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Original Article

A modification of Hardy-Littlewood maximal-function on Lie groups

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ABSTRACT: For a real-valued function f on a metric measure space (X, d, μ) the Hardy-Littlewood centered-ball maximal-function of f is given by the 'supremum-norm':

$$Mf(x) := \sup_{r>0} \frac{1}{\mu(\mathcal{B}_{x,r})} \int_{\mathcal{B}_{x,r}} |f| d\mu.$$

In this note, we replace the supremum-norm on parameters r by $\mathcal{L}_p$-norm with weight w on parameters r and define Hardy-Littlewood integral-function $I_{p,w}f$. It is shown that $I_{p,w}f$ converges pointwise to Mf as $p \to \infty$. Boundedness of the sublinear operator $I_{p,w}$ and continuity of the function $I_{p,w}f$ in case that X is a Lie group, d is a left-invariant metric, and μ is a left Haar-measure (resp. right Haar-measure) are studied.

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1. Introduction

Maximal sublinear operators and their boundedness properties are one of the main tools in various aspects of Fourier Analysis on Euclidean spaces $\mathbb{R}^n$ [9]. The prototype of these operators is the Hardy-Littlewood centered-ball maximal-function M defined by

$$Mf(x) := \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(y)| dy \tag{1}$$

for any locally integrable function f on $\mathbb{R}$. On other hand, there has been many attempts to extend various classical results of Fourier Analysis for general metric measure spaces and in particular for Riemannian manifolds and Lie groups. We only mention a few recent works with different flavors: [1, 3, 4, 5, 6, 8, 10]. One of the problems concerning such extensions, is to define appropriate maximal operators with good boundedness properties. In [7] we considered an abstract and unified approach to (1, 1)-weak type boundedness of Hardy-Littlewood maximal-function operators. The main idea of the present note is to replace 'supremum' in the definitions of maximal operators by

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appropriate integrals on parameter-spaces, in order to find some affable substitutes for maximal operators. In this note we apply this idea to Hardy-Littlewood maximal-function operator on metric measure spaces. For instance, our modified version of (1) becomes

$$I_{p,w}f(x) := \left(\int_0^\infty \Big(\frac{1}{2r} \int_{x-r}^{x+r} |f(y)|dy \Big)^p w(r)dr \right)^{\frac{1}{p}},$$

where w is an integrable function of r and $1 \leq p < \infty$. We call $I_{p,w}f$ the Hardy-Littlewood integral-function. In §2, we give the definition of integral-function operators $I_{p,w}$ and prove that $\lim_{p\to\infty} I_{p,w}f(x) = Mf(x)$. In §3 (resp. §4), we prove that for any (non-compact) Lie group G with a left-invariant metric and a left-invariant measure (resp. right-invariant measure) $I_{p,w}$ is $(\mathcal{L}_q(G), \mathcal{L}_q(G))$-bounded for $1 \leq p \leq q \leq \infty$ (and suitable w). We also show that $I_{p,w}f$ is almost everywhere continuous for $f \in \mathcal{L}_p(G)$ and $1 \leq p < \infty$.

Remark 1.1. *We hope to give some applications of methods and results of this manuscript in future works (see also Remark 2.7):*

(i) To compute explicitly maximal functions of some sort of functions on Euclidean spaces by using Theorem 2.4.
(ii) To prove some regularity properties of maximal functions similar to Theorems 2.6 and 3.4.
(iii) To find boundedness properties of maximal function operators on general Lie groups.
(iv) To define some useful new classes of function spaces.

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2. The main definition

Let X be a metric space with an unbounded distance function denoted by d. The open ball with center $x \in X$ and radius $r > 0$ is denoted by $\mathcal{B}_{x,r}$. We have the following easy lemma.

