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Original Article

Non-classical symmetry and new exact solutions of the Kudryashov-Sinelshchikov and modified KdV-ZK equations

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ABSTRACT: In this paper, by applying the non-classical symmetry method, non-classical symmetries of the Kodryashov-Sinleschikov (K-S) and modified Korteweg-de Vries-Zaharov-Kuznetsov (mKdV-ZK) equations are obtained. Apart from classical symmetries, this theory can be effective in finding a few other solutions for a system of PDEs and ODEs. Also, non-classical symmetries of a system of PDEs can be applied to reduce the number of independent variables. By adding the invariance surface condition to the assumed equations and applying the classical symmetry method for them, non-classical symmetries are calculated. Finally, some of the group invariant solutions and the similarity reduced equations associated to non-classical symmetry are obtained.

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1. Introduction

The study of nonlinear evolution equations has made significant progress in recent years. This is due to the importance of these equations in various sciences including physics and engineering sciences [4, 10, 23]. It is noteworthy that solving these equations is often not easy. The Lie symmetry method, known as the classical Lie method, was initially introduced by Sophus Lie in the 19th century [17]. By this method, we can reduce the order of ordinary differential equations and also reduce PDEs and convert them to ODEs in certain cases [3, 25]. Also, by having symmetry and a specific solution of the equation, we can achieve a wide range of solutions. Indeed, the application of Lie symmetry group theory in solving PDE systems have been one of the widely used branches of study in analyzing and solving these equations [1, 2, 11, 13, 18, 26]. Since Lie classical symmetry method is not able to find all similarity reductions for PDE equations, the motivation for new generalizations of this method arose. The

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history of the non-classical method for similarity reductions goes back to the research of Bluman and Cole in 1969 to obtained another exact solutions for the heat equation [6]. The interpretation of this method is mentioned in [8]. Also, a new approach to detection of non-classical symmetries was propound by Bîlă and Niesen in [5]. Over the past three decades, the theory of non-classical symmetry has been extensively studied, and non-classical symmetry techniques have been applied to find accurate solutions to many partial differential equations derived from physics, mechanics, engineering and so on [12, 19, 24, 27]. Clearly, reducing the number of independent variables of nonlinear equations has been the most significant application of classical and non-classical symmetry groups on these PDEs. In this way, PDEs are converted to ODEs. Also, some symmetries may not reduce the order of the equations. So other exact solutions are obtained by integration [20]. The number of determining equations in non-classical method are less than of classical method, so the number of non-classical symmetries is much more than the classical symmetries. Therefore, any classical symmetry is considered a non-classical symmetry. An important feature of the determining equations corresponding to non-classical symmetries is their nonlinearity. That is, the space of non-classical symmetries is generally not a vector space. In addition, Lie bracket of non-classical vector fields are not usually a non-classical symmetry. In this paper, by applying the above method, non-classical symmetries of the Kodryashov-Sinleschikov equation (K-S)

$$u_t + auu_x + bu_{3x} + cu_{4x} + du_{5x} = eu_{2x}, \tag{1}$$

and modified Korteweg-de Vries-Zakharov-Kuznetsov equation (mKdV-ZK)

$$u_t + \alpha u^2 u_x + u_{xxx} + u_{xyy} + u_{xzz} = 0, \tag{2}$$

are obtained. Also, some of the group invariant solutions and the similarity reduced equations associated to non-classical symmetry are obtained.
This article is set as follows. Section 2, is dedicated to recalling the principal definitions and theorems that are helpful in the after sections. The non-classical symmetries, Lie invariants, similarity solutions and reduced equations of the K-S and mKdV-ZK equations are calculated in sections 3 and 4, respectively.

2. Preliminaries

Suppose $\Delta(x, u^{(n)}) = 0$ be an n'th order system of differential equations consisting of p independent variable $x = (x^1, \ldots, x^p)$ and q dependent variable $u = (u^1, \ldots, u^q)$ where $u^{(i)}$'s denotes all the derivatives of u from order 0 to n. The one-parameter Lie group of infinitesimal symmetries given by

$$(\tilde{x}^i, \tilde{u}^\alpha) \longrightarrow (x^i, u^\alpha) + \varepsilon(\zeta^i(x,u), \varphi^\alpha(x,u)) + O(\varepsilon^2),$$

where $\varepsilon \ll 1$ is a small group parameter and ζ^i and φ^α are the coefficients of infinitesimal transformation. The infinitesimal symmetry operator is defined of the form

