AUT Journal of Mathematics and Computing

AUT J. Math. Com., 4(1) (2023) 39-46
https://doi.org/10.22060/ajmc.2022.21378.1092

Original Article

Reduced designs constructed by Key-Moori Method 2 and their connection with Method 3

Amin Saeidi*[a]

[a]*School of Mathematical and Computer Sciences, University of Limpopo (Turfloop) Sovenga, South Africa*

ABSTRACT: For a 1-(v, k, λ) design $\mathcal{D}$ containing a point x, we study the set I_x, the intersection of all blocks of $\mathcal{D}$ containing x. We use the set I_x together with the Key-Moori Method 2 to construct reduced designs invariant under some families of finite simple groups. We also show that there is a connection between reduced designs constructed by Method 2 and the new Moori Method 3.

Review History:

Received:06 July 2022
Revised:07 December 2022
Accepted:08 December 2022
Available Online:01 February 2023

Keywords:

Reduced design
Key-Moori Method 2
Method 3
Simple group

AMS Subject Classification (2010):

05E18; 05B05

(*Dedicated to Professor Jamshid Moori*)

1. Introduction

Key-Moori Methods 1 and 2 are useful tools to construct 1-designs from simple groups and their maximal subgroups (see [6]). In the first method, the designs are symmetric and constructed by the primitive permutation representations of the groups. In Method 2, the 1-designs can be constructed from a G-conjugacy class, intersecting a maximal subgroup of G. In recent years, many authors have applied the Key-Moori methods to construct 1-designs invariant under the finite simple groups. For example, the methods have been applied to some sporadic simple groups in [1, 2, 8, 9, 12] and some families of finite simple groups in [4, 5, 10, 11, 13, 14]. Recently, a third method has been introduced by Moori (see [7]) to construct designs from the fixed points of a permutation group. In all these three methods, the automorphism groups of the designs contain G, but little is known about the structure of automorphism groups of these designs in general. In [3], the authors obtained some results on the automorphism groups of designs constructed by Key-Moori Methods 1 and 2. In particular, they proved that the automorphism

*Corresponding author.
E-mail addresses: amin.saeidi@ul.ac.za

group of a design constructed by one of these methods contains a normal subgroup which is a direct product of some symmetric groups. The structure of this normal subgroup is related to the following general definition.

Definition 1.1. *Let $\mathcal{D}$ be a 1-(v,k,λ) design. Let x be a point of the design contained in the blocks $B_1,...,B_\lambda$. We define:*

$$I_x = \bigcap_{i=1}^{\lambda} B_i.$$

If $\mathcal{D}$ is a 1-(v,k,λ) design constructed by Key-Moori methods, then by [3, Remark 2.9], we have a $1-(\frac{v}{|I_x|}, \frac{k}{|I_x|}, \lambda)$ design called the reduced design of $\mathcal{D}$ which is denoted by $\mathcal{D}_I$. It is not difficult to show that the size of I_x is independent from the choice of x. In view of this, we denote the size of I_x by $n_{\mathcal{D}}$.

In section 3, we find $n_{\mathcal{D}}$ for two families of simple groups from which the designs using Key-Moori Method 2 are already constructed. The first family is $\mathrm{PSL}_2(q)$, where q is a power of 2. We also consider the Suzuki group $Sz(q)$, where q is an odd power of 2. The designs constructed from these families using Key-Moori methods are given in [13] and [14], respectively.

We pursue three purposes in finding I_x and $n_{\mathcal{D}}$. Our first goal is to construct reduced designs from designs that have been already constructed, namely $\mathrm{PSL}_2(q)$ and $Sz(q)$. Our second goal is to find the parameters of designs invariant under the same families of groups using Method 3. In fact, we show that if $\mathcal{D}$ is a design constructed by Method 2, then finding $n_{\mathcal{D}}$ gives us the parameters of the corresponding designs constructed by Method 3. Our third goal is to use the set I_x to find the automorphism group of each design. It is of course not easy to find the automorphism groups of designs in general, but we show that the set I_x may be helpful in some special cases.

