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Original Article

On the CP exterior product of Lie algebras

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ABSTRACT: In this paper, under certain conditions, we show that the non-abelian CP exterior product distributes over direct product of Lie algebras. Then we present some properties about CP extension of Lie algebras.

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(*Dedicated to Professor Jamshid Moori*)

1. Introduction

During the study of continuous transformation groups in the end of 19th century, Sophus Lie found *Lie algebras* as a new algebraic structure. This new structure played an important role in 19th and 20th centuries mathematical physics (see [17, 23], for more information). *Lie theory* is studying objects like Lie algebras, Lie groups, Root systems, Weyl groups, Linear algebraic groups, etc. and some researches show its emphasis on modern mathematics. (see [5, 17] for more information). Furthermore, it is shown that one can associate a Lie algebra to a continuous Lie group. For example, Lazard introduced a correspondence between some groups and some Lie algebras. (see [16], for more information). So theories of groups and Lie algebras are structurally similar and many concepts related to groups are defined analogously to Lie algebras. In this paper we want to define the Bogomolov multipliers for Lie algebras. This concept is known for groups and it is a group-theoretical invariant introduced as an obstruction to a

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problem in algebraic geometry which is called the rationality problem. This problem can be stated in the following way. Let V be a faithful representation of a group G over a field K. Then G acts naturally on the field of rational functions $K(V)$. Now the rationality problem or Noether's problem can be stated as "is the field of G-invariant functions $K(V)^G$ is rational (purely transcendental) over K?" A question related to the above mentioned is whether there exist independent variables $x_1, ..., x_r$ such that $K(V)^G(x_1, ..., x_r)$ becomes a pure transcendental extension of K? Saltman in [21] gives some examples of groups of order p^9 for which the answer to the Noether's problem was negative, even when taking $K = \mathbb{C}$. He used the notion of the unramified cohomology group $H^2_{nr}(\mathbb{C}(V)^G, \mathbb{Q}/\mathbb{Z})$. Bogomolov in [4] proved that it is canonically isomorphic to

$$B_0(G) = \bigcap \ker\{res^A_G : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(A, \mathbb{Q}/\mathbb{Z})\},$$

where A is an abelian subgroup of G. The group $B_0(G)$ is a subgroup of the Schur multiplier $\mathcal{M}(G) = H^2(G, \mathbb{Q}/\mathbb{Z})$ and Kunyavskii in [15] named it the *Bogomolov multiplier* of G. Thus non-triviality of the Bogomolov multiplier leads to counter-examples to Noether's problem. But it is not always easy to calculate Bogomolov multipliers of groups. Moravec in [19] introduced an equivalent definition of the Bogomolov multiplier. In this sense, he used a notion of the non-abelian exterior square $G \wedge G$ of a group G to obtain a new description of the Bogomolov multiplier. He showed that if G is a finite group, then $B_0(G)$ is non-canonically isomorphic to $\mathrm{Hom}(\tilde{B}_0(G), \mathbb{Q}/\mathbb{Z})$, where the group $\tilde{B}_0(G)$ can be described as a section of the non-abelian exterior square of the group G. Also, he proved that $\tilde{B}_0(G) \cong \mathcal{M}(G)/\mathcal{M}_0(G)$, such that the Schur multiplier $\mathcal{M}(G)$ or the same $H^2(G, \mathbb{Q}/\mathbb{Z})$ is interpreted as the kernel of the commutator homomorphism $G \wedge G \to [G, G]$ given by $x \wedge y \to [x, y]$, and $\mathcal{M}_0(G)$ is the subgroup of $\mathcal{M}(G)$ defined as $\mathcal{M}_0(G) =< x \wedge y \mid [x, y] = 0,\ x, y \in G >$. Thus in the class of finite groups, $\tilde{B}_0(G)$ is non-canonically isomorphic to $B_0(G)$. With this definition and similar to the Schur multiplier, the Bogomolov multiplier can be explained as a measure of the extent to which relations among commutators in a group fail to be consequences of universal relation. Furthermore, Moravec's method relates the Bogomolov multiplier to the concept of commuting probability of a group and shows that the Bogomolov multiplier plays an important role in commutativity preserving central extensions of groups, that are famous cases in K-theory. Now, it is interesting that the analogous theory of commutativity preserving exterior product can be developed to the field of Lie theory. Recently in [2], we introduced a non abelian commutativity preserving exterior product, and the Bogomolov multiplier of Lie algebras. Then we investigated their properties. Moreover we computed the Bogomolov multiplier for some Lie algebras. In this paper we want to introduce some computations of commutativity preserving exterior product of Lie algebras.