$$\mathbf{X} = \sum_{i=1}^{p} \zeta^i(x,u)\partial_{x^i} + \sum_{\alpha=1}^{q} \varphi^\alpha(x,u)\partial_{u^\alpha}, \qquad \zeta^i, \varphi^\alpha \in \mathbb{A}, \tag{3}$$

where $\mathbb{A}$ is the space of differential functions. Also, the Lie characteristic functions of $\mathbf{X}$ described by

$$Q^\alpha(x, u^{(n)}) = \varphi^\alpha(x,u) - \sum_{i=1}^{p} \zeta^i(x,u)\frac{\partial u^\alpha}{\partial x^i}, \quad \alpha = 1, \cdots, q. \tag{4}$$

The n'th prolongation of $\mathbf{X}$ is defined explicitly as

$$\mathbf{X}^{(n)} = \sum_{i=1}^{p} \zeta^i(x,u)\partial_{x^i} + \sum_{\alpha=1}^{q}\sum_{J} \varphi_J^\alpha(x, u^{(n)})\partial_{u_J^\alpha}, \tag{5}$$

where its coefficients are $\varphi^\alpha_{J,i} = D_i\varphi^\alpha_J - \sum\limits_{J=1}^{p} D_i\zeta^J u^\alpha_{J,i}$. Here, $J = (j_1, \cdots, j_k)$, with $1 \le j_k \le p$. The total differential operator with respect to x^i is determined as

$$D_i = \partial/\partial x_i + \sum_{\alpha=1}^{q}\sum_{J} u^\alpha_{J,i}\partial/\partial u^\alpha_J.$$

Also, by (Theorem 2.36 of [22]), $\mathbf{X}^{(n)}$ justify the invariance criterion, that is,

$$\mathbf{X}^{(n)}(\Delta) \equiv 0 \;\; (\text{mod } \Delta = 0).$$

3. Non-Classical Symmetries of the K-S Equation

The Kudryashov-Sinelshchikov (K-S) equation plays a dominant role in studying the process of nonlinear waves in viscoelastic tubes [16]. The K-S equation is

$$u_t + auu_x + bu_{3x} + cu_{4x} + du_{5x} = eu_{2x}, \tag{6}$$

where a, b, c, d and e are non-singular constants. Classical Lie symmetries and some group invariant solutions of this equation have been already obtained, as well as the optimal system of subalgebras and the classification of its invariant solutions by Nadjafikhah and Shirvani [21]. We now compute the non-classical symmetries of the K-S equation that similar to the method applied by Cai and Guoliang et al., to find the non-classical symmetries of the Burgers-Fisher equation [7].

This method is constructed by adding invariance surface condition to the assumed equation and then applying the classical symmetry method to it. This concept can be clearly expressed as follows for (6),

$$X^{(5)}\Delta_1|_{\Delta_1\cap\Delta_2=0} \equiv 0. \tag{7}$$

Assume the symmetry generator for (3) is written as

$$\mathbf{X} = \zeta(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \varphi(x,t,u)\partial_u.$$

Then, the fifth prolongation of $\mathbf{X}$ becomes

$$\begin{aligned}\mathbf{X}^{(5)} = \mathbf{X} &+ \varphi^x\partial_{u_x} + \varphi^t\partial_{u_t} + \varphi^{2x}\partial_{u_{2x}} + \varphi^{xt}\partial_{u_{xt}} \\ &+ \varphi^{2t}\partial_{u_{2t}} + \cdots + \varphi^{xt^4}\partial_{u_{xt^4}} + \varphi^{t^5}\partial_{u_{t^5}}.\end{aligned}$$

Also Δ_1 and Δ_2 are given as follows

$$\Delta_1 := u_t + auu_x + bu_{3x} + cu_{4x} + du_{5x} - eu_{2x}, \qquad \Delta_2 := \zeta u_x + \tau u_t - \varphi. \tag{8}$$

Without losing integrity, we turn to case $\tau = 1$ and $\tau = 0$.

case (1): Assume $\tau = 1$. With this condition, Δ_2 can be revised as $u_t = \varphi - \zeta u_x$. Then, we calculate the total derivation D_t of (6), we have

$$D_t(u_t) = -au_tu_x - auu_{xt} - bu_{3xt} - cu_{4xt} - du_{5xt} + eu_{2xt}. \tag{9}$$