In this paper all groups are finite and the notation is standard. If $G \leq S_\Omega$ is a permutation group, then the point stabilizer of $\alpha \in \Omega$ is denoted by G_α. Also, a subset of Ω of all fixed points of $g \in G$ is denoted by $Fix(g)$. In [14], we defined the set

$$\mathcal{A}_M = \{|M \cap M^g| \mid g \in G\}.$$

This set has been computed for several families of finite simple groups. We will use those results throughout this paper. Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I})$ be an incidence structure, i.e. a triple with point set $\mathcal{P}$, block set $\mathcal{B}$ disjoint to $\mathcal{P}$ and incidence set $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$. If the ordered pair $(p,B) \in \mathcal{I}$ we say that p is incident with B. It is often convenient to assume that the blocks in $\mathcal{B}$ are subsets of $\mathcal{P}$ so $(p,B) \in \mathcal{D}$ if and only if $p \in B$. For a positive integer t, we say that $\mathcal{D}$ a t-design if every block $B \in \mathcal{B}$ is incident with exactly k points and every t distinct points are together incident with λ blocks. In this case we say $\mathcal{D}$ is a $t-(v,k,\lambda)$ design where $v = |\mathcal{P}|$. We say that $\mathcal{D}$ is symmetric if it has the same number of points and blocks.

2. I_x and Key-Moori Methods

We start with the following definition which is a slight generalization of A_x in [3].

Definition 2.1. *Let $G \leq S_\Omega$ be a permutation group . We define*

$$A(x) = \bigcap_{\alpha \in Fix(x)} G_\alpha.$$

Lemma 2.2. *We have $A(x) = \{g \in G{:}Fix(x) \subseteq Fix(g)\}$.*

Proof. First assume that $g \in A(x)$. We will prove that $Fix(x) \subseteq Fix(g)$. Suppose that $\alpha \in Fix(x)$. Since $g \in A(x)$, we conclude that $g \in G_\alpha$. Therefore, $\alpha \in Fix(g)$. Conversely, assume that $g \notin A(x)$. Then we can find some point α such that $g \notin G_\alpha$ and $\alpha \in Fix(\mathrm{x})$. Therefore, $\alpha \in Fix(x) \setminus Fix(g)$ and we have $Fix(x) \not\subseteq Fix(g)$. □

Corollary 2.3. *Let G be a permutation group on Ω. If $x \in G$ then $A(x^g) = A(x)^g$.*

Proof. It follows from Lemma 2.2 and the fact that $Fix(x^g) = Fix(x)^g$. □

Proposition 2.4. *(Key-Moori Method 2.) Let G be a finite simple group and M a maximal subgroup of G. Assume that $B = \{(M \cap x^G)^y | y \in G\}$. Then we have a $1-(|x^G|, |M \cap x^G|, \chi(x))$ design $\mathcal{D}$. The group G acts as an automorphism group on $\mathcal{D}$, primitive on blocks and transitive (not necessarily primitive) on points of $\mathcal{D}$.*

For the purposes of this paper, we denote by $\mathcal{D}_G(M,x)$ the design constructed by Method 2 from a simple group G and a maximal subgroup M containing x. If G and M are known from the context, we simply denote this design by $\mathcal{D}(x)$. We can generalize Method 2 to finite primitive groups (which are not necessarily simple) with a point stabilizer M. In the case that G is a simple group, the action of G by conjugation on the set of conjugates of M is primitive. We denote the permutation character of this action by χ_M. Recall that the permutation character $\chi_M(x)$ is the number of fixed points of x, which in this case is the number of conjugates of M containing x.

Lemma 2.5. *Let $\mathcal{D} = \mathcal{D}_G(M,x)$. Then we have: $I_x = x^G \cap A(x)$. Moreover if $S = N_G(A(x))$, then $I_x = x^S$.*

Proof. The proof follows from [3, Lemma 4.8]. □

By Lemma 2.5, we can see that constructing the reduced designs constructed by Method 2 is related to the subgroup $A(x)$.

