Original Article

A generalization of Taketa's theorem on M-groups II

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ABSTRACT: In the recent paper [A generalization of Taketa's theorem on M-groups, Quaestiones Mathematicae, (2022)], we give an upper bound $5/2$ for the average of non-monomial character degrees of a finite group G, denoted by $\mathrm{acd}_{nm}(G)$, which guarantees the solvability of G. Although the result is true, the example we gave to show that the bound is sharp turns out to be incorrect. In this paper we find a new bound and we give an example to show that this new bound is sharp. Indeed, we prove the solvability of G, by assuming $\mathrm{acd}_{nm}(G) < \mathrm{acd}_{nm}(\mathrm{SL}_2(5)) = 19/7$.

(Dedicated to Professor Jamshid Moori)

1. Introduction

This paper is a corrigendum to our paper [A generalization of Taketa's Theorem on M-groups, Quaestiones Mathematicae, (2022), https://doi.org/10.2989/16073606.2022.2081632], which will refer to as [1], hereafter. In [1], denoting by $\mathrm{acd}_{nm}(G)$ the average of non-monomial irreducible characters of G, we proved that if $\mathrm{acd}_{nm}(G) < 5/2$, then G is solvable. This result is true, however the bound is not sharp. In fact, after the paper was published, we found a mistake in calculating $\mathrm{acd}_{nm}(\mathrm{S}_5)$ which we have miscalculated to be $5/2$, while it is equal to $10/3 > 5/2$. Thus, however the proof of the main result in [1] is correct, the bound is not sharp. In this paper we aim to to find the best possible bound and prove the same result for this new bound. Indeed we prove the solvability of G, by assuming $\mathrm{acd}_{nm}(G) < \mathrm{acd}_{nm}(\mathrm{SL}_2(5)) = 19/7$.

Let N be a normal subgroup of G and $\lambda \in \mathrm{Irr}(N)$. Then $\mathrm{Irr}(G|N)$, $\mathrm{Irr}_{nm}(G|N)$ and $\mathrm{Irr}(G|\lambda)$ denote the set of irreducible characters of G whose kernels do not contain N, the set of the non-monomial irreducible characters of G whose kernels do not contain N and the set of the irreducible characters of G above λ, respectively. By $\mathrm{acd}_{nm}(G|N)$ we mean the average degree of irreducible characters in $\mathrm{Irr}_{nm}(G|N)$. We use the notation $n_d(G)$, $\mathfrak{N}_d(G)$, $n_d(G|N)$ and $\mathfrak{N}_d(G|N)$ for the number of irreducible characters of G of degree d, the number of non-monomial irreducible characters of G of degree d, the number of irreducible characters of G of degree d whose kernels do not contain N and

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the number of non-monomial irreducible characters of G of degree d whose kernels do not contain N, respectively. For the rest of notation, we follow [5].

2. Main Results

In the following we bring up some results and a generalized definition of a monomial group which is taken from [6] and [10].

Definition 2.1. *A non-linear character $\chi \in \mathrm{Irr}(G)$ is called a multiply imprimitive character (or m.i character for short) induced from the pair (U, λ) if there exist a proper subgroup U of G and an irreducible character $\lambda \in \mathrm{Irr}(U)$ such that $\lambda^G = m\chi$ for some nonnegative integer m.*

Clearly, if an irreducible character is not a *m.i* character, then it is primitive (in [10, Definition 3.2] they are called super-primitive character). The proof of the following Lemma is essentially taken from [6].

Lemma 2.2. *If S is a non-abelian simple group, then S has a non-linear irreducible character χ of degree at least 4 which is extendible to $\mathrm{Aut}(S)$ and χ is not an m.i character (χ is a super-primitive character). In particular, if $S \ncong \mathrm{A}_5$, then $\chi(1) \geq 5$.*

Proof. First, let S be a simple group of Lie type, then using [6, Lemma 3.8] the Steinberg character of S is not a *m.i* character and it is extendable to $\mathrm{Aut}(S)$ (see for example [4]), except for the cases $S \cong A_5$, $\mathrm{PSL}_2(7)$ or $S_4(3)$. One can check that in those remaining cases S has an irreducible character of degree 4, 6 and 2^6, respectively, which is not a *m.i* character and it is extendable to $\mathrm{Aut}(S)$ (see [3]), as wanted.

Next, let S be an alternating group of degree $n \geq 7$. By the proof of [6, Proposition 4.1], we see that S has an irreducible character χ of degree $n - 1 \geq 6$ which is extendable to $\mathrm{Aut}(S)$ and it is not a *m.i* character.