By substituting $u_t = \varphi - \zeta u_x$ in (9), we obtain the following relation

$$\begin{aligned}D_t(\varphi - \zeta u_x) = (\zeta u_x - \varphi)au_x - auD_x(\varphi - \zeta u_x) - bD_{xxx}(\varphi - \zeta u_x) \\ - cD_{xxxx}(\varphi - \zeta u_x) - dD_{xxxxx}(\varphi - \zeta u_x) - eD_{xx}(\varphi - \zeta u_x).\end{aligned}$$

By adding ζu_{xt} to both sides, we have

$$\begin{aligned}\varphi^t = a\zeta u_x^2 - a\varphi u_x - au\varphi^x + au\xi u_{2x} - e\varphi^{2x} + e\xi u_{3x} - b\varphi^{3x} \\ + b\xi u_{4x} - c\varphi^{4x} + c\xi u_{5x} - d\varphi^{5x} + d\xi u_{6x} + \zeta u_{xt}.\end{aligned}$$

Due to the

$$\begin{aligned}D_x(u_t) &= D_x(-auu_x - bu_{3x} - cu_{4x} - du_{5x} + eu_{2x}), \\ u_{xt} &= -au_x^2 - auu_{2x} + eu_{3x} - bu_{4x} - cu_{5x} - du_{6x},\end{aligned}$$

we obtain the following equation,

$$\varphi^t + a\varphi u_x + au\varphi^x - e\varphi^{2x} + b\varphi^{3x} + c\varphi^{4x} + d\varphi^{5x} = 0.$$

Also $\varphi^t, \varphi^x, \varphi^{xx}, \varphi^{xxx}, \varphi^{xxxx}$ and φ^{xxxxx} are given by

$$\begin{aligned}\varphi^t &= D_t(\varphi - \zeta u_x - \tau u_t) + \zeta u_{xt} + \tau u_{tt} = D_t(\varphi - \zeta u_x) + \zeta u_{xt}, \\ \varphi^x &= D_x(\varphi - \zeta u_x - \tau u_t) + \zeta u_{xx} + \tau u_{tx} = D_x(\varphi - \zeta u_x) + \zeta u_{xx}, \\ \varphi^{xx} &= D_{xx}(\varphi - \zeta u_x - \tau u_t) + \zeta u_{xxx} + \tau u_{txx} = D_{xx}(\varphi - \zeta u_x) + \zeta u_{xxx}, \\ \varphi^{xxx} &= D_{xxx}(\varphi - \zeta u_x - \tau u_t) + \zeta u_{xxxx} + \tau u_{txxx} = D_{xxx}(\varphi - \zeta u_x) + \zeta u_{xxxx}, \\ \varphi^{xxxx} &= D_{xxxx}(\varphi - \zeta u_x - \tau u_t) + \zeta u_{xxxxx} + \tau u_{txxxx} = D_{xxxx}(\varphi - \zeta u_x) + \zeta u_{xxxxx}, \\ \varphi^{xxxxx} &= D_{xxxxx}(\varphi - \zeta u_x - \tau u_t) + \zeta u_{xxxxxx} + \tau u_{txxxxx} = D_{xxxxx}(\varphi - \zeta u_x) + \zeta u_{xxxxxx}.\end{aligned}$$

Table 1: Lie invariants, Similarity solutions, Reduced equations.

κ_i	$\{r_i, s_i\}$	u_i	Reduced equation
κ_1	$\{t^2-2x, au-t\}$	$\frac{t}{a}+f(r)$	$-32df^{(5)}+16cf^{(4)}-8bf^{(3)}-4ef''+2aff'=-\frac{1}{a}$
κ_2, κ_5	$\{t, u\}$	$f(r)$	$f'(r)=0$
κ_3	$\{t^2+2t-2x, au+t\}$	$\frac{t}{a}+f(r)$	$-32df^{(5)}+16cf^{(4)}-8bf^{(3)}-4ef''+2aff'+2(t+1)f'=-\frac{1}{a}$
κ_4, κ_9	$\{t, atu-x\}$	$\frac{x}{at}+f(r)$	$f(r)=0$
κ_6, κ_8	$\{tx, u\}$	$f(r)$	$aff'-ef''+bf^{(3)}+cf^{(4)}+df^{(5)}=0$
κ_7	$\{t, x+(1+t)au\}$	$\frac{-x}{a(t+1)}+f(r)$	$f(r)=0$
κ_{10}	$\{t-x, u\}$	$f(r)$	$-aff'-ef''-bf^{(3)}+cf^{(4)}-df^{(5)}=0$