2. The commutativity preserving non-abelian exterior product of Lie algebras

In this section, we intend to extend the results of [4, 6, 11, 10, 13, 15, 19] to the theory of Lie algebras. (See our recent article [2], for more information)

Definition 2.1. *[2] Let K be a Lie algebra and M and N be ideals of K. A function $h : M \times N \to K$, is called a Lie-$\tilde{B}_0$-pairing, if we have*

(i) $h(\lambda m, n) = h(m, \lambda n) = \lambda h(m, n)$,
(ii) $h(m + m', n) = h(m, n) + h(m', n)$,
(iii) $h(m, n + n') = h(m, n) + h(m, n')$,
(iv) $h([m, m'], n) = h(m, [m', n]) - h(m', [m, n])$,
(v) $h(m, [n, n']) = h([n', m], n) - h([n, m], n')$,
(vi) $h([n, m], [m', n']) = -[h(m, n), h(m', n')]$,
(vii) *If* $[m, n'] = 0$, *then* $h(m, n') = 0$,

for all $\lambda \in F$, $m, m' \in M$ and $n, n' \in N$.

Definition 2.2. *A Lie-$\tilde{B}_0$-pairing $h : M \times N \to L$ is called universal, if for any Lie-$\tilde{B}_0$-pairing $h' : M \times N \to L'$, there is a unique Lie homomorphism $\theta : L \to L'$ such that $\theta h = h'$.*

Also we extended the concept of CP exterior product in [19] to the theory of Lie algebras.

Definition 2.3. *[2] Let L be a Lie algebra and M and N be ideals of L. The CP exterior product $M \curlywedge N$ is the Lie algebra generated by all symbols $m \curlywedge n$ subject to the following relations*

(i) $\lambda(m \curlywedge n) = \lambda m \curlywedge n = m \curlywedge \lambda n$,
(ii) $(m + m') \curlywedge n = m \curlywedge n + m' \curlywedge n$,
(iii) $m \curlywedge (n + n') = m \curlywedge n + m \curlywedge n'$,

(iv) $[m,m'] \curlywedge n = m \curlywedge [m',n] - m' \curlywedge [m,n]$,
(v) $m \curlywedge [n,n'] = [n',m] \curlywedge n - [n,m] \curlywedge n'$,
(vi) $[(m \curlywedge n),(m' \curlywedge n')] = -[n,m] \curlywedge [m',n']$,
(vii) *If* $[m,n]=0$, *then* $m \curlywedge n = 0$,

for all $\lambda \in F$, $m,m' \in M$ *and* $n,n' \in N$.

In the case $M=N=L$, we call $L \curlywedge L$ the curly exterior product of L.

Proposition 2.4. *[2] The function* $h : M \times N \to M \curlywedge N$ *given by* $(m,n) \longmapsto m \curlywedge n$, *is a universal Lie-*$\tilde{B_0}$*-pairing.*

Theorem 2.5. *[2] Let* L *be a Lie algebra and* M *and* N *be ideals of* L. *Then we have*

$$M \curlywedge N \cong \frac{M \wedge N}{\mathcal{M}_0(M,N)},$$

where $\mathcal{M}_0(M,N) =< m \wedge n \mid m \in M,\ n \in N,\ [m,n]=0 >$.

It is known that $\kappa : M \times N \to [M,N]$ given by $(m,n) \longmapsto [m,n]$ is an exterior pairing. So for all $m \in M$ and $n \in N$, it induces a homomorphism $\tilde{\kappa} : M \wedge N \to [M,N]$, such that $\tilde{\kappa}(m \wedge n) = [m,n]$. Moreover, the kernel of $\tilde{\kappa}$ is denoted by $\mathcal{M}(M,N)$. It can easily seen that $\mathcal{M}_0(M,N) \leq \mathcal{M}(M,N)$, thus there is a homomorphism $\kappa^* : M \wedge N/\mathcal{M}_0(M,N) \to [M,N]$ given by $m \wedge n + \mathcal{M}_0(M,N) \longmapsto [m,n]$, with $\ker \kappa^* \cong \mathcal{M}(M,N)/\mathcal{M}_0(M,N)$. Similar to groups, we denote $\mathcal{M}(M,N)/\mathcal{M}_0(M,N)$ by $\tilde{B_0}(M,N)$, and we call it the Bogomolov multiplier of the pair of Lie algebras (M,N). Therefore, we have an exact sequence

