AUT Journal of Mathematics and Computing

AUT J. Math. Com., 4(2) (2023) 155-160
https://doi.org/10.22060/ajmc.2022.21778.1107

Original Article

Bergman and Dirichlet spaces in the unit ball and symmetric lifting operator

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ABSTRACT: Let $\mathbb{B}_n$ be the open unit ball in $\mathbb{C}^n$ and $\mathbb{B}_n^2 = \mathbb{B}_n \times \mathbb{B}_n$. The symmetric lifting operator which lifts analytic functions from $H(\mathbb{B}_n)$ to $H(\mathbb{B}_n^2)$ is defined as follow

$$L(f)(z,w) = \frac{f(z) - f(w)}{z - w}.$$

In this paper we investigate the action of symmetric lifting operator on the Bergman space in the unit ball. Also, we state a characterization for Dirichlet space and consider symmetric lifting operator on the Dirichlet space in the unit ball.

Review History:

Received:16 September 2022
Revised:09 December 2022
Accepted:09 December 2022
Available Online:20 March 2023

Keywords:

Symmetric lifting operator
Bergman space
Dirichlet space
Pseudo-hyperbolic metric

AMS Subject Classification (2010):

16W25; 46L57; 47B47

1. Introduction

For $z = (z_1, \cdots, z_n)$ and $w = (w_1, \cdots, w_n)$ in $\mathbb{C}^n$, we define $\langle z, w\rangle = z_1\overline{w_1} + \cdots + z_n\overline{w_n}$, where $\overline{w_k}$ is the complex conjugate of w_k. We also write $|z| = \sqrt{\langle z, z\rangle} = \sqrt{|z_1|^2 + \cdots + |z_n|^2}$. Let $\mathbb{B}_n$ denote the open unit ball of $\mathbb{C}^n$, that is

$$\mathbb{B}_n = \{z \in \mathbb{C}^n : |z| < 1\}.$$

Thus for any $a \in \mathbb{B}_n - \{0\}$, we define

$$\varphi_a(z) = \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a\rangle} \qquad z \in \mathbb{B}_n,$$

where $s_a = \sqrt{1 - |a|^2}$, P_a is the orthogonal projection from $\mathbb{C}^n$ onto the subspace $[a]$ generated by a, and Q_a is the orthogonal projection from $\mathbb{C}^n$ onto $\mathbb{C}^n - [a]$. When $a = 0$, write $\varphi_a(z) = -z$. These functions are called

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involutions.
The hyperbolic metric(Bergman metric) is defined by

$$\beta(z,w)=\frac{1}{2}\log\frac{1+|\varphi_z(w)|}{1-|\varphi_z(w)|},\quad z,w\in\mathbb{B}_n.$$

For any $z\in\mathbb{B}_n$ and $r>0$, we denote Bergman metric ball at z by $D(z,r)$. That is

$$D(z,r)=\{w\in\mathbb{B}_n:\beta(z,w)<r\}.$$

Also, pseudo-hyperbolic metric defined as $\rho(z,w)=|\varphi_z(w)|$.
For $\alpha>-1$ let

$$dv_\alpha(z)=c_\alpha(1-|z|^2)^\alpha dv(z),$$

where dv is the Lebesgue volume measure on $\mathbb{B}_n$ and c_α is a positive constant with $v_\alpha(\mathbb{B}_n)=1$. For $0<p<\infty$ and $\alpha>-1$, the weighted Bergman space $A^p_\alpha(\mathbb{B}_n)$ consists of all holomorphic functions in $L^p(\mathbb{B}_n,dv_\alpha)$, that is

$$A^p_\alpha(\mathbb{B}_n)=\left\{f\in H(\mathbb{B}_n):||f||^p_{\alpha,p}=\int_{\mathbb{B}_n}|f(z)|^p dv_\alpha(z)<\infty\right\}.$$

More information about Bergman spaces can be found in [1, 4, 7, 9]. Let $\mathbb{B}^2_n=\mathbb{B}_n\times\mathbb{B}_n$ be the open subset of $\mathbb{C}^2_n=\mathbb{C}_n\times\mathbb{C}_n$ which is

$$\mathbb{B}^2_n=\{(z,w)\in\mathbb{C}^2_n:|z|<1,\quad|w|<1\}.$$

The Bergman space $A^p_\alpha(\mathbb{B}^2_n)$ over the $\mathbb{B}^2_n$ is the space of all holomorphic functions on $\mathbb{B}^2_n$ such that

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}|f(z,w)|^p dv_\alpha(z)dv_\alpha(w)<\infty.$$