Corollary 2.6. *If $\mathcal{D} = \mathcal{D}_G(M,x)$ is a 1-$(v,k,1)$ design, then for each point x of the design, we have $I_x = x^G \cap M$. In particular, $n_{\mathcal{D}} = k$ and the reduced design is a 1-$(\frac{v}{k},1,1)$ design.*

Proof. Since $\lambda = 1$, then we have $Fix(x) = \{M\}$. So $A(x) = M$ and by Lemma 2.5, we have $I_x = x^G \cap M$. Hence $|I_x| = k$.

□

Definition 2.7. *Let H be a subgroup of G. We say that H controls G-fusion in itself if every pair of G-conjugate elements of H are H-conjugate.*

Lemma 2.8. *Let G be a finite group acting 2-transitively on a set of size m. If M is a point stabilizer of G, then we have:*

$$\mathcal{A}_M = \{|M|, \frac{|M|}{m-1}\}.$$

Proof. Since G is 2-transitive, then a point stabilizer M acts transitively on a set of length $m-1$. Hence the length set of suborbits of the action of G equals $\{1, m-1\}$, where $m = |G{:}M|$. In particular, for all $g \in G - M$, we have $M \cap M^g = \frac{|M|}{m-1}$. This completes the proof. □

Proposition 2.9. *Let G be a finite group acting 2-transitively on a set of size m and M a point stabilizer of G. If $\mathcal{D}_G(x,M)$ is a 1-$(v,k,2)$ design, then M is a Frobenius group and $I_x = x^G \cap H$, where H is a Frobenius complement of M.*

Proof. The number of fixed points of x is equal to $\chi_M(x) = \lambda = 2$. Let $Fix(x) = \{\alpha, \beta\}$. Thus, every element of $M = G_\alpha$ fixes at most 1 point in $\Omega - \{\alpha\}$ which implies M is a Frobenius group. Moreover, $H = G_\alpha \cap G_\beta$ is the Frobenius complement of M and is equal to $A(x)$. So by Lemma 2.5, $I_x = x^G \cap H$. The proof is now completed.

□

We are now going to discuss the connection between $n_{\mathcal{D}}$ and Method 3. Notice that $\mathcal{D}$ is a design constructed by Method 2.

Theorem 2.10. *(Method 3, see [7]) Let $G \leq S_\Omega$ be a finite transitive permutation group, $|\Omega| > 1$. For some $g \in G$, let $B = Fix(x)$ and $\mathcal{B} = \{B^y : y \in G\}$. If $S(x) = \{h \in x^G : Fix(x) = Fix(h)\}$, then $|\mathcal{B}| = \frac{|x^G|}{|S(x)|}$ and we have a 1-design $1-(v,k,\lambda)$ with point set Ω and block set $\mathcal{B}$. Moreover, we have $v = |\Omega|$, $k = |Fix(x)|$ and $\lambda = \frac{k \times |x^G|}{v \times |S(x)|}$.*

Lemma 2.11. *Let G be a transitive group and $x \in G$. Then we have $S(x) = A(x) \cap x^G$.*

Proof. Assume that $y \in S(x)$. Then $y \in x^G$ and $Fix(x) = Fix(y)$. So by Lemma 2.2, we have $y \in A(x)$. In particular, $S(x) \subseteq x^G \cap A(x)$. Conversely, assume that $y \in x^G \cap A(x)$. Using Lemma 2.2 again, we have $Fix(x) \subseteq Fix(y)$. On the other hand, $y = x^g$ for some $g \in G$. This implies that $|Fix(x)| = |Fix(y)|$. Hence, $Fix(x) = Fix(y)$ and we get $x^G \cap A(x) \subseteq S$.