$$0 \to \tilde{B_0}(M,N) \to M \curlywedge N \to [M,N] \to 0.$$

In the case $M=N=L$, $\mathcal{M}_0(L,L) =< l \wedge l' \mid l,l' \in L\ ,\ [l,l']=0 >$ and we denote it by $\mathcal{M}_0(L)$.

It is known that the kernel of $\tilde{\kappa} : L \wedge L \to L^2$ given by $l \wedge l' \longmapsto [l,l']$ is the Schur multiplier of L. On the other hand $\mathcal{M}_0(L) \leq \mathcal{M}(L) = \ker \tilde{\kappa}$. So there is a homomorphism $\kappa^* : L \wedge L/\mathcal{M}_0(L) \to L^2$ given by $l \wedge l' + \mathcal{M}_0(L) \longmapsto [l,l']$ and $\ker \kappa^* \cong \mathcal{M}(L)/\mathcal{M}_0(L)$. Similar to groups, we denote $\mathcal{M}(L)/\mathcal{M}_0(L)$ by $\tilde{B_0}(L)$, and we call it the Bogomolov multiplier of the Lie algebra L. So we have an exact sequence

$$0 \to \tilde{B_0}(L) \to L \curlywedge L \to L^2 \to 0.$$

3. Some computations of CP exterior product of Lie algebras

Here, we describe that under certain favourable conditions, the CP exterior product distributes over direct product.

Proposition 3.1. *Let* A,B,C *be Lie algebras, such that*

(i) $a^{(b,c)} = a^b + a^c$, *that is, the direct sum* $B \oplus C$ *acts on* A,
(ii) $(b,c)^a = (b^a, c^a)$, *that is,* A *acts on the direct sum* $B \oplus C$,
(iii) $(b',c')^{(b,c)} = (b'^b, c'^c)$, *that is, the direct sum* $B \oplus C$ *acts on* $B \oplus C$,
(iv) *there exist the trivial actions of* B *on* $A \curlywedge C$ *and* C *on* $A \curlywedge B$,

for all $a \in A$, $b,b' \in B$, $c,c' \in C$. *Then*

$$A \curlywedge (B \oplus C) \cong (A \curlywedge B) \oplus (A \curlywedge C)$$

Proof. Define

$$\alpha : A \times (B \oplus C) \to (A \curlywedge B) \oplus (A \curlywedge C)$$
$$(a,(b,0)) \longmapsto (a \curlywedge b, 0)$$
$$(a,(0,c)) \longmapsto (0, a \curlywedge c)$$

Therefore α is well-defined, and for all $a,a',a'' \in A, b,b',b'' \in B, c,c',c'' \in C$ and $\lambda \in F$ we have

$$\alpha(\lambda a,(b,0)) = (\lambda(a \curlywedge b),0) = \lambda(a \curlywedge b, 0) = \lambda\alpha(a,(b,0)),$$

and

$$\alpha(a,\lambda(b,0)) = \alpha(a,(\lambda b,0)) = (a \curlywedge \lambda b, 0) = \lambda\alpha(a,(b,0)).$$

Similarly, we have

$$\alpha(\lambda a,(0,c)) = \lambda\alpha(a,(0,c)) \quad , \quad \alpha(a,\lambda(0,c)) = \lambda\alpha(a,(0,c)).$$

Also,

$$\begin{aligned}\alpha(a+a',(b,0)) &= ((a+a') \curlywedge b, 0) = ((a \curlywedge b) + (a' \curlywedge b), 0)\\ &= (a \curlywedge b, 0) + (a' \curlywedge b, 0) = \alpha(a,(b,0)) + \alpha(a',(b,0))\end{aligned}$$

and

$$\begin{aligned}\alpha(a+a',(0,c)) &= (0,(a+a') \curlywedge c) = (0,(a \curlywedge c) + (a' \curlywedge c))\\ &= (0, a \curlywedge c) + (0, a' \curlywedge c) = \alpha(a,(0,c)) + \alpha(a',(0,c)).\end{aligned}$$