For $f\in\mathbb{B}_n$, some notations which will be used in this section are:

$$Rf(z)=\sum_{k=1}^n z_k\frac{\partial f}{\partial z_k}(z).$$

Rf is called the radial derivative of f. The (holomorphic) gradient of f at z is

$$|\nabla f(z)|=\left|\left(\frac{\partial f}{\partial z_1}(z),\cdots,\frac{\partial f}{\partial z_n}(z)\right)\right|=\left(\sum_{k=1}^n\left|\frac{\partial f}{\partial z_k}(z)\right|^2\right)^{1/2}.$$

Also invariant gradient of f at z defined by

$$|\widetilde{\nabla}f(z)|=|\nabla(f\circ\varphi_z)(0)|.$$

The Dirichlet space $D^p_\alpha(\mathbb{B}_n)$ is the space of the analytic functions $f:\mathbb{B}_n\to\mathbb{C}$ such that $R(f)\in A^p_\alpha(\mathbb{B}_n)$. In a similar way we can define $D^p_\alpha(\mathbb{B}^2_n)$.

If $f\in H(\mathbb{B}_n)$ then $L(f)\in H(\mathbb{B}^2_n)$. So direct calculation shows that

$$R(Lf)(z,w)=L(Rf)(z,w)+\sum_{k=1}^n\frac{(z_k-w_k)(f(w)-f(z))}{(z-w)^2}. \tag{1}$$

The operator we use here is the well-known symmetric lifting operator which lifts analytic functions from $\mathbb{B}_n$ to $\mathbb{B}^2_n$ and defined by

$$L:H(\mathbb{B}_n)\to H(\mathbb{B}^2_n)$$

$$L(f)(z,w)=\frac{f(z)-f(w)}{z-w},$$

where $H(\mathbb{B}^2_n)$ is the space of all analytic functions on $\mathbb{B}^2_n$.

Wulan and Zhu in [8] characterized the Bergman spaces in the unit disk and unit ball in terms of Lipschitz type conditions also studied the action of operator $L:H(\mathbb{D})\to H(\mathbb{D}^2)$ on A^p_α. Double integral characterization for $A^p_\alpha(\mathbb{B}_n)$ can be found for example in [3, 5, 6]. In [3] the authors use the double integral characterization for proving the action of the operator L on A^p_α where $p=\alpha+2$. Recently, the first author of this paper and Sohrabi investigated the symmetric lifting operator on the Bloch type spaces in [2]. In this work, we investigate the action symmetric lifting operator $L:A^p_\alpha(\mathbb{B}_n)\to A^p_\alpha(\mathbb{B}^2_n)$ and $L:D^p_\alpha(\mathbb{B}_n)\to D^p_\alpha(\mathbb{B}^2_n)$.

2. Symmetric lifting operator from $A^p_\alpha(\mathbb{B}_n)$ into $A^p_\alpha(\mathbb{B}^2_n)$

In this section we first bring some lemmas which are needed for proving the main results.

Theorem 2.1. *[8] Suppose that $p > 0$, $\alpha > -1$, and f is analytic in $\mathbb{B}_n$. Then the following conditions are equivalent.*

(i) $f \in A^p_\alpha$.

(ii) There exists a continuous function g in $L^p(\mathbb{B}_n, dv_\alpha)$ such that

$$|f(z) - f(w)| \leq \rho(z,w)(g(z) + g(w)), \quad z, w \in \mathbb{B}_n.$$

(iii) There exists a continuous function g in $L^p(\mathbb{B}_n, dv_\alpha)$ such that

$$|f(z) - f(w)| \leq \beta(z,w)(g(z) + g(w)), \quad z, w \in \mathbb{B}_n.$$

(iv) There exists a continuous function g in $L^p(\mathbb{B}_n, dv_{p+\alpha})$ such that

$$|f(z) - f(w)| \leq |z-w|(g(z) + g(w)), \quad z, w \in \mathbb{B}_n.$$

Lemma 2.2. *([5]) Let $r > 0$. Then*

$$1 - |z|^2 \sim 1 - |w|^2 \sim |1 - \langle z, w\rangle|^2$$

for all $z \in \mathbb{B}_n$ and $w \in D(z,r)$. Furthermore, there exists a positive constant C such that

$$(1 - |z|^2)^p |\nabla f(z)|^p \leq \frac{C}{(1-|z|^2)^{n+1}} \int_{D(z,r)} |f(w) - f(z)|^p dv(w)$$

for all $z \in \mathbb{B}_n$ and $f \in H(\mathbb{B}_n)$.

