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Original Article

Finite non-solvable groups with few 2-parts of co-degrees of irreducible characters

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ABSTRACT: For a character χ of a finite group G, the number $\chi^c(1) = \frac{[G:\ker\chi]}{\chi(1)}$ is called the co-degree of χ. Let $\mathrm{Sol}(G)$ denote the solvable radical of G. In this paper, we show that if G is a finite non-solvable group with $\{\chi^c(1)_2 : \chi \in \mathrm{Irr}(G)\} = \{1, 2^m\}$ for some positive integer m, then $G/\mathrm{Sol}(G)$ has a normal subgroup $M/\mathrm{Sol}(G)$ such that $M/\mathrm{Sol}(G) \cong \mathrm{PSL}_2(2^n)$ for some integer $n \geq 2$, $[G : M]$ is odd and $G/\mathrm{Sol}(G) \lesssim \mathrm{Aut}(\mathrm{PSL}_2(2^n))$.

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1. Introduction

Throughout this paper, G is a finite group, $\mathrm{Fit}(G)$ is the Fitting subgroup of G, $\mathrm{Sol}(G)$ is the solvable radical of G and p is a prime number. Let $\mathrm{Irr}(G)$ be the set of irreducible characters of G. For a normal subgroup N of G, set $\mathrm{Irr}(G|N) = \mathrm{Irr}(G) - \mathrm{Irr}(G/N)$. For $\psi \in \mathrm{Irr}(N)$, let $\mathrm{Irr}(G|\psi)$ be the set of irreducible constituents of the induced character ψ^G and let $I_G(\psi)$ denote the inertia group of ψ in G. If n is a positive integer, we use n_p to show the p-part of n.

For a character χ of G, the number $\chi^c(1) = \frac{[G:\ker\chi]}{\chi(1)}$ is called the co-degree of χ (see [17]). Set $\mathrm{Codeg}(G) = \{\chi^c(1) : \chi \in \mathrm{Irr}(G)\}$. In [4, 6, 5, 3, 2, 1, 7, 9, 8], [14] and [18], it has been shown how the co-degrees of irreducible characters of G can explain the structure of G.

Let N be a normal subgroup of G. By [2, Theorem 1.6], if the co-degrees of elements of $\mathrm{Irr}(G|N)$ are square-free, then N is a super-solvable group of derived length at most 2. In [10], the finite groups with non-trivial Fitting

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subgroups such that the co-degrees of their irreducible characters whose kernels do not contain the Fitting subgroups are cube-free have been studied. If G is a finite p-solvable group, then [2] and [9] show that the p-length of G is at most $\mathrm{Min}\{|A|, \log_p(B)\}$, where $A = \{\chi^c(1) : \chi \in \mathrm{Irr}(G), p \mid \chi^c(1)\}$ and $B = \mathrm{Max}\{\chi^c(1)_p : \chi \in \mathrm{Irr}(G)\}$. In [15], the previous result obtained in [9] has been improved and it has been shown that if G is a p-solvable group, then the p-length of G is at most $2\log_2(\log_p(B)) + 3$. In [1], we have found the upper for the p-length of a p-solvable group in terms of the number of its irreducible character co-degrees which are divisible by p. In this paper, we prove the following theorem:

Theorem 1.1. *Let G be a finite non-solvable group. If $\{\chi^c(1)_2 : \chi \in \mathrm{Irr}(G)\} = \{1, 2^m\}$ for some positive integer m, then $G/\mathrm{Sol}(G)$ has a normal subgroup $M/\mathrm{Sol}(G)$ such that $M/\mathrm{Sol}(G) \cong \mathrm{PSL}_2(2^n)$ for some integer $n \geq 2$, $[G : M]$ is odd and $G/\mathrm{Sol}(G) \lesssim \mathrm{Aut}(\mathrm{PSL}_2(2^n))$.*

2. The proofs of the main theorems

We first bring some lemmas that will be used in the proof of the main theorem.

Lemma 2.1. [17, Lemma 2.1] *Let N be a normal subgroup of G. Then,* $\mathrm{Codeg}(G/N) \subseteq \mathrm{Codeg}(G)$. *Also, if $\theta \in \mathrm{Irr}(N)$, then for every $\chi \in \mathrm{Irr}(G|\theta)$, $\theta^c(1) \mid \chi^c(1)$.*

Lemma 2.2. [17, Theorem A] *Every prime divisor p of $|G|$ divides some elements of* $\mathrm{Codeg}(G)$.