Moreover,

$$\begin{aligned}\alpha(a,(b,0)+(b',0)) &= \alpha(a,(b+b',0)) = (a \curlywedge (b+b'), 0)\\ &= ((a \curlywedge b) + (a \curlywedge b'), 0)\\ &= (a \curlywedge b, 0)(a^b \curlywedge b'^b, 0)\\ &= \alpha(a,(b,0)) + \alpha(a,(b',0))\end{aligned}$$

and

$$\begin{aligned}\alpha(a,(0,c)+(0,c')) &= \alpha(a,(0,c+c')) = (0, a \curlywedge (c+c'))\\ &= (0,(a \curlywedge c) + (a \curlywedge c'))\\ &= (0,(a \curlywedge c)) + (0, a \curlywedge c'))\\ &= \alpha(a,(0,c)) + \alpha(a,(0,c'))\end{aligned}$$

Also,

$$\begin{aligned}\alpha([a,a'],(b,0)) &= ([a,a'] \curlywedge b, 0) = (a \curlywedge [a',b] - a' \curlywedge [a,b], 0)\\ &= (a \curlywedge [a',b], 0) - (a' \curlywedge [a,b], 0).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\alpha(a,[a',(b,0)]) - \alpha(a',[a,(b,0)]) &= \alpha(a,[a',b]) - \alpha(a',[a,b])\\ &= \alpha(a,([a',b],0)) - \alpha(a',([a,b],0))\\ &= (a \curlywedge [a',b], 0) - (a' \curlywedge [a,b], 0).\end{aligned}$$

So,

$$\alpha([a,a'],(b,0)) = \alpha(a,[a',(b,0)]) - \alpha(a',[a,(b,0)]),$$

and

$$\begin{aligned}\alpha([a,a'],(0,c)) &= (0,[a,a'] \curlywedge c) = (0, a \curlywedge [a',c] - a' \curlywedge [a,c])\\ &= (0, a \curlywedge [a',c]) - (0, a' \curlywedge [a,c]).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\alpha(a,[a',(0,c)]) - \alpha(a',[a,(0,c)]) &= \alpha(a,[a',c]) - \alpha(a',[a,c])\\ &= \alpha(a,(0,[a',c])) - \alpha(a',(0,[a,c]))\\ &= (0, a \curlywedge [a',c]) - (0, a' \curlywedge [a,c]).\end{aligned}$$

So,

$$\alpha([a,a'],(0,c)) = \alpha(a,[a',(0,c)]) - \alpha(a',[a,(0,c)]).$$

In addition,

$$\begin{aligned}\alpha(a,[(b,0),(b',0)]) &= \alpha(a,([b,b'],0)) = (a \curlywedge [b,b'], 0)\\ &= ([b,b'] \curlywedge b - [b,a] \curlywedge b', 0) = ([b',a] \curlywedge b, 0) - ([b,a] \curlywedge b', 0)\\ &= \alpha([b',a],(b,0)) - \alpha([b,a],(b',0))\\ &= \alpha([(b',0),a],(b,0)) - \alpha([(b,0),a],(b',0)),\end{aligned}$$

and

$$\begin{aligned}\alpha(a,[(0,c),(0,c')]) &= \alpha(a,(0,[c,c'])) = (0, a \curlywedge [c,c'])\\ &= (0,[c',a]\curlywedge c - [c,a]\curlywedge c')\\ &= (0,[c',a]\curlywedge c) - (0,[c,a]\curlywedge c')\\ &= \alpha([c',a],(0,c)) - \alpha([c,a],(0,c'))\\ &= \alpha([(0,c'),a],(0,c)) - \alpha([(0,c),a],(0,c')).\end{aligned}$$

Moreover,

$$\begin{aligned}\alpha([a,(b,0)],[a',(b',0)]) &= \alpha([a,b],[a',b']) = \alpha([a,b],([a',b'],0))\\ &= ([a,b]\curlywedge[a',b'],0) = ([a\curlywedge b, a'\curlywedge b'],0)\\ &= ([b\curlywedge a, b'\curlywedge a'],0).\end{aligned}$$