□

Corollary 2.12. *We have $|S(x)| = n_{\mathcal{D}}$, where $\mathcal{D}$ is a design constructed by Method 2.*

Proof. It follows directly from Lemma 2.5 and Lemma 2.11. □

In [15, Proposition 3.8], we proved that having the parameters of Method 3, we can directly compute the parameters of Method 2. The following result shows that the converse is true if $n_{\mathcal{D}}$ is given.

Proposition 2.13. *Let G be a group acting primitively on a set Ω. Suppose that for some $x \in G$, the design $\mathcal{D}$ constructed by applying Method 2 is a $1-(v,k,\lambda)$ design. Then the design $\mathcal{D}'$ constructed by applying Method 3 to a point stabilizer M containing x is of type*

$$1-(\frac{\lambda v}{k}, \lambda, \frac{k}{n_{\mathcal{D}}}).$$

Proof. By the construction of Method 3 and Corollary 2.12, we have $\mathcal{D}'$ is a $1-(|\Omega|, \lambda, \frac{\lambda \times v}{|\Omega| \times n_{\mathcal{D}}})$ design. On the other hand, by [14, Lemma 4.2], we have $|\Omega| = |G{:}M| = \frac{\lambda \times v}{k}$. The proof is now completed. □

Corollary 2.14. *Let $\mathcal{D}$ be $1-(v,k,\lambda)$ reduced design of a design constructed by Method 2. Then the corresponding design constructed by Method 3 is a $1-(\frac{\lambda v}{k}, \lambda, k)$ design.*

Proof. We have $n_{\mathcal{D}} = 1$, so the result follows from Proposition 2.13. □

3. Constructing new designs from $\mathrm{PSL}_2(q)$ and $Sz(q)$

Throughout this section, we assume that $G_1 = \mathrm{PSL}_2(q)$ and $G_2 = Sz(q)$ (where in both cases, q is even) containing maximal subgroups M_1 and M_2, respectively. We aim to find $|I_x|$ for all maximal subgroups of G_1 and G_2, so we can find the parameters of the reduced designs as well as the designs constructed by Method 3. Note that if $\lambda = 1$, then by Corollary 2.6, we have $|I_x| = k$.

Theorem 3.1. *For $i \in \{1,2\}$, suppose that the action of G_i on the set of conjugates of M_i is 2-transitive. Then we have the following*

1. $|I_x| = q-1$ *if* $o(x) = 2$;
2. $|I_x| = \frac{q(q-1)}{2}$ *if* $o(x) = 4$;
3. $|I_x| = 2$ *if* $o(x) | q-1$.

Proof. By [13, Proposition 3.3] and [14, Table 2], if $o(x) = 2$ or 4, then we have $\lambda = 1$. So, assume that $o(x)$ divides $q-1$. As $\lambda = 2$, we conclude by Lemma 2.9 that $I_x = x^G \cap H$, where H is a subgroup of a dihedral group of index 2 containing x. Since H is TI-subgroup, we have $x^G \cap H = \{x, x^{-1}\}$. Therefore, $|I_x| = 2$. This completes the proof. □

Remark 3.2. *Let $\mathcal{D}$ be a 1-(v,k,λ) design constructed by Method 2. If $\lambda = 1$, then the reduced design is a trivial design. Also if $|I_x| = 1$, then the reduced design is the same as the original design. So, we are mainly interested in designs with $\lambda > 1$ and $n_{\mathcal{D}} > 1$. Let us call them interesting reduced designs.*

Corollary 3.3. *For $i = 1$ or 2, let $\mathcal{D}_i$ be an interesting reduced design from the group G_i and a maximal subgroup of order $q^i(q+1)$. Then $\mathcal{D}_i$ is a $1-(\frac{q^i(q^i+1)}{2}, q^i, 2)$ design. Also, the corresponding designs $\mathcal{D}_i'$ constructed by Method 3 are of type $1-(q^i+1, 2, q^i)$.*

Proof. The result follows by [13, Proposition 3.3], [14, Table 2] and Theorem 3.1. □

Proposition 3.4. *For $i \in \{1,2\}$, if M_i is not of index q^i+1, then M_i controls G_i-fusion in itself.*

Proof. See [13, 14]. □

Proposition 3.5. *For $i \in \{1,2\}$, assume that M_i is a dihedral group. Then we have*

1. $|I_x| = 1$ *if* $o(x) = 2$;
2. $|I_x| = 2$ *if* $o(x) > 2$.