Lemma 2.3. ([16] and [11, Theorem 2.1 and Lemma 2.2]) *Let $M = S^n$ be a non-abelian minimal normal subgroup of a group G. Then, there exists a non-trivial irreducible character $\chi = \alpha \times \cdots \times \alpha \in \mathrm{Irr}(M)$ of p'-degree such that $[G : I_G(\chi)]$ is a p'-number and χ extends to $I_G(\chi)$.*

Lemma 2.4. [13, Theorem 1] *Let G be a non-solvable group such that $\chi(1)_2 = 1$ or $|G|_2$ for every $\chi \in \mathrm{Irr}(G)$. Then, there exists a minimal normal subgroup N of G such that $N \cong \mathrm{PSL}_2(2^n)$ and G/N is an odd order group.*

Proof of Theorem 1.1. We complete the proof by induction on $|G|$. First let $\mathrm{Sol}(G) \neq 1$. Since $G/\mathrm{Sol}(G)$ is non-solvable, $2 \mid |G/\mathrm{Sol}(G)|$. It follows from Lemma 2.2 that $2 \mid \chi^c(1)$ for some $\chi \in \mathrm{Irr}(G/\mathrm{Sol}(G))$. On the other hand, Lemma 2.1 guarantees that $\mathrm{Codeg}(G/\mathrm{Sol}(G)) \subseteq \mathrm{Codeg}(G)$. This shows that $G/\mathrm{Sol}(G)$ satisfies the assumption of the theorem. Note that $\mathrm{Sol}(G/\mathrm{Sol}(G)) = 1$. So, the theorem follows from the induction on $|G|$. Now suppose that $\mathrm{Sol}(G) = 1$. Let M be a minimal normal subgroup of G. Then, M is non-abelian. Hence, $M = S^t$ for some non-abelian simple group S and a positive integer t. Thus, Lemma 2.3 forces the existence of a non-principal character $\chi \in \mathrm{Irr}(M)$ of odd degree such that $2 \nmid [G : I_G(\chi)]$ and χ is extendible to $I_G(\chi)$. Note that $C_G(M) \leq I_G(\chi)$ and $M \cap C_G(M) = 1$. Hence, we can assume that χ is extendible to $\mu \in \mathrm{Irr}(I_{G/C_G(M)}(\chi))$. Set $\psi = \mu^{G/C_G(M)}$. Then, $\psi \in \mathrm{Irr}(G/C_G(M))$ and $2 \nmid \psi(1)$. We have $G/C_G(M) \lesssim \mathrm{Aut}(M)$ and $M \cong MC_G(M)/C_G(M)$ is the unique minimal normal subgroup of $G/C_G(M)$. If $\ker\psi \neq 1$, then since $\ker\psi \trianglelefteq G/C_G(M)$, $MC_G(M)/C_G(M) \leq \ker\psi$, a contradiction. This shows that $\ker\psi = 1$. Hence, $|G/C_G(M)|_2 = \psi^c(1)_2$. If $2 \mid |G/MC_G(M)|$, then Lemma 2.2 guarantees the existence of $\varphi \in \mathrm{Irr}(G/MC_G(M))$ such that $2 \mid \varphi^c(1)$. As $\varphi^c(1) \mid |G/MC_G(M)|$ and $2 \mid |MC_G(M)/C_G(M)|$, we get that $1 < \varphi^c(1)_2 < \psi^c(1)_2$. However, $\mathrm{Codeg}(G/C_G(M)) \subseteq \mathrm{Codeg}(G)$. So, $\varphi^c(1), \psi^c(1) \in \mathrm{Codeg}(G)$ with distinct and non-trivial 2-parts. This is a contradiction. This yields that $2 \nmid |G/MC_G(M)|$. Consequently, $2 \nmid \theta(1)$ for every $\theta \in \mathrm{Irr}(G/MC_G(M))$. Also, $MC_G(M)/C_G(M)$ is the unique minimal normal subgroup of $G/C_G(M)$. So, if $\chi \in \mathrm{Irr}(G/C_G(M)|MC_G(M)/C_G(M))$, then $\ker\chi = 1$. By the assumption of the theorem, $\chi^c(1)_2 = 1$ or

$$\chi^c(1)_2 = \psi^c(1)_2 = |G/C_G(M)|_2 = |MC_G(M)/C_G(M)|_2 = |M|_2. \tag{1}$$