At last, let S be a sporadic simple group or the Tits group. Then according to the the proof of [6, Proposition 5.1] if for each S listed in the first column of Table 2 of [6] we take χ to be the character in the third column of Table 2 of [6], then χ satisfies the hypothesis of the Lemma. □

Lemma 2.3. *(See [2, Lemma 5]) Let N be a minimal normal subgroup of G such that $N = S_1 \times \cdots \times S_t$, where $S_i \cong S$, a non-abelian simple group. If $\sigma \in \mathrm{Irr}(S)$ extends to $\mathrm{Aut}(S)$, then $\sigma \times \cdots \times \sigma \in \mathrm{Irr}(N)$ extends to G.*

Lemma 2.4. *Assume $N \cong S^k$ is a non-abelian minimal normal subgroup of a finite group G, for some simple group S and integer k. Let either $\phi \in \mathrm{Irr}(G/N)$ be primitive and $\sigma \in \mathrm{Irr}(S)$ be a super-primitive character; or $\phi \in \mathrm{Irr}(G/N)$ be linear and $\sigma \in \mathrm{Irr}(S)$ be primitive character. If χ is an extension of the product of all different conjugate of σ to G, then $\chi\phi$ is primitive.*

Proof. Note that N and G satisfies [10, Hypothesis 1.1] and χ is an extension of θ the product of all distinct G-conjugate of σ (see the explanation before [10, Lemma 3.1]). By applying [10, Corollary 3.4], $\chi\phi$ is primitive. □

Lemma 2.5. *(See [8, Lemma 2.8].) Let $1 < N$ be a normal subgroup of G with $N = N'$, and suppose that a non-trivial $\theta \in \mathrm{Irr}(N)$ extends to $\chi \in \mathrm{Irr}(G)$. If $\chi = \lambda^G$, where $\lambda \in \mathrm{Irr}(H)$ for some subgroup H of G, then the following results are true:*

  (1) $HN = G$, and $\theta = (\lambda_{H \cap N})^N$, where $\lambda_{H \cap N}$ is irreducible.
  (2) If in addition $\theta(1)$ is minimal among all non-trivial irreducible character degrees of N, then χ is primitive.

Theorem 2.6. *Let G be a finite group. If $\mathrm{acd}_{nm}(G) < 19/7$, then G is solvable.*

Proof. Assume, on the contrary, G is an example with minimal order, such that G is non-solvable and $\mathrm{acd}_{nm}(G) < 19/7$. Thus,

$$\mathrm{acd}_{nm}(G) = \frac{\sum\limits_{\chi \in \mathrm{Irr}_{nm}(G)} \chi(1)}{|\mathrm{Irr}_{nm}(G)|} < 19/7.$$

Then, $\sum\limits_{\chi \in \mathrm{Irr}_{nm}(G)} \chi(1) = \sum\limits_{d \geq 1} d\mathfrak{N}_d(G)$ and $|\mathrm{Irr}_{nm}(G)| = \sum\limits_{d \geq 1} \mathfrak{N}_d(G)$. So by the above inequality we have

$$\sum_{d \geq 3} (7d - 19)\mathfrak{N}_d(G) < 12\mathfrak{N}_1(G) + 5\mathfrak{N}_2(G). \qquad (*)$$

First, we claim that there is no non-solvable minimal normal subgroup of G contained in G'. On the contrary, let $M \leq G'$ be a non-solvable minimal normal subgroup of G. Then M is a direct product of k copies of a non-abelian finite simple group S, for some integer k. By the hypothesis M is contained in the kernel of every linear character

of G. We show that M is contained in the kernel of every irreducible character of G of degree 2. Let $\chi \in \mathrm{Irr}(G)$ such that $\chi(1) = 2$. Since, non-abelian finite simple groups do not have any irreducible character of degree 2 and the only linear character of a simple group is the principle character, then $\chi_M = 2.1_M$. Therefore M lies in the kernel of χ, as wanted. Hence $n_d(G) = n_d(G/M)$, for $d = 1, 2$ and so $\mathfrak{N}_d(G) = \mathfrak{N}_d(G/M)$ for $d = 1, 2$.