Lemma 2.3. *[5] The involutive automorphism φ_z has the following properties:*

$$1 - |\varphi_z(w)|^2 = \frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2}$$

and

$$|\varphi_z(w)|^2 = \frac{|z-w|^2 + |\langle z,w\rangle|^2 - |z|^2|w|^2}{|1-\langle z,w\rangle|^2}.$$

Consequently

$$|\varphi_z(w)| \leq \frac{|z-w|}{|1-\langle z,w\rangle|}$$

and

$$\frac{(1-|z|^2)(1-|w|^2)}{|1-\langle z,w\rangle|^2} \leq \frac{1-|\varphi_z(w)|^2}{|\varphi_z(w)|^2}.$$

In the next three theorems, the symmetric lifting operator $L : A^p_\alpha(\mathbb{B}_n) \to A^p_\alpha(\mathbb{B}^2_n)$ will be considered.

Theorem 2.4. *Suppose $\alpha > -1$ and $0 < p < \alpha + n + 1$. Then the symmetric lifting operator L maps $A^p_\alpha(\mathbb{B}_n)$ boundedly into $A^p_\alpha(\mathbb{B}^2_n)$.*

Proof. Suppose that $f \in A^p_\alpha(\mathbb{B}_n)$. Theorem 2.1 implies that there exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that

$$|f(z) - f(w)| \leq \rho(z,w)(g(z) + g(w)), \qquad z, w \in \mathbb{B}_n.$$

Since $\rho(z,w) = |\varphi_z(w)| \leq \frac{|z-w|}{|1-\langle z,w\rangle|}$, we have

$$\frac{|f(z) - f(w)|}{|z-w|} \leq \frac{g(z)}{|1-\langle z,w\rangle|} + \frac{g(w)}{|1-\langle z,w\rangle|}, \qquad z, w \in \mathbb{B}_n.$$

So

$$|L(f)(z,w)|^p \leq C\left(\frac{g(z)^p}{|1-\langle z,w\rangle|^p} + \frac{g(w)^p}{|1-\langle z,w\rangle|^p}\right).$$

Therefore

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}|L(f)(z,w)|^p dv_\alpha(z)dv_\alpha(w) \le 2C\int_{\mathbb{B}_n} g(z)^p dv_\alpha(z)\int_{\mathbb{B}_n}\frac{dv_\alpha(w)}{|1-\langle z,w\rangle|^p}.$$

If $p < n+1+\alpha$ then from Theorem 1.12 of [9] we have the boundedness of the internal integral. So

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}|L(f)(z,w)|^p dv_\alpha(z)dv_\alpha(w) \le C\int_{\mathbb{B}_n} g(z)^p dv_\alpha(z).$$

We get $L(f) \in A^p_\alpha(\mathbb{B}^2_n)$ and L maps $A^p_\alpha(\mathbb{B}_n)$ into $A^p_\alpha(\mathbb{B}^2_n)$. The proof of boundedness of $L : A^p_\alpha(\mathbb{B}_n) \to A^p_\alpha(\mathbb{B}^2_n)$ comes from the closed graph theorem. Suppose that $(f,g) \in \overline{G}$ where G is the graph of L. Then there exists a sequence (f_n, Lf_n) of G such that $(f_n, Lf_n) \to (f,g)$ which results $f_n \to f$ and $Lf_n \to g$. One can check that using the definition of norm of the Bergman space, $Lf_n \to Lf$. So $Lf = g$ which means that the graph of L is closed and the operator L is bounded. □

Theorem 2.5. *Suppose $\alpha > -1$ and $p > \alpha+n+1$. Then the symmetric lifting operator L maps $A^p_\alpha(\mathbb{B}_n)$ boundedly into $A^p_\beta(\mathbb{B}^2_n)$ where $\beta = (p+\alpha-n-1)/2$.*

The proof of the above theorem is similar to the previous one by using Theorem 2.1 and Theorem 1.12[9].

Theorem 2.6. *Suppose $\alpha > -1$ and $p = \alpha+n+1$. Then the symmetric lifting operator L maps $A^p_\alpha(\mathbb{B}_n)$ boundedly into $A^p_\gamma(\mathbb{B}^2_n)$ for any $\gamma > \alpha$.*

Proof. If $f \in A^p_\alpha$, then by Theorem 2.1 there exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that

$$|f(z)-f(w)| \le \rho(z,w)(g(z)+g(w)) \le \frac{|z-w|}{|1-\langle z,w\rangle|}(g(z)+g(w)).$$