Then, we gain the determining equations of K-S equation. By solving the resulted system, applying the classical symmetry method, non-classical symmetries for the K-S equation are concluded as follows,

$$\zeta = c_1 t + c_2, \qquad \varphi = \frac{c_1}{a}. \tag{10}$$

In the following, we will analyze the possible cases:
Assume $c_1 = 1$ and $c_2 \neq 1$, the characteristics of symmetries are given by

$$\kappa_1 = \frac{1}{a} - tu_x - u_t, \qquad \kappa_2 = u_x.$$

Assume $c_1 = c_2 = 1$, then we have

$$\kappa_3 = \frac{1}{a} - tu_x - u_x - u_t.$$

Assume $c_1 \neq 1$, $c_2 \neq 1$ and $c_1 \neq c_2$, the characteristics of symmetries are

$$\kappa_4 = \frac{1}{a} - tu_x, \qquad \kappa_5 = u_x, \qquad \kappa_6 = u_t.$$

Assume $c_1 \neq 1$, $c_2 \neq 1$ and $c_1 = c_2$, then we have

$$\kappa_7 = \frac{1}{a} - (t+1)u_x, \qquad \kappa_8 = u_t.$$

Assume $c_1 \neq 1$, $c_2 = 1$, the characteristics of symmetries are

$$\kappa_9 = \frac{1}{a} - tu_x, \qquad \kappa_{10} = u_x + u_t.$$

Using the obtained characteristic equations, Lie invariants, similarity solutions and reduced equations for K-S equation are computed. The results are shown in Table (1).

case (2): Let $\tau = 0$. Without reducing the totality, suppose $\zeta = 1$. Then we have $u_x = \varphi$. Also, assume

$$A(x,t,u) = -au\varphi - b\varphi_{xx} - c\varphi_{xxx} - d\varphi_{xxxx} + e\varphi_{xx}.$$

By substituting $A(x,t,u)$ in the following determining equation,

$$A\varphi_u + \varphi_t - A_u\varphi - A_x = 0,$$

we obtain

$$\begin{aligned}&\varphi_t+2a\varphi^2+au\varphi_u\varphi+b\varphi_{2xu}\varphi+c\varphi_{3xu}\varphi+d\varphi_{4xu}\varphi+e\varphi_{xu}\varphi-au\varphi\varphi_u-b\varphi_{2x}\varphi_u\\&-c\varphi_{3x}\varphi_u-d\varphi_{4x}\varphi_u-e\varphi_x\varphi_u+au\varphi_x+b\varphi_{xxx}+c\varphi_{xxxx}+d\varphi_{xxxxx}-e\varphi_{xx}=0.\end{aligned}$$

Suppose $\varphi=\varphi(x,t)$. Therefore, the above equation is written as follows

$$\varphi_t+2a\varphi^2+au\varphi_x+b\varphi_{xxx}+c\varphi_{xxxx}+d\varphi_{xxxxx}-e\varphi_{xx}=0.$$

Solving this equation we have $\varphi=\dfrac{1}{2at+c}$. So the following solution is obtained

$$u(x,t)=\frac{x}{2at+c}+F(t).$$

4. Non-Classical Symmetries of the mKdV-ZK Equation

the mKdV-ZK equation is

$$u_t+\alpha u^2u_x+u_{xxx}+u_{xyy}+u_{xzz}=0, \tag{11}$$

where α is a scattering coefficient.

The mKdV-ZK equation has a significant role in the study of physical phenomena and the evolution of acoustic ion disorders in magnetic plasma [9]. Lie classical symmetries and some invariant solutions of this equation have been obtained in some studies [14, 15]. Using the non-classical symmetry method for it, we have

$$X^{(3)}\Delta_1|_{\Delta_1=0,\Delta_2=0}=0, \tag{12}$$

where X is defined as

$$\mathbf{X}=\zeta(x,y,z,t,u)\partial_x+\eta(x,y,z,t,u)\partial_y+\psi(x,y,z,t,u)\partial_z+\tau(x,y,z,t,u)\partial_t+\varphi(x,y,z,t,u)\partial_u,$$

and Δ_1 and Δ_2 are given as follows

$$\Delta_1:=u_t+\alpha u^2u_x+u_{xxx}+u_{xyy}+u_{xzz}=0, \qquad \Delta_2:=\varphi-\zeta u_x-\eta u_y-\psi u_z-\tau u_t. \tag{13}$$