Lemma 2.1. *Let ν be a Borel measure on X which is finite on bounded subsets. Then the function $(x,r) \mapsto \nu(\mathcal{B}_{x,r})$ from $X \times (0,\infty)$ into $[0,\infty)$ is lower semi-continuous and the function $r \mapsto \nu(\mathcal{B}_{x,r})$ is left continuous. If for every x, r we have $\nu\{y : d(x,y) = r\} = 0$ (e.g. X is a Riemannian manifold and d, ν are the canonical metric and measure on X) then the function $(x,r) \mapsto \nu(\mathcal{B}_{x,r})$ is continuous.*

Proof. Let $(x_n)_n$ and $(r_n)_n$ be sequences respectively in X and $(0,\infty)$ such that $x_n \to x$ and $r_n \to r > 0$. We have $\cap_n \cup_{k\geq n} (\mathcal{B}_{x,r} \setminus \mathcal{B}_{x_k,r_k}) = \emptyset$ and hence $\nu(\mathcal{B}_{x,r} \setminus \mathcal{B}_{x_n,r_n}) \to 0$. Thus if $\epsilon > 0$ then for sufficiently large n we have

$$\nu(\mathcal{B}_{x,r}) - \epsilon < \nu(\mathcal{B}_{x,r} \cap \mathcal{B}_{x_n,r_n}) \leq \nu(\mathcal{B}_{x_n,r_n}).$$

This shows the desired lower semi-continuity. Since $r \mapsto \nu(\mathcal{B}_{x,r})$ is an increasing function, the lower semi-continuity implies the desired left continuity. We have

$$\cap_n \cup_{k\geq n} (\mathcal{B}_{x_k,r_k} \setminus \mathcal{B}_{x,r}) \subseteq \{y : d(x,y) = r\}.$$

Thus if $\nu\{y : d(x,y) = r\} = 0$ then $\nu(\mathcal{B}_{x_n,r_n} \setminus \mathcal{B}_{x,r}) \to 0$ and hence for sufficiently large n we have

$$\nu(\mathcal{B}_{x_n,r_n}) - \epsilon < \nu(\mathcal{B}_{x_n,r_n} \cap \mathcal{B}_{x,r}) \leq \nu(\mathcal{B}_{x,r}).$$

□

Let μ be a Borel measure on X with $\mu(X) = \infty$ and such that for any nonempty bounded open subset U of X, $0 < \mu(U) < \infty$. We denote by $\mathcal{F}_{loc}(X)$ the set of measurable functions f on X such that $\int_U |f|d\mu < \infty$ for every bounded Borel subset U. For any $f \in \mathcal{F}_{loc}(X)$ the averaging-function Af of f is defined by

$$Af : X \times (0,\infty) \to [0,\infty), \quad Af(x,r) := \frac{1}{\mu(\mathcal{B}_{x,r})} \int_{\mathcal{B}_{x,r}} |f|d\mu.$$

By Lemma 2.1 the functions $(x,r) \mapsto \int_{\mathcal{B}_{x,r}} |f|d\mu$ and $(x,r) \mapsto \frac{1}{\mu(\mathcal{B}_{x,r})}$ are measurable. Thus Af is measurable. The Hardy-Littlewood maximal-function Mf of f is a measurable function on X defined by

$$Mf : X \to [0,\infty], \quad Mf(x) := \sup_{r>0} Af(x,r).$$

Thus $Mf(x)$ is just the supremum-norm of the function $r \mapsto Af(x,r)$. Our main idea is to replace the supremum-norm by an $\mathcal{L}_p$-norm:

Definition 2.2. *Let X, d, μ be as above. Let w denote a nonnegative measurable function on $(0,\infty)$ with*

$$\|w\| := \int_0^\infty w(r)dr < \infty$$

and such that w is also almost everywhere nonzero. We call w a radius-weight. Denote by $\hat{w}$ the finite measure on $(0,\infty)$ with density w. For any $f \in \mathcal{F}_{loc}(X)$ the Hardy-Littlewood integral-function $I_{p,w}f$ of type (p,w), $1 \leq p \leq \infty$, is defined to be the measurable function on X given by

$$I_{p,w}f : X \to [0,\infty], \quad I_{p,w}f(x) := \|r \mapsto Af(x,r)\|_{\mathcal{L}_p(\hat{w})}.$$

More explicitly, for $1 \leq p < \infty$ we have

$$I_{p,w}f(x) := \Big(\int_0^\infty w(r)(Af(x,r))^p dr\Big)^{1/p}.$$

By Lemma 2.1, $r \mapsto Af(x,r)$ is left continuous. Thus for $p = \infty$ we have

$$I_{\infty,w}f(x) = Mf(x).$$

Since A is sublinear, $I_{p,w}$ is sublinear on $\mathcal{F}_{loc}(X)$.