By Lemmas 2.2, 2.3 and 2.4, M has a primitive irreducible character θ with degree $d_0 \geq 4$ which is extendable to a primitive irreducible character of G. Note that if $\phi \in \mathrm{Irr}_{nm}(G/M)$ and $\phi(1) = p$ for some prime p then ϕ is primitive. Hence the number of primitive characters of degree p of G/M, for some prime p, is $\mathfrak{N}_p(G/M)$. Then, by Lemma 2.4, we have $\mathfrak{N}_1(G) + \mathfrak{N}_2(G) = \mathfrak{N}_1(G/M) + \mathfrak{N}_2(G/M) \leq \mathfrak{N}_{d_0}(G|M) + \mathfrak{N}_{2d_0}(G|M)$ and so $\mathfrak{N}_1(G) + \mathfrak{N}_2(G) \leq \sum_{d_0|d} \mathfrak{N}_d(G|M)$.

Therefore,

$$\mathfrak{N}_1(G) + \mathfrak{N}_2(G) \leq \sum_{d_0|d} \mathfrak{N}_d(G).$$

Hence,

$$\sum_{d_0|d}(7d - 19)\mathfrak{N}_d(G) \geq (7d_0 - 19)(\mathfrak{N}_2(G) + \mathfrak{N}_1(G)). \qquad (**)$$

If $M \not\cong \mathrm{A}_5$, then $d_0 \geq 5$ by Lemma 2.2, and the non-equality $(**)$ contradicts $(*)$. Now, let $M \cong \mathrm{A}_5$ and set $C = C_G(M)$. We know that $MC/C \leq G/C \leq \mathrm{Aut}(MC/C) \cong \mathrm{S}_5$. First, let $G = MC \cong C \times M$. Then, M contains three non-linear primitive characters, two of degree 3 and one of degree 4. Note that in this case $(**)$ holds with $d_0 = 4$, which means that

$$\sum_{4|d}(7d - 19)\mathfrak{N}_d(G) \geq 9(\mathfrak{N}_2(G) + \mathfrak{N}_1(G)).$$

On the other hand, by Lemma 2.4, the extensions of irreducible characters of degree 3 of M to G are primitive, yielding that $\mathfrak{N}_3(G) \geq \mathfrak{N}_3(G|M) = 2\mathfrak{N}_1(G/M) = 2\mathfrak{N}_1(G)$. Hence,

$$\begin{aligned}
\sum_{d\geq 3}(7d - 19)\mathfrak{N}_d(G) &\geq \sum_{d\geq 4}(7d - 19)\mathfrak{N}_d(G) + 2\mathfrak{N}_3(G) \\
&\geq 9(\mathfrak{N}_2(G) + \mathfrak{N}_1(G)) + 4\mathfrak{N}_1(G) \\
&\geq 5\mathfrak{N}_2(G) + 12\mathfrak{N}_1(G),
\end{aligned}$$

which is contradicting $(*)$. So, we may assume $G \neq CM$, implying that $G/C \cong \mathrm{S}_5$. Recall that M contains a character $\chi \in \mathrm{Irr}(M)$ of degree 5 which is extendable to G. We show that all extensions of χ to G are primitive. On the contrary, assume $\chi_0 \in \mathrm{Irr}(G)$ is an extension of χ that is not primitive, which means that there exists a subgroup $H < G$ and a linear character $\eta \in \mathrm{Irr}(H)$ such that $\chi_0 = \eta^G$. Remark that $G/C \cong \mathrm{S}_5$ has a primitive extension of χ, say ψ. By Gallagher's theorem [5, Theorem 6.17], $\psi = \chi_0\lambda$ for some linear character $\lambda \in \mathrm{Irr}(G/M)$. As $\psi = \chi_0\lambda = \eta^G\lambda = (\eta\lambda_H)^G$, (see [5, Problem 5.3]), we get that ψ is not primitive, a contradiction. Hence, all extensions of χ to G are primitive. Thus, $\mathfrak{N}_5(G) \geq \mathfrak{N}_5(G|M) = \mathfrak{N}_1(G/M) = \mathfrak{N}_1(G)$. Again using, $(**)$, we have

$$\begin{aligned}
\sum_{d\geq 4}(7d - 19)\mathfrak{N}_d(G) &\geq \sum_{4|d}(7d - 19)\mathfrak{N}_d(G) + 16\mathfrak{N}_5(G) \\
&\geq 9(\mathfrak{N}_2(G) + \mathfrak{N}_1(G)) + 16\mathfrak{N}_1(G) \\
&\geq 5\mathfrak{N}_2(G) + 12\mathfrak{N}_1(G),
\end{aligned}$$

which is a contradiction.