Therefore, $\chi(1)_2 = |G/C_G(M)|_2$ or $\chi(1)_2 = 1$, for every $\chi \in \mathrm{Irr}(G/C_G(M))$. So, Lemma 2.4 shows that

$$M \cong MC_G(M)/C_G(M) \cong \mathrm{PSL}_2(2^n) \tag{2}$$

for some integer $n \geq 2$, $[G/C_G(M) : MC_G(M)/C_G(M)]$ is odd and $G/C_G(M) \lesssim \mathrm{Aut}(\mathrm{PSL}_2(2^n))$. If $C_G(M) \neq 1$, then we can assume that G has a minimal normal subgroup N such that $N \leq C_G(M)$. Arguing by analogy as above, $N \cong \mathrm{PSL}_2(2^l)$ for some integer $l \geq 2$ and for every $\theta \in \mathrm{Irr}(G/C_G(N)|NC_G(N)/C_G(N))$, $\theta^c(1)_2 = 1$ or $|N|_2$. This forces $|N|_2 = |M|_2$ and hence, $l = n$. We have $M \times N \trianglelefteq G$. As M and N are non-solvable, there exist non-principal characters $\mu_1 \in \mathrm{Irr}(M)$ and $\mu_2 \in \mathrm{Irr}(N)$ such that $2 \nmid \mu_1(1), \mu_2(1)$, by Ito-Michler's theorem. Then, $\eta = \mu_1\mu_2 \in \mathrm{Irr}(M \times N)$ and $\ker\eta = 1$. So, $|M|_2^2 = |MN|_2 = \eta^c(1)_2$. It follows from Lemma 2.1 that $|M|_2^2 \mid \beta^c(1)$ for every $\beta \in \mathrm{Irr}(G|\eta)$. Using (1) and the assumption of the theorem which says that $\{\chi^c(1)_2 : \chi \in \mathrm{Irr}(G)\} = \{1, 2^m\}$, we deduce that $|M|_2^2 \leq |M|_2$, a contradiction. This forces $C_G(M) = 1$ and hence, (2) implies that $\mathrm{PSL}_2(2^n) \cong M \trianglelefteq G$, $[G : M]$ is odd and $G = G/C_G(M) \lesssim \mathrm{Aut}(\mathrm{PSL}_2(2^n))$, as desired.

□

References

[1] N. Ahanjideh, *On the p-solvable groups with few p-parts of irreducible character co-degrees.* Submitted.

[2] ——, *The Fitting subgroup, p-length, derived length and character table*, Mathematische Nachrichten, 294 (2021), pp. 214–223.

[3] ——, *The one-prime hypothesis on the co-degrees of irreducible characters*, Communications in Algebra, 49 (2021), pp. 4016–4020.

[4] ——, *Co-degree graphs and order elements*, Bulletin of the Malaysian Mathematical Sciences Society, 45 (2022), pp. 2653–2664.

[5] ——, *Finite groups admitting at most two irreducible characters having equal co-degrees*, Journal of Algebra and Its Applications, (2022), p. 2350098.

[6] ——, *Nondivisibility among irreducible character co-degrees*, Bulletin of the Australian Mathematical Society, 105 (2022), pp. 68–74.

[7] Z. Akhlaghi, M. Ebrahimi, and M. Khatami, *On the multiplicities of the character codegrees of finite groups*, Algebras and Representation Theory, (2022), pp. 1–16.

[8] R. Bahramian and N. Ahanjideh, *p-divisibility of co-degrees of irreducible characters*, Bull. Aust. Math. Soc., 103 (2021), pp. 78–82.

[9] ——, *p-parts of co-degrees of irreducible characters*, Comptes Rendus. Mathématique, 359 (2021), pp. 79–83.

[10] R. Bahramian, N. Ahanjideh, and A. R. Naghipour, *Groups with some cube-free irreducible character co-degrees*, Communications in Algebra, (2022), pp. 1–5.

[11] N. N. Hung, *Characters of p'-degree and Thompson's character degree theorem*, Revista matemática iberoamericana, 33 (2017), pp. 117–138.

[12] I. M. Isaacs, *Character theory of finite groups*, Dover Publications, Inc., New York, 1994. Corrected reprint of the 1976 original [Academic Press, New York].

[13] Y. Liu, *Nonsolvable groups whose irreducible character degrees have special 2-parts*, Frontiers of Mathematics in China, (2021), pp. 1–6.

[14] A. Moretó, *A dual version of Huppert's ρ-σ conjecture for character codegrees*, Forum Math., 34 (2022), pp. 425–430.

[15] A. Moretó and N. Rizo, *Kernels of p'-degree irreducible characters*, Mediterr. J. Math., 19 (2022), pp. Paper No. 121, 8.

[16] G. Navarro and P. H. Tiep, *Degrees of rational characters of finite groups*, Adv. Math., 224 (2010), pp. 1121–1142.

[17] G. Qian, Y. Wang, and H. Wei, *Co-degrees of irreducible characters in finite groups*, J. Algebra, 312 (2007), pp. 946–955.

[18] Y. Yang and G. Qian, *The analog of Huppert's conjecture on character codegrees*, J. Algebra, 478 (2017), pp. 215–219.

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