Proof. First assume that $o(x) = 2$ and $M = M_i$ for $i = 1$ or 2. Then by the proof of Theorem 3.4 (Step 3) in [14], we have $|M \cap M^g| \leq 2$. In particular, if x belongs to both M and M^g, then $M \cap M^g \cap x^G = \{x\}$. That is, $|I_x| = 1$. Next suppose that $o(x) > 2$ and $\mathcal{D}(G_i, M_i)$ is a 1-(v,k,λ) design. By [13, Corollary 3.5] and [14, Table 2], we have $k = 2$ and $\lambda = 1$. Therefore by Lemma 2.6, we have $|I_x| = 2$. □

Proposition 3.6. *Let M be a maximal subgroup of G_2 and $|M| = 4(q^2 \pm 2q + 1)$. Then we have*

1. $|I_x| = 1$ *if* $o(x) = 2$;
2. $|I_x| = 2$ *if* $o(x) = 4$;
3. $|I_x| = 4$ *if* $o(x)|q^2 + 1$.

Proof. If $o(x)|q^2 + 1$, then by [14, Table 2] we have $k = 4$ and $\lambda = 1$. So we can assume that $o(x) = 2$ or 4. By the results of [14], we have $\mathcal{A}_M = \{1, 2, 4, |M|\}$. Hence if M contains x and $M \neq M^g$ for some $g \in G$, then $|M \cap M^g| = 2$ or 4. In particular if $o(x) = 2$, then $|I_x| = 1$. Now let $|M \cap M^g| = 4$. It is clear that $M \cap M^g$ is a cyclic group of order 4. Therefore, $A(x) = \langle x \rangle$. Since we have a single conjugacy class of elements of order 4, both x and x^{-1} lie in $I_x = A(x) \cap x^G$. Hence, $|I_x| = 2$.

□

Corollary 3.7. *There are no interesting reduced designs constructed from the maximal subgroups of $PSL_2(q)$ or $Sz(q)$ of dihedral type. If $G = Sz(q)$ and M is a maximal subgroup of order $4(q^2 \pm \sqrt{2q} + 1)$, then we have that the intersecting reduced designs $\mathcal{D}$ are of type*

$$1 - (\frac{q(q^2 + 1)(q - 1)}{4}, \frac{q^2 \pm \sqrt{2q} + 1)}{2}, \frac{q}{2}).$$

Also, the corresponding designs $\mathcal{D}'$ constructed by Method 3 are of type

$$1 - (\frac{q^2(q - 1)(q \mp \sqrt{2q} + 1)}{4}, \frac{q}{2}, \frac{q^2 \pm \sqrt{2q} + 1}{2}).$$

Proof. The result follows by [14, Table 2] and Proposition 3.6. □

Theorem 3.8. *Let $M_1 \cong PSL_2(q_0)$ and $M_2 \cong Sz(q_0)$. If $x \in M_1$ then:*

1. $|I_x| = q_0 - 1$ *if* $o(x) = 2$;
2. $|I_x| = 2$ *if* $o(x)|q_0 - 1$.

If $x \in M_2$ then:

1. $|I_x| = q_0 - 1$ *if* $o(x) = 2$.
2. $|I_x| = \frac{q_0^2 - 1}{2}$ *if* $o(x) = 4$;
3. $|I_x| = 4$ *if* $o(x)|q_0^2 \pm \sqrt{2q_0} + 1$;
4. $|I_x| = 2$ *if* $o(x)|q_0 - 1$.