Lemma 2.3. *Let θ be a finite measure on a measurable space T and let $\phi : T \to [0,\infty)$ be measurable. Then*

$$\|\phi\|_{\mathcal{L}_\infty(\theta)} = \lim_{p\to\infty} \|\phi\|_{\mathcal{L}_p(\theta)}.$$

Proof. We denote $\|\cdot\|_{\mathcal{L}_p(\theta)}$ by $\|\cdot\|_p$. Suppose $\|\phi\|_\infty < \infty$. Without loss of generality assume $\|\phi\|_\infty = 1$ and $\theta(T) = 1$. We have $\limsup_p \|\phi\|_p \leq 1$. For $\epsilon > 0$ let $S_\epsilon := \{t : \phi(t) > 1-\epsilon\}$. Then $(1-\epsilon)\theta(S_\epsilon)^{\frac{1}{p}} \leq \|\phi\|_p$. Since $\theta(S_\epsilon) \neq 0$ we have $(1-\epsilon) \leq \liminf_p \|\phi\|_p$, and hence $1 \leq \liminf_p \|\phi\|_p$. Thus $\|\phi\|_\infty = \lim_p \|\phi\|_p$. Now suppose $\|\phi\|_\infty = \infty$. Let $S'_n := \{t : \phi(t) \leq n\}$. By the first part of the proof we have $\|\phi|_{S'_n}\|_\infty = \lim_p \|\phi|_{S'_n}\|_p$. Thus $\|\phi|_{S'_n}\|_\infty \leq \liminf_p \|\phi\|_p$. Since $\sup_n \|\phi|_{S'_n}\|_\infty = \infty$ we have $\lim_p \|\phi\|_p = \infty$. □

Theorem 2.4. *For any $f \in \mathcal{F}_{loc}(X)$ and every $x \in X$ we have*

$$\lim_{p\to\infty} I_{p,w}f(x) = Mf(x).$$

Proof. It follows from Lemma 2.3, with $T = (0,\infty)$, $\theta = \hat{w}$, $\phi = Af(x,\cdot)$. □

Theorem 2.5. *For $q \in [1,\infty)$ the following statements are equivalent:*

(i) M is $(\mathcal{L}_q(X), \mathcal{L}_q(X))$-bounded.
(ii) The family $\{I_{p,w}\}_{1\leq p<\infty}$ is uniformly $(\mathcal{L}_q(X), \mathcal{L}_q(X))$-bounded.
(iii) There exists a sequence $(p_n)_n$ in $[1,\infty)$ such that $p_n \to \infty$ and such that the family $\{I_{p_n,w}\}_n$ is uniformly $(\mathcal{L}_q(X), \mathcal{L}_q(X))$-bounded.

Proof. Since $Af(x,r) \leq Mf(x)$ we have $I_{p,w}f(x) \leq \|w\|^{\frac{1}{p}} Mf(x)$. Thus $\|I_{p,w}f\|_{\mathcal{L}_q(X)} \leq \|w\|^{\frac{1}{p}} \|Mf\|_{\mathcal{L}_q(X)}$. This shows (i)⇒(ii). By Theorem 2.4 and Fatou's Lemma we have