Therefore, our claim is proved. Hence, we may assume that every minimal normal subgroup of G contained in G' is solvable. Let $M \trianglelefteq G$ be minimal such that M is non-solvable. Notice that M is a perfect group contained in the last term of derived series of G. Let $T \leq M$, such that T is a minimal normal subgroup of G. In addition, if $[M, R] \neq 1$, we assume $T \leq [M, R]$, where R is the radical solvable of M. Therefore $T \leq M' \leq G'$, so T is solvable and then G/T is non-solvable. As G is a counterexample of minimal order, we have $\mathrm{acd}_{nm}(G/T) \geq 19/7$ and so it follows by arguing exactly as in the first paragraph of the proof that

$$\sum_{d\geq 3}(7d - 19)\mathfrak{N}_d(G/T) \geq 5\mathfrak{N}_2(G/T) + 12\mathfrak{N}_1(G/T). \qquad (***)$$

Noting $\mathfrak{N}_1(G/T) = \mathfrak{N}_1(G)$, we get that $\mathfrak{N}_2(G/T) < \mathfrak{N}_2(G)$ from $(*)$ and $(***)$. Hence $\mathrm{Irr}_{nm}(G|T)$ contains a character of degree 2, say χ. If $K = \ker(\chi)$, then G/K is a primitive linear group of degree 2 (see [5, Chapter 14]). By the classification of the non-solvable primitive linear groups of degree 2 (see [5, Theorem 14.23]) we have

$G/C \cong A_5$, where $C/K = Z(G/K)$. This implies that $G = MC$. Recall that $M/(M \cap C) \cong MC/C = G/C$ and $M \cap C \trianglelefteq M$ is a proper subgroup of M. Therefore, $M \cap C$ is a subgroup of radical solvable subgroup of M by the minimality of M. Since $M/(M \cap C)$ is simple, we obtain that $M \cap C = R$, where R is the radical solvable subgroup of M, and hence $R \leq C$. Thus $[M, R] \leq K$. But $T \not\leq K$, so we have $T \not\leq [M, R]$. By the choice of T, we have $R = \mathbf{Z}(M)$. Therefore, M is a perfect central cover of the simple group $M/\mathbf{Z}(M) = M/(M \cap C) \cong G/C$. Since C and M are both normal in G, we have $[M, C] \leq C \cap M = \mathbf{Z}(M)$, and so $[C, M, M] = [M, C, M] = 1$. By the three subgroups lemma, we deduce that $[M, M, C] = 1$ and hence $[M, C] = 1$ as M is perfect. We conclude that $G = MC$ is a central product with a central subgroup $M \cap C = \mathbf{Z}(M) \neq 1$. Thus, by the choice of T, we get that $T = M \cap C = \mathbf{Z}(M)$. As, $\mathbf{Z}(M)$ lies in the Schur multiplier of $M/\mathbf{Z}(M) \cong A_5$, we deduce that $\mathbf{Z}(M) = T \cong C_2$. So $M \cong \mathrm{SL}_2(5)$ and $\mathrm{Irr}(G|T) = \mathrm{Irr}(G|\lambda)$, where λ is the only non-trivial character of T. By [3], $\mathrm{Irr}(M|\lambda)$ contains two primitive characters of degree 2, one primitive character of degree 4, and one character of degree 6 and $\mathrm{Irr}_{nm}(G|M)$ contains a primitive character χ of degree 2, which is an extension of one of the irreducible characters of degree 2 of M. By [7, Lemma 2], $\chi(1) = \beta(1)\alpha(1)$ where $\beta \in \mathrm{Irr}(C|T)$ and $\alpha \in \mathrm{Irr}(M|T)$. As $\mathrm{Irr}(M|T)$ does not contain any linear character, β is a linear character which means λ extends to C. Applying [7, Lemma 2], for every $\psi \in \mathrm{Irr}(M|T)$, we have $\mathrm{Irr}(G|T)$ contains a character of degree $\psi(1)\beta(1)$, which clearly is the extension of ψ. By Lemma 2.5(1), we deduce that every extension of ψ is primitive, if $\psi(1) \in \{2, 4\}$. Therefore, $\mathfrak{N}_4(G|T) = \mathfrak{N}_1(G/M) = n_1(G/M)$ and $\mathfrak{N}_2(G|T) = 2\mathfrak{N}_1(G/M) = 2n_1(G/M)$. Then

$$\mathrm{acd}_{nm}(G|T) = \frac{4n_1(G/M) + 4n_1(G/M) + \sum\limits_{d \geq 6} d\mathfrak{N}_d(G|T)}{2n_1(G/M) + n_1(G/M) + \sum\limits_{d \geq 6} \mathfrak{N}_d(G|T)}$$

$$= \frac{8n_1(G/M) + \sum\limits_{d \geq 6} d\mathfrak{N}_d(G|T)}{3n_1(G/M) + \sum\limits_{d \geq 6} \mathfrak{N}_d(G|T)}.$$