Proof. First assume that $M = M_1$. By [13], we have $A_M = \{1, q_0, q_0 \pm 1, |M|\}$. Let $o(x) = 2$. Then x lies in a unique Sylow 2-subgroup Q_1 of M (the uniqueness is due to the fact that the Sylow 2-subgroups of M are TI-subgroups). Now let Let $\{M^{g_1}, ..., M^{g_k}\}$ be distinct conjugates of M containing x with $g_1 = 1$. Then, $M \cap M^{g_i}$ $(2 \leq i \leq k)$ is a Sylow 2-subgroup of M containing x. Therefore all these conjugates of M intersect in Q_1 and we get $A(x) = Q_1$. Since G has only one class of involutions, we conclude that $x^G \cap Q_1 = Q_1 - \{1\}$. That is, $|I_x| = q_0 - 1$. If $o(x)|q_o \pm 1$, then using a similar argument we get $A(x)$ is a subgroup of order $q_0 \pm 1$. Since this subgroup lies in a dihedral group, we conclude that $A(x) \cap x^G = \{x, x^{-1}\}$. Therefore, $|I_x| = 2$.
Next suppose that $M = M_2$. Then we have

$$A_M = \{1, q_0, q_0^2, q_0 - 1, q_0^2 \pm \sqrt{2q_0} + 1, |M|\}.$$

If $o(x) = 2$ or 4, then $A(x)$ lies in a unique Sylow 2-subgroup Q_2 of M_2. In particular, $|A(x)| \in \{q_0, q_0^2\}$. We know that $|Z(Q_2)| = q_0$ and $Z(Q_2)$ contains all involutions of Q_2. So if $o(x) = 2$, then $I_x = A(x) \cap x^G$. Hence, $|I_x| = q_0 - 1$. Also if $o(x) = 4$, then $A(x) = Q_2$. Since we have two classes of elements of order 4, we conclude that $|I_x| = |A(x) \cap x^G| = \frac{q_0^2 - 1}{2}$. If $o(x)|q_0 - 1$, then using a similar argument to the previous case we have $|I_x| = 2$. So assume that $o(x)|q_0^2 + 1$. Hence, $A(x) = q_0^2 \pm \sqrt{2q_0} + 1$. Now we can write

$$|M \cap x^G| = |x^M| = |M{:}C_M(x)| = \frac{|M|}{|A(x)|}.$$

Also, $|A(x) \cap x^G|$ equals $|M \cap x^G|/t$, where t is the number of conjugates of $A(x)$ in M. On the other hand we have:

$$t = |M{:}N_M(A(x))| = \frac{|M|}{4|A(x)|}.$$

Hence,

$$|I_x| = |A(x) \cap x^G| = \frac{|M|}{|A(x)|} / \frac{|M|}{4|A(x)|} = 4$$

□

Corollary 3.9. *The followings are the parameters of interesting designs from the groups $G_1 = PSL_2(q)$ and $G_2 = Sz(q)$ and their maximal subgroups $M_1 = \mathrm{PSL}_2(q_0)$ and $M_2 = Sz(q_0)$, respectively.*

- $1-(\frac{q^2-1}{q_0-1}, q_0+1, \frac{q}{q_0})$;
- $1-(\frac{q^2+q}{2}, \frac{q_0^2+q_0}{2}, \frac{q-1}{q_0-1})$;
- $1-(\frac{q^2-q}{2}, \frac{q_0^2-q_0}{2}, \frac{q+1}{q_0+1})$ *or* $1-(\frac{q^2+q}{2}, \frac{q_0^2-q_0}{2}, \frac{q-1}{q_0+1})$;
- $1-(\frac{(q^2+1)(q-1)}{q_0-1}, q_0^2+1, \frac{q^2}{q_0^2})$;
- $1-(\frac{q(q^2+1)(q-1)}{q_0^2-1}, \frac{q_0(q_0^2+1)}{q_0+1}, \frac{q}{q_0})$;
- $1-(\frac{q^2(q^2+1)}{2}, \frac{q_0^2(q_0^2+1)}{2}, \frac{q-1}{q_0-1})$;
- $1-(\frac{q^2(q-1)(q\pm\sqrt{2q}+1)}{4}, \frac{q_0^2(q_0-1)(q_0\pm\sqrt{2q_0}+1)}{4}, \frac{(q\mp\sqrt{2q}+1)}{(q_0\mp\sqrt{2q_0}+1)})$.