$$\int_X (Mf)^q d\mu \leq \liminf_p \int_X (I_{p,w}f)^q d\mu.$$

This shows (iii)⇒(i). (ii)⇒(iii) is trivial. □

It is not hard to see that the statement of Theorem 2.5 is valid if the term '$(\mathcal{L}_q(X), \mathcal{L}_q(X))$-bounded' is replaced by '$(\mathcal{L}_q(X), \mathcal{L}_q(X))$-weak-bounded'. In the case that $X = \mathbb{R}^n$, d the standard Euclidean distance, and μ the n-dimensional Lebesgue-measure, it is well-known that M is $(\mathcal{L}_q(\mathbb{R}^n), \mathcal{L}_q(\mathbb{R}^n))$-bounded for $1 < q \leq \infty$ and also $(\mathcal{L}_1(\mathbb{R}^n), \mathcal{L}_1(\mathbb{R}^n))$-weak-bounded ([9]). Thus the latter statement is valid with M replaced by $I_{p,w}$. We will see from Theorem 3.3 that $I_{1,w}$ is also $(\mathcal{L}_1(\mathbb{R}^n), \mathcal{L}_1(\mathbb{R}^n))$-bounded. The proof of the next result follows from the definition of $I_{p,w}$, and is omitted.

Theorem 2.6. *In the case that $X = \mathbb{R}^n$, for any nonnegative Schwartz test-function f on $\mathbb{R}^n$ and every $p \in [1,\infty)$, $I_{p,w}f$ is continuously $[p]$ times differentiable, where $[p]$ denotes the greatest integer $\leq p$.*

We will see from Theorem 3.4 that for any $p \in [1,\infty)$ and every $f \in \mathcal{L}_p(\mathbb{R}^n)$, the function $I_{p,w}f$ is almost everywhere continuous.

Remark 2.7. *It is clear that the above formalism of 'replacing supremum-norm by $\mathcal{L}_p$-norm on parameter-space' may be applied to almost all maximal sublinear operators of any kind. One can also work in an abstract framework as in [7]. In this note we only consider the formalism for centered-ball Hardy-Littlewood maximal-function operators.*

3. $I_{p,w}$ on Lie groups (I)

With the notations X, d, μ, w as in §2, suppose that $X = G$ is a non-compact Lie group and suppose that d and $\mu = \lambda$ denote the distance function and the measure canonically induced by a left-invariant Riemannian metric on G. Thus d is a left-invariant metric and λ is a left Haar-measure on G. The space $\mathcal{F}_{loc}(\lambda) = \mathcal{F}_{loc}(G)$ coincides with the vector space of locally integrable functions on G with respect to λ. By Lemma 2.1, we know that for any $f \in \mathcal{F}_{loc}(\lambda)$ the function $Af : G \times (0, \infty) \to [0, \infty)$ is continuous.

Lemma 3.1. *For any $f \in \mathcal{F}_{loc}(\lambda), r \in (0, \infty), p \in [1, \infty)$ we have*

$$\|Af(\cdot, r)\|_{\mathcal{L}_p(\lambda)} \leq \Big(\frac{1}{\lambda(\mathcal{B}_{e,r})} \int_{\mathcal{B}_{e,r}} \Delta(y^{-1}) d\lambda(y)\Big)^{\frac{1}{p}} \|f\|_{\mathcal{L}_p(\lambda)}. \qquad (2)$$

Also we have $\|Af(\cdot, r)\|_{\mathcal{L}_\infty(\lambda)} \leq \|f\|_{\mathcal{L}_\infty(\lambda)}$.

Here Δ denotes the modular function of G ([2]), i.e. the unique mapping $\Delta : G \to (0, \infty)$ satisfying $\lambda(Bx) = \Delta(x)\lambda(B)$ for every $x \in G$ and every Borel subset B of G. Note that Δ is a continuous group-homomorphism. Thus, it follows from the relatively-compactness of $\mathcal{B}_{e,r}$ in G, that the integral in right-hand side of (2) is finite. For unimodular groups (e.g. abelian groups) $\Delta \equiv 1$. Thus for unimodular G, (2) becomes

$$\|Af(\cdot, r)\|_{\mathcal{L}_p(\lambda)} \leq \|f\|_{\mathcal{L}_p(\lambda)}.$$