There exists a positive constant C such that

$$\begin{aligned}\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}|L(f)(z,w)|^p dv_\gamma(z)dv_\gamma(w) &\le 2C\int_{\mathbb{B}_n} g(z)^p dv_\gamma(z)\int_{\mathbb{B}_n}\frac{dv_\gamma(w)}{|1-\langle z,w\rangle|^p}\\ &= 2C\int_{\mathbb{B}_n} g(z)^p dv_\gamma(z)\int_{\mathbb{B}_n}\frac{dv_\gamma(w)}{|1-\langle z,w\rangle|^{n+1+\alpha}}.\end{aligned}$$

Since $\gamma > \alpha$, the last integral is bounded. Then

$$\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}|L(f)(z,w)|^p dv_\gamma(z)dv_\gamma(w) \le 2C\int_{\mathbb{B}_n} g(z)^p dv_\gamma(z) < 2C\int_{\mathbb{B}_n} g(z)^p dv_\alpha(z) < \infty.$$

□

3. Symmetric lifting operator from $D^p_\alpha(\mathbb{B}_n)$ into $D^p_\alpha(\mathbb{B}^2_n)$

In this section we first state a characterization for Dirichlet space using pseudo-hyperbolic metric and then study the symmetric lifting operator L from $D^p_\alpha(\mathbb{B}_n)$ into $D^p_\alpha(\mathbb{B}^2_n)$.

The following characterization for Bergman space is crucial for our main results, [9].

Lemma 3.1. *Suppose that $p > 0$, $\alpha > -1$, and $f \in H(\mathbb{B}_n)$. Then the following conditions are equivalent.*

(a) $f \in A^p_\alpha(\mathbb{B}_n)$

(b) $|\widetilde{\nabla} f(z)|$ *is in* $L^p(\mathbb{B}_n, dv_\alpha)$.

(c) $(1-|z|^2)|\nabla f(z)|$ *is in* $L^p(\mathbb{B}_n, dv_\alpha)$.

(d) $(1-|z|^2)|Rf(z)|$ *is in* $L^p(\mathbb{B}_n, dv_\alpha)$.

Theorem 3.2. *Suppose that $p > 0$ and $\alpha > -1$. If $f \in D^p_\alpha(\mathbb{B}_n)$ then there exists a continuous function $g \in L^p(\mathbb{B}_n, dv_\alpha)$ such that*

$$|f(z)-f(w)| \le \rho(z,w)(g(z)+g(w)).$$

Proof. It can be seen that

$$f(z) - f(0) = \int_0^1 \frac{Rf(tz)}{t} dt$$

for all $z \in \mathbb{B}_n$. Fix $r \in (0,1)$. Then we have

$$|f(z) - f(0)| \leq |z| \sup\{|\nabla f(w)| : w \in D(0,r)\}$$

where $D(0,r)$ is the pseudo-hyperbolic disk centered at 0 with radius r. Since the Euclidean metric is comparable to the pseudo-hyperbolic metric in the relatively compact set $D(0,r)$ and $|\nabla f(w)|$ is comparable to $|\widetilde{\nabla} f(w)|$ in this set, there exists a positive constant C such that

$$|f(z) - f(0)| \leq C\rho(z,0) \sup\{|\widetilde{\nabla} f(w)| : w \in D(0,r)\}.$$

Put $f \circ \varphi_w$ in place of f, $\rho(z,w) < r$, and $\varphi_w(z)$ in place of z. Then the Mobius invariancy of pseudo-hyperbolic metric and invariant gradient (see [9]) implies that

$$|f(z) - f(w)| \leq C\rho(z,w) \sup\{|\widetilde{\nabla} f(u)| : u \in D(z,r)\}.$$

Set

$$h(z) = \sup\{|\widetilde{\nabla} f(u)| : u \in D(z,r)\}.$$

So

$$|f(z) - f(w)| \leq C\rho(z,w)(h(z) + h(w)),$$

for z and w with $\rho(z,w) < r$. If $\rho(z,w) \geq r$, then we set

$$g(z) = \frac{|Rf(z)|}{r} + Ch(z).$$

Clearly

$$|f(z) - f(w)| \leq C\rho(z,w)(g(z) + g(w)),$$

for all z, w. It just remains that showing $g \in L^p(\mathbb{B}_n, dv_\alpha)$. But $Rf \in L^p(\mathbb{B}_n, dv_\alpha)$. So we need to show that $h \in L^p(\mathbb{B}_n, dv_\alpha)$ which is a similar argument as in the proof of Theorem 5.1 of [8]. □

Lemma 3.3. *Suppose $\alpha > -1$ and $0 < p < \alpha + n + 1$. Then the symmetric lifting operator L maps $D^p_\alpha(\mathbb{B}_n)$ boundedly into $A^p_\alpha(\mathbb{B}^2_n)$.*

The proof is the same as proof of Theorem 2.4 using the previous theorem.