On the other hand,

$$\begin{aligned}[\alpha(a,(b,0)),\alpha(a',(b',0))] &= [(a\curlywedge b,0),(a'\curlywedge b',0)] = [a\curlywedge b, a'\curlywedge b']\\ &= ([a\curlywedge b, a'\curlywedge b'],0) = ([b\curlywedge a, b'\curlywedge a'],0).\end{aligned}$$

Thus,

$$\alpha([a,(b,0)],[a',(b',0)]) = [\alpha(a,(b,0)),\alpha(a',(b',0))],$$

and

$$\begin{aligned}\alpha([a,(0,c)],[a',(0,c')]) &= \alpha([a,c],[a',c']) = \alpha([a,c],(0,[a',c']))\\ &= (0,[a,c]\curlywedge[a',c']) = (0,[a\curlywedge c, a'\curlywedge c'])\\ &= (0,[c\curlywedge a, c'\curlywedge a']).\end{aligned}$$

On the other hand,

$$\begin{aligned}[\alpha(a,(0,c)),\alpha(a',(0,c'))] &= [(0,a\curlywedge c),(0,a'\curlywedge c')] = [a\curlywedge c, a'\curlywedge c']\\ &= (0,[a\curlywedge c, a'\curlywedge c']) = (0,[c\curlywedge a, c'\curlywedge a']).\end{aligned}$$

So,

$$\alpha([a,(0,c)],[a',(0,c')]) = [\alpha(a,(0,c)),\alpha(a',(0,c'))].$$

Finally, if $[a,(b,0)] = 0$ then $[a,b] = 0$, and hence $a \curlywedge b = 0$. So,

$$\alpha(a,(0,c)) = (a\curlywedge c, 0) = 0,$$

and if $[a,(0,c)] = 0$ then $[a,c] = 0$, and hence $a\curlywedge c = 0$. So,

$$\alpha(a,(b,0)) = (a\curlywedge b,0) = 0.$$

Therefore α is Lie $\tilde{B}_0$-pairing map, and α determines a unique homomorphism of Lie algebras

$$\bar{\alpha} : A\curlywedge(B\oplus C) \to (A\curlywedge B)\oplus(A\curlywedge C)$$

$$(a\curlywedge(b,0)) \longmapsto (a\curlywedge b,0)$$

$$(a\curlywedge(0,c)) \longmapsto (0,a\curlywedge c),$$

for all $a\in A, b\in B, c\in C$.

Now, for introducing the inverse map of $\bar{\alpha}$, we define $\beta_1 : A\times B \to A\curlywedge(B\oplus C)$ given by $(a,b)\longmapsto a\curlywedge(b,0)$ and $\beta_2 : A\times C\to A\curlywedge(B\oplus C)$ given by $(a,c)\longmapsto a\curlywedge(0,c)$. The maps β_1 and β_2 are well-defined, and Lie $\tilde{B}_0$-pairing.

Therefore β_1 and β_2 determine homomorphisms $\bar{\beta_1} : A\curlywedge B \to A\curlywedge(B\oplus C)$ given by $(a\curlywedge b)\longmapsto a\curlywedge(b,0)$ and $\bar{\beta_2} : A\curlywedge C\to A\curlywedge(B\oplus C)$ given by $(a\curlywedge c)\longmapsto a\curlywedge(0,c)$, respectively. Also, $Im\bar{\beta_1}$ and $Im\bar{\beta_1}$ are ideals of $A\curlywedge(B\oplus C)$, and $Im\bar{\beta_1}\cap Im\bar{\beta_2} = a\curlywedge(0,0) = 0$. So, the images of $a\curlywedge b$ under $\bar{\beta_1}$ and of $a\curlywedge c$ under $\bar{\beta_2}$, commute. Therefore we can define a homomorphism β as

$$\bar{\beta} : (A\curlywedge B)\times(A\curlywedge C)\to A\curlywedge(B\oplus C)$$

$$(a \curlywedge b, 0) \longmapsto \bar{\beta}_1(a \curlywedge b)$$
$$(0, a \curlywedge c) \longmapsto \bar{\beta}_2(a \curlywedge c).$$

Also, $\bar{\alpha}\bar{\beta}$ and $\bar{\beta}\bar{\alpha}$ are identity maps. Hence

$$A \curlywedge (B \oplus C) \cong (A \curlywedge B) \oplus (A \curlywedge C).$$

□

The distributive law for curly exterior squares over direct products is a straightforward consequence of Proposition 3.1.