Moreover, the corresponding designs constructed by Method 3 have the following parameters.

- $1-(\frac{q^3-q}{q_0^3-q_0^3}, \frac{q}{q_0}, q_0+1)$;
- $1-(\frac{q^3-q}{q_0^3-q_0^3}, \frac{q-1}{q_0-1}, \frac{q_0^2+q_0}{2})$;
- $1-\frac{q^3-q}{q_0^3-q_0^3}, \frac{q+1}{q_0+1}, \frac{q_0^2-q_0}{2})$ *or* $1-(\frac{q^2+q}{2}, \frac{q-1}{q_0+1}, \frac{q_0^2-q_0}{2})$;
- $1-(\frac{q^2(q^2+1)(q-1)}{q_0^2(q_0^2+1)(q_0-1)}, \frac{q^2}{q_0^2}, q_0^2+1)$;
- $1-(\frac{q^2(q^2+1)(q-1)}{q_0^2(q_0^2+1)(q_0-1)}, \frac{q}{q_0}, \frac{q_0(q_0^2+1)}{q_0+1})$;
- $1-(\frac{q^2(q^2+1)(q-1)}{q_0^2(q_0^2+1)(q_0-1)}, \frac{q-1}{q_0-1}, \frac{q_0^2(q_0^2+1)}{2})$;
- $1-(\frac{q^2(q^2+1)(q-1)}{q_0^2(q_0^2+1)(q_0-1)}, \frac{(q\mp\sqrt{2q}+1)}{(q_0\mp\sqrt{2q_0}+1)}, \frac{q_0^2(q_0-1)(q_0\pm\sqrt{2q_0}+1)}{4})$.

Proof. The result follows by [13, Proposition 3.3], [14, Table 2] and Theorem 3.8. □

4. Some results on automorphism groups

If $\mathcal{D}$ is a 1-$(v,k,1)$, then it is easy to see that the automorphism group of $\mathcal{D}$ equals $S_k \wr S_b$. If $\lambda = 2$, the structure of the automorphism group of the design could be more complicated. However, in some special cases, we might be able to find the structure of the automorphism groups of the designs.

Lemma 4.1. *Let $\mathcal{D}$ be a 1-$(v,k,2)$ design with b blocks. Assume that for every two distinct blocks B_1 and B_2, we have $|B_1 \cap B_2| = m$. Then we have:*

1. $|n_{\mathcal{D}}| = m$;
2. $\mathcal{D}$ *is of type* $1-(\binom{b}{2}m, (b-1)m, 2)$.
3. $\mathcal{D}_I$ *is of type* $1-(\binom{b}{2}, b-1, 2)$.

4. $Aut(\mathcal{D}) \cong (S_m)^{b(b-1)/2}{:}S_b$.

Proof. We have $\lambda = 2$, so every point x lies in exactly two blocks. Hence, $|I_x| = m$. Now we can write:

$$v = |\bigcup_{i=1}^{b} B_i| - |\bigcup_{i=1}^{b-1}\bigcup_{j=i+1}^{b}(B_i \cap B_j)| = bk - \binom{b}{2}m.$$