In the cases $p > \alpha + n + 1$ and $p = \alpha + n + 1$ we have the following lemma which are stated without proof.

Lemma 3.4. *Suppose $\alpha > -1$ and $p > \alpha + n + 1$. Then the symmetric lifting operator L maps $D^p_\alpha(\mathbb{B}_n)$ boundedly into $A^p_\beta(\mathbb{B}^2_n)$ where $\beta = (p + \alpha - n - 1)/2$.*

Lemma 3.5. *Suppose $\alpha > -1$ and $p = \alpha + n + 1$. Then the symmetric lifting operator L maps $D^p_\alpha(\mathbb{B}_n)$ boundedly into $A^p_\gamma(\mathbb{B}^2_n)$ for any $\gamma > \alpha$.*

Now we are ready to state the main result of this section.

Theorem 3.6. *Suppose $\alpha > -1$. Then*

(a) For $0 < p < \alpha + n + 1$, the symmetric lifting operator L maps $D^p_\alpha(\mathbb{B}_n)$ boundedly into $D^p_\alpha(\mathbb{B}^2_n)$.

(b) For $p > \alpha + n + 1$, the symmetric lifting operator L maps $D^p_\alpha(\mathbb{B}_n)$ boundedly into $D^p_\beta(\mathbb{B}^2_n)$ where $\beta = (p + \alpha - n - 1)/2$.

(c) For $p = \alpha + n + 1$, the symmetric lifting operator L maps $D^p_\alpha(\mathbb{B}_n)$ boundedly into $D^p_\gamma(\mathbb{B}^2_n)$ for any $\gamma > \alpha$.

Proof. Suppose that $f \in D^p_\alpha(\mathbb{B}_n)$. We need to prove that $Lf \in D^p_\alpha(\mathbb{B}^2_n)$ or equivalently $R(Lf) \in A^p_\alpha(\mathbb{B}^2_n)$. $f \in D^p_\alpha(\mathbb{B}_n)$ implies that $Rf \in A^p_\alpha(\mathbb{B}_n)$ and using Theorem 2.4, we get $L(Rf) \in A^p_\alpha(\mathbb{B}^2_n)$. From (1), it will be sufficient to prove that $J \in A^p_\alpha(\mathbb{B}^2_n)$ where

$$J = \sum_{k=1}^{n} \frac{(z_k - w_k)(f(w) - f(z))}{(z - w)^2}.$$

Using triangle inequality and direct calculation we obtain for some positive constant C

$$\begin{aligned}\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}|J|^p dv_\alpha(z)dv_\alpha(w) &= \int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\left|\sum_{k=1}^{n}\frac{(z_k-w_k)(f(w)-f(z))}{(z-w)^2}\right|^p dv_\alpha(z)dv_\alpha(w)\\ &\leq C\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\sum_{k=1}^{n}\frac{|z_k-w_k|^p|f(z)-f(w)|^p}{|z-w|^{2p}}dv_\alpha(z)dv_\alpha(w)\\ &= C\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\sum_{k=1}^{n}\frac{|z_k-w_k|^p}{|z-w|^p}\frac{|f(z)-f(w)|^p}{|z-w|^p}dv_\alpha(z)dv_\alpha(w)\\ &\leq Cn^p\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}\frac{|f(z)-f(w)|^p}{|z-w|^p}dv_\alpha(z)dv_\alpha(w)\\ &= Cn^p\int_{\mathbb{B}_n}\int_{\mathbb{B}_n}|Lf(z,w)|^p dv_\alpha(z)dv_\alpha(w)\\ &< \infty.\end{aligned}$$

The last line of the above equation comes from Lemma 3.3. The proof of part (a) is completed. In the other parts we have the similar argument. □

One can see that for $\alpha > -1$ and $p > 0$, the symmetric lifting operator L maps $A^p_\alpha(\mathbb{B}_n)$ boundedly into $A^p_{\alpha+p}(\mathbb{B}^2_n)$ and also maps $D^p_\alpha(\mathbb{B}_n)$ boundedly into $D^p_{\alpha+p}(\mathbb{B}^2_n)$.

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Please cite this article using:

Mostafa Hassanlou, Ebrahim Abbasi, Bergman and Dirichlet spaces in the unit ball and symmetric lifting operator, AUT J. Math. Com., 4(2) (2023) 155-160
https://doi.org/10.22060/ajmc.2022.21778.1107