Lemma 3.2. *Let L_1 and L_2 be Lie algebras. Then*

$$(L_1 \oplus L_2) \curlywedge (L_1 \oplus L_2) \cong (L_1 \curlywedge L_1) \oplus (L_2 \curlywedge L_2).$$

Proof. Since L_1 and L_2 act trivially on each other, L_1 acts trivially on $L_2 \curlywedge L_2$ and $L_1 \curlywedge L_2$. So it acts trivially on $(L_1 \curlywedge L_2) \oplus (L_2 \curlywedge L_2) \cong (L_1 \oplus L_2) \curlywedge L_2$. Similarly L_2 acts on $(L_1 \oplus L_2) \curlywedge L_1$ trivially. Therefore all the conditions of the previous proposition are held, so

$$\begin{aligned}(L_1 \oplus L_2) \curlywedge (L_1 \oplus L_2) &\cong ((L_1 \oplus L_2) \curlywedge L_1) \oplus ((L_1 \oplus L_2) \curlywedge L_2)\\ &\cong (L_1 \curlywedge L_1) \oplus (L_2 \curlywedge L_1) \oplus (L_1 \curlywedge L_2) \times (L_2 \curlywedge L_2).\end{aligned}$$

Also, $L_1 \curlywedge L_2 = L_2 \curlywedge L_1 = 0$. Therefore

$$(L_1 \oplus L_2) \curlywedge (L_1 \oplus L_2) \cong (L_1 \curlywedge L_1) \oplus (L_2 \curlywedge L_2).$$

□

Lemma 3.3. *Let L_1 and L_2 be Lie algebras. Then*

$$\mathcal{M}_0(L_1 \oplus L_2) \cong \mathcal{M}_0(L_1) \oplus \mathcal{M}_0(L_2) \oplus {L_1}^{ab} \otimes {L_2}^{ab}.$$

Proof. Consider the following epimorphism

$$\eta : \mathcal{M}_0(L_1 \oplus L_2) \to \mathcal{M}_0(L_1) \oplus \mathcal{M}_0(L_2)$$
$$((l_1, l_2) \wedge (l_1', l_2')) \longmapsto (l_1 \wedge l_1', l_2 \wedge l_2').$$

We show that

$$\ker \eta = K =< g \wedge h \mid g \in L_1, h \in L_2 > .$$

Also, we know $\bigtriangledown(L_1, L_2) = 0$. Hence

$$\ker \eta = K =< g \otimes h \mid g \in L_1, h \in L_2 > .$$

It should be noted

$$\eta(g \otimes h) = \eta(g \wedge h) = \eta((g + 0) \wedge (0 + h)) = (g \wedge 0, 0 \wedge h) = 0.$$

Therefore $K \subseteq \ker \eta$, hence we have the following epimorphism

$$\phi : \mathcal{M}_0(L_1 \oplus L_2)/K \to \mathcal{M}_0(L_1) \oplus \mathcal{M}_0(L_2),$$

such that $\ker \phi = \ker \eta/K$. Now, we show that ϕ has a left inverse. We define

$$\Psi : \mathcal{M}_0(L_1) \oplus \mathcal{M}_0(L_2) \to \mathcal{M}_0(L_1 \oplus L_2)/K$$
$$(l_1 \wedge l_1', l_2 \wedge l_2') \longmapsto ((l_1, l_2) \wedge (l_1', l_2')) + K,$$

such that $[l_1, l_1'] = [l_2, l_2'] = 0$. One can see that $\Psi\Phi = 0$. So Φ is one-to-one and $\ker \eta/K = 1$. Thus $\ker \eta = K$ and

$$\mathcal{M}_0(L_1 \oplus L_2)/K \cong \mathcal{M}_0(L_1) \oplus \mathcal{M}_0(L_2).$$

Hence we have the following exact sequence

$$0 \to K \to \mathcal{M}_0(L_1 \oplus L_2) \to \mathcal{M}_0(L_1) \oplus \mathcal{M}_0(L_2) \to 0.$$