Proof. Suppose that $f \geq 0$. For $1 \leq p < \infty$, by Jensen's inequality we have

$$\begin{aligned}
\int_G (Af(x,r))^p d\lambda(x) &\leq \int_G \Big(\frac{1}{\lambda(\mathcal{B}_{x,r})} \int_{\mathcal{B}_{x,r}} f^p(y) d\lambda(y)\Big) d\lambda(x) \\
&= \int_G \Big(\frac{1}{\lambda(x\mathcal{B}_{e,r})} \int_{\mathcal{B}_{e,r}} f^p(xy) d\lambda(y)\Big) d\lambda(x) \\
&= \frac{1}{\lambda(\mathcal{B}_{e,r})} \int_{\mathcal{B}_{e,r}} \Big(\int_G f^p(xy) d\lambda(x)\Big) d\lambda(y) \\
&= \frac{1}{\lambda(\mathcal{B}_{e,r})} \int_{\mathcal{B}_{e,r}} \Big(\Delta(y^{-1}) \int_G f^p(x) d\lambda(x)\Big) d\lambda(y) \\
&= \frac{\|f\|^p_{\mathcal{L}_p(\lambda)}}{\lambda(\mathcal{B}_{e,r})} \int_{\mathcal{B}_{e,r}} \Delta(y^{-1}) d\lambda(y).
\end{aligned}$$

The case $p = \infty$ is trivial. □

Definition 3.2. *With the above assumptions, the G-norm of any radius-weight w is denoted by $\|w\|_G$ and is defined by*

$$\|w\|_G := \int_0^\infty \frac{w(r)}{\lambda(\mathcal{B}_{e,r})} \Big(\int_{\mathcal{B}_{e,r}} \Delta(y^{-1}) d\lambda(y)\Big) dr.$$

If G is unimodular then we have $\|w\|_G = \|w\| < \infty$. It is clear that for any G there exist radius-weights with finite G-norm. For instance:

$$w(r) = \begin{cases} \dfrac{e^{-r^2} \lambda(\mathcal{B}_{e,r})}{\int_{\mathcal{B}_{e,r}} \Delta(y^{-1}) d\lambda(y)} & \text{if } 1 < \frac{1}{\lambda(\mathcal{B}_{e,r})} \int_{\mathcal{B}_{e,r}} \Delta(y^{-1}) d\lambda(y) \\ e^{-r^2} & \text{otherwise} \end{cases}$$

Theorem 3.3. *With assumptions of this section on G, suppose that w is a radius-weight with finite G-norm. Then we have*

$$\|I_{p,w} f\|_{\mathcal{L}_q(\lambda)} \leq \|w\|^{\frac{q-p}{qp}} \|w\|_G^{\frac{1}{q}} \|f\|_{\mathcal{L}_q(\lambda)}, \qquad (f \in \mathcal{F}_{loc}(\lambda), 1 \leq p \leq q \leq \infty).$$

(Note that, for $q = \infty$ we let $\frac{q-p}{qp} := \frac{1}{p}$ and $\frac{1}{q} := 0$.)

Proof. By Jensen's Inequality and Lemma 3.1, for $q \neq \infty$ we have

$$\begin{aligned}
\|I_{p,w}f\|^q_{\mathcal{L}_q(\lambda)} &= \int_G \Big(\int_0^\infty w(r)(Af(x,r))^p dr \Big)^{\frac{q}{p}} d\lambda(x) \\
&= \|w\|^{\frac{q}{p}} \int_G \Big(\int_0^\infty \frac{w(r)}{\|w\|}(Af(x,r))^p dr \Big)^{\frac{q}{p}} d\lambda(x) \\
&\leq \|w\|^{\frac{q}{p}} \int_G \int_0^\infty \frac{w(r)}{\|w\|}(Af(x,r))^q dr d\lambda(x) \\
&= \|w\|^{\frac{q-p}{p}} \int_0^\infty w(r) \Big(\int_G (Af(x,r))^q d\lambda(x) \Big) dr \\
&\leq \|w\|^{\frac{q-p}{p}} \int_0^\infty \frac{w(r)\|f\|^q_{\mathcal{L}_q(\lambda)}}{\lambda(\mathcal{B}_{e,r})} \Big(\int_{\mathcal{B}_{e,r}} \Delta(y^{-1}) d\lambda(y) \Big) dr \\
&= \|w\|^{\frac{q-p}{p}} \|w\|_G \|f\|^q_{\mathcal{L}_q(\lambda)}.
\end{aligned}$$

