phi|_J\neq 0$. *It is easy to see that Connes amenability of A implies that A is left ϕ-amenable (or A is left ϕ-contractible), see [7]. So by similar method as in [6, Lemma 3.1] we have J is left ϕ-contractible. That is there is an element* $m=\begin{pmatrix}0&b_0\\0&c_0\end{pmatrix}$ *in J such that* $jm=\phi(j)m$ *and* $\phi(m)=1$ *for each* $j\in J$, *where* $b_0,c_0\in\mathbb{C}$. *Suppose that* $j=\begin{pmatrix}0&j_1\\0&j_2\end{pmatrix}$ *is an arbitrary element of J, where j_1 and j_2 in $\mathbb{C}$. Thus*

$$jm=\begin{pmatrix}0&j_1\\0&j_2\end{pmatrix}\begin{pmatrix}0&b_0\\0&c_0\end{pmatrix}=\begin{pmatrix}0&j_1c_0\\0&j_2c_0\end{pmatrix}=\phi(j)m=j_2\begin{pmatrix}0&b_0\\0&c_0\end{pmatrix}=\begin{pmatrix}0&j_2b_0\\0&j_2c_0\end{pmatrix}$$

and $\phi(\begin{pmatrix}0&b_0\\0&c_0\end{pmatrix})=c_0=1$. *It follows that for each j_1 and j_2 in $\mathbb{C}$ we have* $j_2b_0=j_1$. *Put* $j_2=0$ *and* $j_1=1$. *Then contradiction reveals.*
Since for each element i_1 and i_2 in I, we have $i_1i_2=0$, *Lemma 2.3 follows that I is not Connes biprojective.*
To show that A is I-Connes biprojective define $\rho:I\to A\otimes_p A\subseteq(\sigma wc(A\otimes_p A)^*)^*$ *by*

$$\rho(\begin{pmatrix}0&b\\0&0\end{pmatrix})=\begin{pmatrix}0&b\\0&0\end{pmatrix}\otimes\begin{pmatrix}0&1\\0&1\end{pmatrix},\quad(b\in\mathbb{C})$$

Clearly ρ is a bounded A-bimodule morphism and $\pi_{\sigma wc}\circ\rho(i)=i$ *for all* $i\in I$.

3. Applications for Lipschitz algebras

Let X be a compact metric space and $\alpha>0$. The space of complex valued function on X is denoted by $Lip_\alpha(X)$ which

$$p_\alpha(f)=\sup\{\frac{|f(x)-f(y)|}{d(x,y)^\alpha}:x,y\in X,x\neq y\}$$

is finite. Also

$$\ell ip_\alpha(X)=\{f\in Lip_\alpha(X):\frac{|f(x)-f(y)|}{d(x,y)^\alpha}\to 0\quad\text{as}\quad d(x,y)\to 0\}.$$

Define

$$||f||_\alpha = ||f||_\infty + p_\alpha(f).$$

With $||\cdot||_\alpha$ and the pointwise operationes $Lip_\alpha(X)$ and $\ell ip_\alpha(X)$ become Banach algebras. It is known that for $0 < \alpha < 1$, $\ell ip_\alpha(X)^{**}$ is isometricaly isomorphism with $Lip_\alpha(X)$. Also $Lip_\alpha(X)$ and $\ell ip_\alpha(X)$ are Arens regular Banach algebras for more details, see [2]. Recently Minapour and Zivari-Kazmpour showed that $Lip_\alpha(X)$ is a dual Banach algebra, [9].

Theorem 3.1. *Let X be a compact metric space and $0 < \alpha < 1$. Then $Lip_\alpha(X)$ is Connes biprojective if and only if X is finite.*

Proof. Suppose that $Lip_\alpha(X)$ is Connes biprojective. Since $Lip_\alpha(X)$ posses an identity, by [16, Theorem 2.2] $Lip_\alpha(X)$ is Connes amenable. Thus $Lip_\alpha(X) \cong (\ell ip_\alpha(X))^{**}$ is Connes amenable. It is easy to see that $\ell ip_\alpha(X)$ is a closed ideal of $Lip_\alpha(X)$. Applying [12, Theorem 4.4.8] follows that $\ell ip_\alpha(X)$ is amenable. By the main result of [3] X is finite.
Converse is clear. □

A Banach algebra A is called biflat if there exists a bounded A-bimodule morphism $\rho : A \to (A \otimes_p A)^{**}$ such that $\pi_A^{**} \circ \rho(a) = a$ for each $a \in A$ [12].

Theorem 3.2. *Let X be a compact metric space and $0 < \alpha < 1$. Then $Lip_\alpha(X)$ is $\ell ip_\alpha(X)$-biprojective if and only if X is finite.*

Proof. Suppose that $Lip_\alpha(X)$ is $\ell ip_\alpha(X)$-biprojective. Then by [14, Lemma 3.5] $\ell ip_\alpha(X)$ is biflat. Clearly $\ell ip_\alpha(X)$ posses an identity. Thus $\ell ip_\alpha(X)$ is amenable. Applying [3] X is finite.
For converse, let X be finite. Then by [1, Corollary 2.2] $Lip_\alpha(X)$ seperates the point of X. Applying [1, Theorem 3.2] follows that $Lip_\alpha(X)$ is biprojective. Then there exists a bounded $Lip_\alpha(X)$-bimodule morphism ρ from $Lip_\alpha(X)$ into $Lip_\alpha(X) \otimes_p Lip_\alpha(X)$ such that $\pi_{Lip_\alpha(X)} \circ \rho(a) = a$ for all $a \in Lip_\alpha(X)$. Restrict ρ on $\ell ip_\alpha(X)$ finishes the proof. □

Let X be a metric space. A subalgebra A of $C_b(X)$ (Banach algebra of bounded and continuous functions) is called strongly separating the points of X, if for each $x, y \in x$ with $x \neq y$, there exists $f \in A$ such that $f(x) \neq f(y)$.

Proposition 3.3. *Let G be a metric space which is a compact group and $\alpha > 0$. Suppose that $Lip_\alpha(G)$ is strongly separating the points of G. Let I be a non-zero closed ideal of $Lip_\alpha(G)$. Then $Lip_\alpha(G)$ is I-biprojective if and only if G finite.*

Proof. Since I is a non-zero closed ideal of $Lip_\alpha(G)$, semisimplicity of $Lip_\alpha(G)$ gives that there exists a non-zero multiplicative linear functional ϕ_g on $Lip_\alpha(G)$ such that $\phi_g|_I \neq 0$. By some modifications of the arguments as in Proposition 2.4, I-biprojectivity of $Lip_\alpha(G)$ implies that $Lip_\alpha(G)$ is left ϕ_g-contractible. So there exists $m \in Lip_\alpha(G)$ such that $fm = \phi_g(f)m$ and $\phi_g(m) = m(g) = 1$ for all $f \in Lip_\alpha(G)$. Let $y \neq g$ be an arbitrary element of G. Since $Lip_\alpha(G)$ is strongly separating the element of G, by [1, Proposition 2.1] there exists a $f_0 \in Lip_\alpha(G)$ such that $f_0(g) = 1$ and $f_0(y) = 0$. we know that $m = \phi_g(f_0)m = f_0 m$. Thus $m(y) = f_0 m(y) = f_0(y)m(y) = 0$ and $m(g) = f_0 m(g) = f_0(g)m(g) = 1$. So $m = \chi_{\{g\}} \in Lip_\alpha(G)$, the charcteristic function at g. Since m is a continuous function on G, it gives that G is discrete (and compact). Then G is finite.
The converse is similar to the only if part of previous Theorem. □

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