Now, we define

$$\eta' : \mathcal{M}_0(L_1) \oplus \mathcal{M}_0(L_2) \to \mathcal{M}_0(L_1 \oplus L_2)$$
$$(l_1 \wedge l_1', l_2 \wedge l_2') \longmapsto ((l_1, l_2) \wedge (l_1', l_2')),$$

such that $[l_1, l_1'] = [l_2, l_2'] = 0$. So $[l_1 l_2, l_1' l_2'] = 0$, η' is well-defined and $\eta'\eta = 0$. Therefore the above sequence splits, so

$$\mathcal{M}_0(L_1 \oplus L_2) \cong \mathcal{M}_0(L_1) \oplus \mathcal{M}_0(L_2) \oplus K,$$

in which

$$K =< g \otimes h \mid g \in L_1, h \in L_2 >\cong L_1 \otimes L_2 \cong {L_1}^{ab} \otimes {L_2}^{ab}.$$

□

3.1. Hopf-type formula for the Bogomolov multiplier of Lie algebras

Let L be a Lie algebra with a free presentation $L \cong F/R$. By the well-known Hopf formula [8], we have an isomorphism $\mathcal{M}(L) \cong (R \cap F^2)/[R, F]$. In [2] we gave the similar formula for $\tilde{B_0}(L)$. We proved the next proposition in [2], where $K(F)$ denotes $\{[x, y] \mid x, y \in F\}$.

Proposition 3.4. *[2] Let L be a Lie algebra with the free presentation $L \cong F/R$, then*

$$\tilde{B_0}(L) \cong \frac{R \cap F^2}{< K(F) \cap R >}.$$

4. CP extension of Lie algebras

For groups, the Schur multiplier is a universal object of central extensions. Recently, parallel to the classical theory of central extensions, Jezernik and Moravec in [11, 10] developed a version of extension that preserves commutativity. They showed that the Bogomolov multiplier is also the universal object parametrizing such extensions for a given group. Also in [2], we introduced a similar notion for Lie algebras.

Definition 4.1. *[2] Let L, M and C be Lie algebras. An exact sequence of Lie algebras $0 \to M \xrightarrow{\chi} C \xrightarrow{\pi} L \to 0$ is called a commutativity preserving extension (CP extension) of M by L, if commuting pairs of elements of L have commuting lifts in C. A special type of CP extension with the central kernel is named a central CP extension.*

Proposition 4.2. *[2] Let $e : 0 \to M \xrightarrow{\chi} C \xrightarrow{\pi} L \to 0$ be a central extension. Then e is a CP extension if and only if $\chi(M) \cap K(C) = 0$.*

Definition 4.3. *[2] An abelian ideal M of a Lie algebra L is called a CP Lie subalgebra of L if the extension $0 \to M \to L \to \dfrac{L}{M} \to 0$ is a CP extension.*

Moreover, by using Proposition 4.2 an abelian ideal M of a Lie algebra L is a CP Lie subalgebra of L if $M \cap K(L) = 0$.

Proposition 4.4. *Let L be a Lie algebra and N be a central Lie subalgebra of L. Then the following conditions are equivalent.*

1. *N is a CP Lie subalgebra of L,*
2. *The canonical map $\psi : \mathcal{M}_0(L) \to \mathcal{M}_0(L/N)$ is surjective,*
3. *The canonical map $\varphi : L \curlywedge L \to L/N \curlywedge L/N$ is an isomorphism.*

Proof. Let $L = F/R$ and $N = S/R$ be free presentation of L and N. We have

$$\psi : \frac{< K(F) \cap R >}{[R, F]} \to \frac{< K(F) \cap S >}{[S, F]}$$

$$x + [R, F] \mapsto x + [S, F].$$

Now, as N is CP Lie subalgebra of L, then by using the Proposition 4.2 and Definition 4.3, $N \cap K(L) = 0$. So, $S \cap K(F) \subseteq R$. Hence, $< K(F) \cap R >=< K(F) \cap S >$. Thus $Im\psi =< K(F) \cap S > /[S, F]$ and ψ is surjective. On the other hand, if ψ is surjective, then $< K(F) \cap S >=< K(F) \cap R >$. So, $K(F) \cap S \subseteq R$. Thus $N \cap K(L) = 0$ and N is CP Lie subalgebra of L. Therefore (i) and (ii) are equivalent.