For $q = \infty$ the desired inequality is easily concluded. □

Theorem 3.4. *With assumptions of this section on G, suppose that w is an arbitrary radius-weight. Let $f \in \mathcal{L}_p(\lambda)$ with $1 \leq p < \infty$. Then for any $x \in G$ such that f is essentially bounded on a neighborhood of x, $I_{p,w}f$ is continuous at x. In particular, $I_{p,w}f$ is continuous almost everywhere.*

Proof. Without loss of generality, suppose that $f \geq 0$. Let $\epsilon > 0$ be arbitrary and fixed. Choose a positive real number a such that $\int_0^a w(r)dr < \epsilon$ and such that $M := \operatorname{ess\,sup} f|_{\mathcal{B}(x,2a)} < \infty$. Then, for any $y \in G$ with $d(x,y) < a$ we have

$$\begin{aligned}
\int_0^a w(r) \big| Af(x,r) - Af(y,r) \big|^p dr &= \int_0^a w(r) \Big| \frac{1}{\lambda(\mathcal{B}_{e,r})} \int_{\mathcal{B}_{e,r}} (f(xz) - f(yz)) d\lambda(z) \Big|^p dr \\
&\leq \int_0^a w(r) \Big(\frac{1}{\lambda(\mathcal{B}_{e,r})} \int_{\mathcal{B}_{e,r}} |f(xz) - f(yz)| d\lambda(z) \Big)^p dr \\
&\leq \int_0^a w(r) 2^p M^p dr < 2^p M^p \epsilon.
\end{aligned} \tag{3}$$

Choose a positive real number b such that $\frac{1}{\lambda(\mathcal{B}_{e,b})} < \epsilon$. Then, by Jensen's Inequality, for any $y \in G$ we have

$$\begin{aligned}
\int_b^\infty w(r) \big| Af(x,r) - Af(y,r) \big|^p dr &\leq \int_b^\infty w(r) \Big(\frac{1}{\lambda(\mathcal{B}_{e,r})} \int_{\mathcal{B}_{e,r}} |f(xz) - f(yz)| d\lambda(z) \Big)^p dr \\
&\leq \int_b^\infty \frac{w(r)}{\lambda(\mathcal{B}_{e,r})} \int_{\mathcal{B}_{e,r}} |f(xz) - f(yz)|^p d\lambda(z) dr \\
&\leq \int_b^\infty \frac{w(r)}{\lambda(\mathcal{B}_{e,r})} \int_G |f(x\cdot) - f(y\cdot)|^p d\lambda dr \\
&\leq \int_b^\infty \frac{w(r)}{\lambda(\mathcal{B}_{e,r})} 2^p \|f\|^p_{\mathcal{L}_p(\lambda)} dr < 2^p \|w\| \|f\|^p_{\mathcal{L}_p(\lambda)} \epsilon.
\end{aligned} \tag{4}$$

Since Af is continuous there exists $\delta > 0$ such that for any $y \in \mathcal{B}_{x,\delta}$:

$$|Af(x,r) - Af(y,r)| < \epsilon, \quad (a \leq r \leq b),$$

and hence

$$\int_a^b w(r) \big| Af(x,r) - Af(y,r) \big|^p dr \leq \|w\| \epsilon^p. \tag{5}$$

If $d(x,y) < \min\{a, \delta\}$ then by (3),(4),(5) we have

$$\begin{aligned}
\big| I_{p,w}f(x) - I_{p,w}f(y) \big|^p &\leq \int_0^\infty w(r) \big| Af(x,r) - Af(y,r) \big|^p dr \\
&\leq \big(2^p M^p \epsilon \big) + \big(\|w\| \epsilon^p \big) + \big(2^p \|w\| \|f\|^p_{\mathcal{L}_p(\lambda)} \epsilon \big).
\end{aligned}$$

The proof is complete. □

4. $I_{p,w}$ on Lie Groups (II)

With the notations X, d, μ, w as in §2, suppose that $X = G$ is a non-compact Lie group. Consider two Riemannian metrics on G such that one of them is left-invariant and another one is right-invariant, and such that the two metrics coincide on Lie-algebra of G. Let d denote the distance function on G induced by the left-invariant metric and let $\mu = \rho$ denote the measure on G induced by the right-invariant metric. Thus ρ is a right Haar-measure on G. If λ as in §3 denotes the measure induced by the left-invariant metric then we have $\lambda(B) = \rho(B^{-1})$ and $\rho(xB) = \Delta(x^{-1})\rho(B)$ for every $x \in G$ and Borel subset B of G.