On the other hand, $2v = bk$. Therefore, $v = \binom{b}{2}m$, and

$$k = 2v/b = (b-1)m.$$

In particular, $\mathcal{D}_I$ is of type $1-(\binom{b}{2}, (b-1), 2)$. We also have $S(I) = (S_m)^t$, where $t = v/m = \binom{b}{2}$. Hence,

$$Aut(\mathcal{D})/(S_m)^t \cong Aut(\mathcal{D}_I).$$

Now consider the design $\mathcal{D}_I$ and note that the intersection of every two blocks is a singleton. So it is easy to see that $Aut(\mathcal{D}_I) = S_b$. This completes the proof. □

As an application of Lemma 4.1 and the concept of reduced designs, we state the following

Corollary 4.2. *Suppose that $\mathcal{D}$ is a 1-$(q^2+q, 2q, 2)$ design invariant under $PSL_2(q)$ and $\mathcal{D}'$ is a 1-$(q^2(q^2+1), q^2, 2)$ design invariant under $Sz(q)$ constructed by Method 2, using elements of orders dividing $q-1$. Then we have*

$$Aut(\mathcal{D}) \cong 2^{\frac{q^2(q^2+1)}{2}}{:}S_{q^i+1}.$$

Acknowledgement

The author wishes to thank both referees for their constructive comments and suggestions.

References

[1] M. R. Darafsheh, A. Iranmanesh, and R. Kahkeshani, *Some designs and codes invariant under the groups S_9 and A_8*, Des. Codes Cryptogr., 51 (2009), pp. 211–223.

[2] J. D. Key and J. Moori, *Codes, designs and graphs from the Janko groups J_1 and J_2*, J. Combin. Math. Combin. Comput., 40 (2002), pp. 143–159.

[3] T. Le and J. Moori, *On the automorphisms of designs constructed from finite simple groups*, Des. Codes Cryptogr., 76 (2015), pp. 505–517.

[4] X. Mbaale and B. G. Rodrigues, *Symmetric 1-designs from* $\mathrm{PGL}_2(q)$*, for q an odd prime power*, Glas. Mat. Ser. III, 56(76) (2021), pp. 1–15.

[5] X. Mbaale and B. G. Rodrigues, *Symmetric 1-designs from* $\mathrm{PSL}_2(q)$*, for q a power of an odd prime*, Trans. Comb., 10 (2021), pp. 43–61.

[6] J. Moori, *Finite groups, designs and codes*, in Information security, coding theory and related combinatorics, vol. 29 of NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur., IOS, Amsterdam, 2011, pp. 202–230.

[7] J. Moori, *Designs and codes from fixed points of finite groups*, Comm. Algebra, 49 (2021), pp. 706–720.

[8] J. Moori and B. G. Rodrigues, *Some designs and codes invariant under the simple group* Co_2, J. Algebra, 316 (2007), pp. 649–661.

[9] ——, *On some designs and codes invariant under the Higman-Sims group*, Util. Math., 86 (2011), pp. 225–239.

[10] J. Moori, B. G. Rodrigues, A. Saeidi, and S. Zandi, *Some symmetric designs invariant under the small Ree groups*, Comm. Algebra, 47 (2019), pp. 2131–2148.

[11] ——, *Designs from maximal subgroups and conjugacy classes of Ree groups*, Adv. Math. Commun., 14 (2020), pp. 603–611.

[12] J. Moori and A. Saeidi, *Some designs and codes invariant under the Tits group*, Adv. Math. Commun., 11 (2017), pp. 77–82.

[13] ——, *Constructing some designs invariant under $PSL_2(q)$, q even*, Comm. Algebra, 46 (2018), pp. 160–166.

[14] ——, *Some designs invariant under the Suzuki groups*, Util. Math., 109 (2018), pp. 105–114.

[15] A. Saeidi, *Designs and codes from fixed points of alternating groups*, Comm. Algebra, 50 (2022), pp. 2215–2222.

Please cite this article using:

Amin Saeidi, Reduced designs constructed by Key-Moori Method 2 and their connection with Method 3, AUT J. Math. Com., 4(1) (2023) 39-46
https://doi.org/10.22060/ajmc.2022.21378.1092