Without losing integrity, we choose $\tau=1$. so $\Delta_2=0$ will become

$$u_t=\varphi-\zeta u_x-\eta u_y-\psi u_z.$$

First, the total differentiation D_t of above equation gives

$$D_t(\varphi-\zeta u_x-\eta u_y-\psi u_z)=-2\alpha u_tu_xu-\alpha u^2u_{xt}-u_{xxxt}-u_{xyyt}-u_{xzzt}.$$

Then we have

$$\begin{aligned}D_t(u_t)=&-2\alpha(\varphi-\zeta u_x-\eta u_y-\psi u_z)u_xu-\alpha u^2D_x(\varphi-\zeta u_x-\eta u_y-\psi u_z)\\&-D_{xxx}(\varphi-\zeta u_x-\eta u_y-\psi u_z)-D_{xyy}(\varphi-\zeta u_x-\eta u_y-\psi u_z)\\&-D_{xzz}(\varphi-\zeta u_x-\eta u_y-\psi u_z).\end{aligned}$$

Substituting $\zeta u_{xt},\eta u_{yt}$ and ψu_{zt} to both sides, we get

$$\begin{aligned}\varphi^t=&-2\alpha\varphi uu_x+2\alpha\zeta uu_x^2+2\alpha uu_xu_y\eta+2\alpha\psi uu_zu_x-\alpha u^2\varphi^x-\varphi^{xxx}-\varphi^{xyy}\\&-\varphi^{xzz}+\alpha u^2\zeta u_{xx}+\alpha u^2\eta u_{xy}+\alpha u^2\psi u_{xz}+\zeta u_{xxxx}+\eta u_{xxxy}+\psi u_{xxxz}\\&+\zeta u_{xxyy}+\eta u_{xyyy}+\psi u_{xyyz}+\zeta u_{xxzz}+\eta u_{xzzy}+\psi u_{xzzz}.\end{aligned} \tag{14}$$

Due to

$$\begin{aligned}D_x(u_t)&=D_x(-\alpha u^2u_x-u_{xxx}-u_{xyy}-u_{xzz}),\\D_y(u_t)&=D_y(-\alpha u^2u_x-u_{xxx}-u_{xyy}-u_{xzz}),\\D_z(u_t)&=D_z(-\alpha u^2u_x-u_{xxx}-u_{xyy}-u_{xzz}),\end{aligned}$$

we have

$$\begin{aligned} u_{xt} &= -2\alpha u_x^2 u - \alpha u^2 u_{2x} - u_{xxxx} + u_{xxyy} - u_{xxzz}, \\ u_{yt} &= -2\alpha u_y u_x u - \alpha u^2 u_{xy} - u_{xxxy} - u_{xyyy} - u_{xyzz}, \\ u_{zt} &= -2\alpha u_z u_x u - \alpha u^2 u_{xz} - u_{xxxz} - u_{xzyy} - u_{xzzz}. \end{aligned}$$

By substituting the above relations in (14), we have

$$\varphi^t + \alpha u^2 \varphi^x + \alpha \varphi^2 u_x + \varphi^{xxx} - \varphi^{xyy} - \varphi^{xzz} = 0. \tag{15}$$

Also $\varphi^t, \varphi^x, \varphi^{xxx}, \varphi^{xyy}$ and φ^{xzz} are given by

$$\begin{aligned} \varphi^t &= D_t(\varphi - \zeta u_x - \eta u_y - \psi u_z) + \zeta u_{xt} + \eta u_{yt} + \psi u_{zt}, \\ \varphi^x &= D_x(\varphi - \zeta u_x - \eta u_y - \psi u_z) + \zeta u_{xx} + \eta u_{xy} + \psi u_{zx}, \\ \varphi^{xxx} &= D_{xxx}(\varphi - \zeta u_x - \eta u_y - \psi u_z) + \zeta u_{xxxx} + \eta u_{xxxy} + \psi u_{xxxz}, \\ \varphi^{xyy} &= D_{xyy}(\varphi - \zeta u_x - \eta u_y - \psi u_z) + \zeta u_{xxyy} + \eta u_{xyyy} + \psi u_{xyyz}, \\ \varphi^{xzz} &= D_{xzz}(\varphi - \zeta u_x - \eta u_y - \psi u_z) + \zeta u_{xxzz} + \eta u_{xzzy} + \psi u_{xzzz}. \end{aligned}$$