Lemma 4.1. *For any $f \in \mathcal{F}_{loc}(\rho), r \in (0,\infty), p \in [1,\infty]$ we have*

$$\|Af(\cdot,r)\|_{\mathcal{L}_p(\rho)} \leq \|f\|_{\mathcal{L}_p(\rho)}.$$

.

Proof. Suppose that $f \geq 0$. For $1 \leq p < \infty$, by Jensen's inequality we have

$$\begin{aligned}
\int_G (Af(x,r))^p d\rho(x) &\leq \int_G \Big(\frac{1}{\rho(\mathcal{B}_{x,r})}\int_{\mathcal{B}_{x,r}} f^p(y)d\rho(y)\Big)d\rho(x)\\
&= \int_G \Big(\frac{1}{\Delta(x^{-1})\rho(\mathcal{B}_{e,r})}\int_{x\mathcal{B}_{e,r}} f^p(y)d\rho(y)\Big)d\rho(x)\\
&= \frac{1}{\rho(\mathcal{B}_{e,r})}\int_G\int_{\mathcal{B}_{e,r}} f^p(xy)d\rho(y)d\rho(x)\\
&= \frac{1}{\rho(\mathcal{B}_{e,r})}\int_{\mathcal{B}_{e,r}}\int_G f^p(xy)d\rho(x)d\rho(y)\\
&= \frac{1}{\rho(\mathcal{B}_{e,r})}\int_{\mathcal{B}_{e,r}} \|f\|^p_{\mathcal{L}_p(\rho)}d\rho(y)\\
&= \|f\|^p_{\mathcal{L}_p(\rho)}.
\end{aligned}$$

The case $p = \infty$ is trivial. □

The proof of the following theorem is omitted. It is similar to the proof of Theorem 3.3 but uses Lemma 4.1.

Theorem 4.2. *With assumptions of this section on G, suppose that w is an arbitrary radius-weight. We have*

$$\|I_{p,w}f\|_{\mathcal{L}_q(\rho)} \leq \|w\|^{\frac{1}{p}}\|f\|_{\mathcal{L}_q(\rho)}, \qquad (f \in \mathcal{F}_{loc}(\rho), 1 \leq p \leq q \leq \infty).$$

The statements of Theorem 3.4 remain valid with the new assumptions of this section on G. The proof is also similar to the proof of Theorem 3.4. The only thing that may need an explanation is the relevant modification of (4): We have $\|f\|^p_{\mathcal{L}_p(\rho)} = \Delta(x^{-1})\int_G f^p(x\cdot)d\rho$. Thus if we get y so close to x such that $\Delta(y) \leq 2\Delta(x)$ then we have

$$\begin{aligned}
\int_G |f(x\cdot) - f(y\cdot)|^p d\rho &= \|f(x\cdot) - f(y\cdot)\|^p_{\mathcal{L}_p(\rho)}\\
&\leq \Big(\|f(x\cdot)\|_{\mathcal{L}_p(\rho)} + \|f(y\cdot)\|_{\mathcal{L}_p(\rho)}\Big)^p\\
&= \Big(\Delta(x)^{\frac{1}{p}} + \Delta(y)^{\frac{1}{p}}\Big)^p\|f\|^p_{\mathcal{L}_p(\rho)}\\
&\leq 3^p\Delta(x)\|f\|^p_{\mathcal{L}_p(\rho)}.
\end{aligned}$$

Hence we replace the last line of (4) by

$$\leq \int_b^\infty \frac{w(r)}{\lambda(\mathcal{B}_{e,r})}3^p\Delta(x)\|f\|^p_{\mathcal{L}_p(\rho)}dr < 3^p\Delta(x)\|w\|\|f\|^p_{\mathcal{L}_p(\rho)}\epsilon.$$

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