Now, let N be a CP Lie subalgebra of L, then $N \cap K(L) = 0$. So, $S \cap K(F) \subseteq R$. By using Proposition 3.4, $L \curlywedge L \cong F^2 / < K(F) \cap R >$ and for all $x \in F^2$, we have

$$\varphi : \frac{F^2}{< K(F) \cap R >} \to \frac{F^2}{< K(F) \cap S >}$$

$$x+ < K(F) \cap R > \mapsto x+ < K(F) \cap S > .$$

Also,

$$\ker \varphi = \{x+ < K(F) \cap R > \mid x \in < K(F) \cap S >\} = \frac{< K(F) \cap S >}{< K(F) \cap R >}.$$

Since $S \cap K(F) \subseteq R$, then $< S \cap K(F) > \leq < R \cap K(F) >$. So $\ker \varphi = 1$ and φ is injective. Also, $Im\varphi = F^2 / < K(F) \cap S >$. Thus, φ is surjective. Hence φ is an isomorphism. On the other hand, if φ is an isomorphism, then $< K(F) \cap S > \leq < K(F) \cap R >$ and $K(F) \cap S \subseteq R$. Thus $N \cap K(L) = 0$. Hence N is a CP Lie subalgebra of L. □

Definition 4.5 ([1]). *Let C and $\tilde{B_0}$ be Lie algebras. We call a pair of Lie algebras $(C, \tilde{B_0})$, a commutativity preserving defining pair (CP defining pair) for L, if*

(i) $L \cong C/\tilde{B_0}$
(ii) $\tilde{B_0} \subseteq Z(C) \cap C^2$
(iii) $\tilde{B_0} \cap K(C) = 0$.

In other words, every stem central CP extension $0 \to \tilde{B_0} \to C \xrightarrow{\pi} L \to 0$ with $L \cong C/\tilde{B_0}$, $(C, \tilde{B_0})$ is termed a CP defining pair.

Lemma 4.6. *[1] Let L be a Lie algebra of finite dimension n and C be the first term in a CP defining pair for L. Then* $\dim C \leq n(n+1)/2$.

A pair $(C, \tilde{B_0})$ is called a maximal CP defining pair if the dimension of C is maximal.

Definition 4.7. *[1] For a maximal CP defining pair $(C, \tilde{B_0})$, C is called a commutativity preserving cover or (CP cover) for L.*

Note that in [1], we showed that if (K, C) is a maximal CP defining pair of L, then $K \cong \tilde{B_0}(L)$.

Lemma 4.8. *Let $0 \to N \xrightarrow{\chi} C \xrightarrow{\pi} L \to 0$ be a central CP extension. Then $\pi(Z(C)) = Z(L)$ and $Z(C) \cong N \oplus Z(L)$.*

Proof. Let $y \in \pi(Z(C))$, then there is a $z \in Z(C)$ such that $y = \pi(z)$. Also for every element $l \in L$, there exists $c' \in C$ such that $l = \pi(c')$. So we have

$$[y, l] = [\pi(z), \pi(c')] = \pi([z, c']) = \pi(0) = 0.$$

Thus $y \in Z(L)$ and $\pi(Z(C)) \subseteq Z(L)$. On other hand let $y \in Z(L)$, then for each element $l \in L$, $[y, l] = 0$. Also, there are $c, c' \in C$ such that $l = \pi(c')$ and $y = \pi(c)$. So, we have

$$[y, l] = [\pi(c), \pi(c')] = \pi([c, c']).$$

Hence, $\pi([c, c']) = 0$ and $[c, c'] \in \ker \pi = Im\chi = \chi(N) \cong N$. Since $\chi(N) \cap K(C) = 0$, then $[c, c'] = 0$. Thus $y \in \pi(Z(C))$. Hence $Z(L) \subseteq \pi(Z(C))$ and $\pi(Z(C)) = Z(L)$. Therefore the map $\pi|_{Z(C)} : Z(C) \to Z(L)$ is surjective. Hence $\frac{Z(C)}{\ker \pi \cap Z(C)} \cong Z(L)$. On other hand $0 \to N \xrightarrow{\chi} C \xrightarrow{\pi} L \to 0$ is a central CP extension, so $N \cong \ker \pi \subseteq Z(C)$. Thus $\frac{Z(C)}{\ker \pi} \cong Z(L)$ and $Z(C) \cong N \oplus Z(L)$. □

Lemma 4.9. *Let (N, C) be a maximal CP defining pair of L. Then $Z(C) \cong Z(L) \oplus \tilde{B_0}(L)$.*

Proof. The proof is completed by using Lemma 4.8 and the fact that $N \cong \tilde{B_0}(L)$. □

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