On the other hand, $G/T \cong C/T \times M/T \cong C/T \times \mathrm{PSL}_2(5)$. Then using Lemma 2.5(1) all irreducible characters $\lambda \times \mu \in \mathrm{Irr}(C/T \times \mathrm{PSL}_2(5))$ are non-monomial, provided that $\mu \in \mathrm{Irr}(\mathrm{PSL}_2(5))$ is a non-monomial character and λ is a linear character of C/T, which means μ is one of those irreducible characters of $\mathrm{PSL}_2(5)$ of degree 3 or 4. Also, by Lemma 2.4, if μ is the only super-primitive character of M/T of degree 4 and λ is a primitive character of C/T, then $\lambda \times \mu$ is primitive. We denote by $\mathfrak{M}_d(G/T)$ the number of non-monomial irreducible characters of G/T of degree d in form of $\lambda \times \mu$, where either $\mu \in \mathrm{Irr}(\mathrm{PSL}_2(5))$ has degree 1 or 5; μ has degree 3 and λ is not linear; or μ has order 4 and $\lambda(1) > 2$. Clearly $\mathfrak{M}_d(G/T) = \mathfrak{N}_d(G/T) = \mathfrak{N}_d(G/M)$ for $d = 1, 2$ and $\mathfrak{N}_1(G/M) = n_1(G/M)$. Therefore, by the above argument,

$$\mathrm{acd}_{nm}(G) = \frac{\sum\limits_{d \geq 1} d\mathfrak{N}_d(G|T) + \sum\limits_{d \geq 1} d\mathfrak{N}_d(G/T)}{|\mathrm{Irr}_{nm}(G|T)| + |\mathrm{Irr}_{nm}(G/T)|}$$

$$= \frac{10n_1(G/M) + 8\mathfrak{N}_2(G/M) + \sum\limits_{1 \leq d \leq 2} d\mathfrak{M}_d(G/M) + \sum\limits_{d \geq 3} d\mathfrak{M}_d(G/T) + 8n_1(G/M) + \sum\limits_{d \geq 6} d\mathfrak{N}_d(G|T)}{3n_1(G/M) + \mathfrak{N}_2(G/M) + \sum\limits_{1 \leq d \leq 2} \mathfrak{N}_d(G/M) + \sum\limits_{d \geq 3} \mathfrak{M}_d(G/T) + 3n_1(G/M) + \sum\limits_{d \geq 6} \mathfrak{N}_d(G|T)}$$

$$= \frac{19n_1(G/M) + 10\mathfrak{N}_2(G/M) + \sum\limits_{d \geq 3} d\mathfrak{M}_d(G/T) + \sum\limits_{d \geq 6} d\mathfrak{N}_d(G|T)}{7n_1(G/M) + 2\mathfrak{N}_2(G/M) + \sum\limits_{d \geq 3} \mathfrak{M}_d(G/T) + \sum\limits_{d \geq 6} \mathfrak{N}_d(G|T)}.$$

Therefore,

$$7\left(19n_1(G/M) + 10\mathfrak{N}_2(G/M) + \sum\limits_{d \geq 3} d\mathfrak{M}_d(G/T) + \sum\limits_{d \geq 6} d\mathfrak{N}_d(G|T)\right)$$

$$\geq 7\left(19n_1(G/M) + 10\mathfrak{N}_2(G/M) + \sum\limits_{d \geq 3} 3\mathfrak{M}_d(G/T) + \sum\limits_{d \geq 6} 6\mathfrak{N}_d(G|T)\right)$$

$$\geq 19\left(7n_1(G/M) + 2\mathfrak{N}_2(G/M) + \sum\limits_{d \geq 3} \mathfrak{M}_d(G/T) + \sum\limits_{d \geq 6} \mathfrak{N}_d(G|T)\right),$$

which means $\mathrm{acd}_{nm}(G) \geq 19/7$. This contradiction proves the theorem. □

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