By substituting these coefficients in (15) we obtain the determining equations. By solving these equations, the non-classical symmetries of equation (11) are obtained as follows

$$\begin{aligned} &\zeta(x,y,z,t,u) = c_4, && \eta(x,y,z,t,u) = c_1 z + c_2, \\ &\psi(x,y,z,t,u) = -c_1 y + c_3, && \varphi(x,y,z,t,u) = 0. \end{aligned} \tag{16}$$

If $c_i = 1, i = 1, \cdots, 4$, then the associated symmetry is obtained as

$$\kappa_1 = -u_x - (z+1)u_y - (1-y)u_z - u_t.$$

If $c_i \neq 1$, then the associated symmetries are obtained as

$$\kappa_2 = -u_x - (z+1)u_y - (1-y)u_z, \qquad \kappa_3 = u_t.$$

If $c_3 = c_4 = 1$ and $c_1 = c_2 \neq 1$, then the associated symmetries are obtained as

$$\kappa_4 = -u_x - u_t - u_z, \qquad \kappa_5 = -(z+1)u_y + yu_z.$$

If $c_1 = c_2 = 1$ and $c_3 = c_4 \neq 1$, then the associated symmetries are obtained as

$$\kappa_6 = -u_x - u_z, \qquad \kappa_7 = -(z+1)u_y + yu_z - u_t.$$

If $c_1 = c_3 = 1$ and $c_2 = c_4 \neq 1$, then the associated symmetries are obtained as

$$\kappa_8 = -u_x - u_y, \qquad \kappa_9 = -zu_y - (1-y)u_z - u_t.$$

If $c_1 = c_3 \neq 1$ and $c_2 = c_4 = 1$, then the associated symmetries are obtained as

$$\kappa_{10} = -u_x - u_y - u_t, \qquad \kappa_{11} = -zu_y - (1-y)u_z.$$

If $c_4 = 1$ and $c_1 = c_2 = c_3 \neq 1$,then the associated symmetries are obtained as

$$\kappa_{12} = -u_x - u_t, \qquad \kappa_{13} = -(z+1)u_y - (1-y)u_z.$$

If $c_1 = c_2 = c_3 = 1$ and $c_4 \neq 1$, then the associated symmetries are obtained as

$$\kappa_{14} = -u_x, \qquad \kappa_{15} = (z+1)u_y + (1-y)u_z + u_t.$$

If $c_2 = c_3 = c_4 \neq 1$ and $c_1 = 1$, then the associated symmetries are obtained as

$$\kappa_{16} = -u_x - u_y - u_z, \qquad \kappa_{17} = -zu_y + yu_z - u_t.$$

If $c_2 = c_3 = c_4 = 1$ and $c_1 \neq 1$, then the associated symmetries are obtained as

$$\kappa_{18} = -u_x - u_y - u_z - u_t, \qquad \kappa_{19} = -zu_y + yu_z.$$

Because all the possible states for c_i, $i = 1, \ldots, 4$ have been considered, there is no other state left to investigate. Next, Using the resulted symmetries, the differential invariants and reduced equations for the mKdV-ZK equation are obtained. These results are listed in Tables (2) and (3).

where $\rho = \dfrac{1-y}{\sqrt{z^2+2z+1}}$. where $\alpha = (1+z)^2 + (1-y)^2$, $\beta = (1+z)^2 + y^2$, $\gamma = z^2 + (1-y)^2$ and $\eta = y^2 + z^2$.

Table 2: Lie invariants for symmetries of the mKdV-ZK equation

κ_j	$\{w_j, r_j, k_j\}$
κ_1	$\{y^2+z^2+2z-2y, x-\arctan\rho, t-\arctan\rho\}$
κ_2	$\{z^2-y^2+2z-2y, x-\arctan\rho, t\}$
κ_3	$\{x, y, z\}$
κ_4	$\{x-z, y, z-t\}$
κ_5	$\{x, y^2+z^2+2z, t\}$
κ_6	$\{x-z, y, t\}$
κ_7	$\{x, y^2+z^2+2z, t-\arctan\dfrac{y}{\sqrt{1+2z+z^2}}\}$
κ_8	$\{x-y, t, z\}$
κ_9	$\{x, y^2+z^2-2y, t-\arctan\dfrac{1-y}{z}\}$
κ_{10}	$\{x-y, y-t, z\}$
κ_{11}	$\{x, y^2+z^2+2y, t\}$
κ_{12}	$\{x-t, y, z\}$
κ_{13}	$\{x, y^2+z^2+2z-2y, t)$
κ_{14}	$u=F(t)=c$
κ_{15}	$\{x, y^2+z^2+2z-2y, t-\arctan\rho\}$
κ_{16}	$\{x-y, y-z, t)$
κ_{17}	$\{x, z^2+y^2, t+\arctan\dfrac{y}{z}\}$
κ_{18}	$\{x-y, y-z, z-t\}$
κ_{19}	$\{x, y^2+z^2, t\}$

Table 3: Similarity reductions of the mKdV-ZK equation

κ_j	Similarity reduced equations
κ_1	$f_t+\alpha f^2 f_r+f_{3r}+4\alpha f_{3w}+4\alpha f_{2w}+\dfrac{1}{\alpha}(f_{w2r}+f_{w2k})=0$
κ_2	$f_k+\alpha f^2 f_r+f_{3r}+4\alpha f_{3w}+\dfrac{1}{\alpha}f_{w2r}=0$
κ_3	$\alpha f^2 f_w+f_{3w}+f_{w2r}+f_{w2k}=0$
κ_4	$-f_k+\alpha f^2 f_w+2f_{3w}+f_{w2r}+f_{w2k}=0$
κ_5	$f_k+\alpha f^2 f_w+f_{3w}+4\beta f_{w2r}=0$
κ_6	$f_k+\alpha f^2 f_w+2f_{3w}+f_{w2r}=0$
κ_7	$f_k+\alpha f^2 f_w+f_{3w}+4(1+y^2)f_{w2r}+2f_{wr}+\dfrac{1}{\beta}f_{w2k}=0$
κ_8	$f_r+\alpha f^2 f_w+2f_{3w}+f_{w2k}=0$
κ_9	$f_k+\alpha f^2 f_w+f_{3w}+4\gamma f_{w2r}+4f_{wr}+\dfrac{1}{\gamma}f_{w2k}+4\dfrac{z(1-y)}{\gamma^2}f_{wk}=0$
κ_{10}	$-f_r+\alpha f^2 f_w+2f_{3w}+f_{w2r}+f_{w2k}=0$
κ_{11}	$f_k+\alpha f^2 f_w+f_{3w}+4((1+y)^2+z^2)f_{w2r}+4f_{wr}=0$
κ_{12}	$-f_w+\alpha f^2 f_w+f_{3w}+f_{w2r}+f_{w2k}=0$
κ_{13}	$f_k+\alpha f^2 f_w+f_{3w}+4\alpha f_{w2r}+4f_{wr}=0$
κ_{14}	$f_k=0$
κ_{15}	$f_k+\alpha f^2 f_w+f_{3w}+4\alpha f_{w2r}+4f_{wr}+\dfrac{1}{\alpha}f_{w2k}=0$
κ_{16}	$f_k+\alpha f^2 f_w+f_{3w}+f_{2w}+f_{2r}+f_{w2r}=0$
κ_{17}	$f_k+\alpha f^2 f_w+f_{3w}+4\eta f_{w2r}+4f_{wr}+\dfrac{1}{\eta}f_{w2k}=0$
κ_{18}	$-f_k+\alpha f^2 f_w+2f_{3w}+f_{w2r}+f_{w2k}=0$
κ_{19}	$f_k+\alpha f^2 f_w+f_{3w}+4\eta f_{w2r}+4f_{wr}=0$

Remark 4.1. Compared to the previous works on these two equations, it can be said that the present study has investigated more symmetries of the equations, which results in finding new solutions of the equations. In fact, non-classical symmetries include a wider group of symmetries of the equation, and this leads to finding more group

invariant solutions of the equation.

5. Conclusions

In this article, using the non-classical symmetry method, which is a kind of generalization of Lie symmetries, we have obtained the non-classical symmetries of K-S and mKdV-ZK equations, which are two important equations in physics. In fact, these symmetries do not constitute an algebra because their determining equations are nonlinear. Also, using these resulted symmetries, some group invariant solutions and the similarity reduced equations have been investigated. Some of these invariant solutions are not obtained from the classical Lie method, and this highlights the importance of these symmetries.

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