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Original Article

On a maximal subgroup of the Symplectic group $Sp(4,4)$

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ABSTRACT: This paper is dealing with a split extension group of the form $2^6{:}(3\times A_5)$, which is the largest maximal subgroup of the Symplectic group $Sp(4,4)$. We refer to this extension by $\overline{G}$. We firstly determine the conjugacy classes of $\overline{G}$ using the coset analysis technique. The structures of inertia factor groups were determined. We then compute the Fischer matrices of $\overline{G}$ and apply the Clifford-Fischer theory to calculate the ordinary character table of this group. The Fischer matrices of $\overline{G}$ are all integer valued, with sizes ranging from 1 to 4. The full character table of $\overline{G}$ is 26×26 complex valued matrix and is given at the end of this paper.

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(*Dedicated to Professor Jamshid Moori*)

1. Introduction

The Symplectic group $Sp(4,4)$ is a classical simple group of order 979200. From the Atlas [16] we can see that $Sp(4,4)$ has 7 conjugacy classes of maximal subgroups. The two largest maximal subgroups are two isomorphic non-conjugate groups of the form $2^6{:}(3\times A_5)$. For the purpose of this paper, it will not make difference to which group we deal with. Thus we refer to such a group of the form $2^6{:}(3\times A_5)$ by $\overline{G}$ and clearly it has order $64\times 3\times |A_5| = 11520$ and index 85 in $Sp(4,4)$. Using GAP [2] we were able to construct this split extension group in terms of permutations of 85 points. We used the coset analysis technique to construct the conjugacy classes of $\overline{G}$, where correspond to the 15 classes of $3\times A_5$, we obtained 26 classes of $\overline{G}$. Then by looking at the maximal subgroups of $3\times A_5$, we were able to determine the structures of the inertia factor groups. Then we computed the Fischer matrices of the extension and we found to be integer valued matrices with sizes ranging from 1 to 4. Finally we were able to compute the ordinary character table of $\overline{G}$ using Clifford-Fischer theory and we supplied it at the end

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of this paper. The character table of $\overline{G}$ is a 26×26 complex valued matrix and partitioned into 60 parts correspond to the 15 classes of $3 \times A_5$ and the four inertia factor groups.

The character table of any finite group extension $\overline{G} = N{\cdot}G$ (here N is the kernel of the extension and G is isomorphic to $\overline{G}/N$) produced by Clifford-Fischer Theory is in a special format that could not be achieved by direct computations using GAP or Magma [15]. Also there is an interesting interplay between the coset analysis and Clifford-Fischer Theory. Indeed the size of each Fischer matrix is $c(g_i)$, the number of $\overline{G}$-classes corresponding to $[g_i]_G$ obtained via the coset analysis technique. That is computations of the conjugacy classes of $\overline{G}$ using the coset analysis technique will determine the sizes of all Fischer matrices.

For the notation used in this paper and the description of Clifford-Fischer theory technique, we follow [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14].

By the electronic Atlas of Wilson [21], we can see that $Sp(4,4)$ can be represented in terms of 85 points. We used the two generators given at the Atlas to generate $Sp(4,4)$ in GAP. In fact $Sp(4,4)$ has 7 conjugacy classes of maximal subgroups. The first two largest maximal subgroups are groups of the form $2^6{:}(3 \times A_5)$ (two isomorphic non-conjugate copies) with order 11520 and index 85 in $Sp(4,4)$. Using $(c_1, c_2, \cdots, c_k)$ to denote the k-cycle $(c_1\ c_2\ \cdots\ c_k)$, the group $\overline{G}$ is generated by the below two elements a and b :

$$\begin{aligned} a &= (1,56,29,66,23,82,80,83,40,43,18,15,12,62,84)(2,5,37,60,38,69,41,4,39,30,51,75,3,67,77)\\ &\quad (6,81,47,57,78,71,52,68,55,27,42,61,21,45,53)(7,79,72,35,22)(8,64,24,49,31,17,25,14,9,\\ &\quad 50,33,28,36,63,26)(10,20,76,73,46)(11,34,65,32,85,44,59,13,16,19,48,58,74,54,70),\\ b &= (1,18)(2,34)(3,85)(4,73)(5,84)(6,75)(7,72)(8,71)(10,19)(11,67)(12,74)(13,41)(14,17)(15,58)\\ &\quad (16,63)(20,44)(21,78)(23,57)(24,29)(25,47)(26,62)(27,80)(28,51)(30,54)(31,59)(33,43)(35,79)\\ &\quad (36,55)(37,70)(38,52)(39,46)(40,68)(42,83)(45,65)(48,60)(49,61)(50,53)(56,64)(66,69)(76,82), \end{aligned}$$

with $o(a) = 15$, $o(b) = 2$ and $o(ab) = 15$.

Having $\overline{G}$ being constructed in GAP, it is easy to obtain all its normal subgroups. In fact $\overline{G}$ possesses four proper non-trivial normal subgroups of orders 4, 64, 192 and 3840. The normal subgroup of order 64 is an elementary abelian group isomorphic to N.

In GAP one can check for the complements of N in $\overline{G}$, where in our case we obtained only one complement isomorphic to $3 \times A_5$ and together with N gives the split extension in consideration.

2. The conjugacy classes of $\overline{G}$

In this section we compute the conjugacy classes of the group $\overline{G}$ using the coset analysis technique (see Basheer [3], Basheer and Moori [5, 6, 8, 7] or Moori [17] and [18] for more details) as we are interested to organize the classes of $\overline{G}$ corresponding to the classes of $3 \times A_5$. Firstly note that A_5 has 5 conjugacy classes (see the Atlas) and thus $3 \times A_5$ will have 15 conjugacy classes. Corresponding to these 15 classes of $3 \times A_5$, we obtained 26 classes in $\overline{G}$.

In Table 1, we list the conjugacy classes of $\overline{G}$, where in this table:

- k_i is the number of orbits $Q_{i1}, Q_{i2}, \cdots, Q_{ik_i}$ for the action of N on the coset $N\overline{g}_i = Ng_i$, where g_i is a representative of a class of the complement ($\cong 3 \times A_5$) of N in $\overline{G}$. In particular, the action of N on the identity coset N produces 64 orbits, where each orbit consists of a singleton. Thus for $\overline{G}$, we have $k_1 = 64$.
- f_{ij} is the number of orbits fused together under the action of $C_G(g_i)$ on $Q_1, Q_2, \cdots, Q_k$. In particular, the action of $C_G(1_G) = G$ on the orbits $Q_1, Q_2, \cdots, Q_k$ affords four orbits of lengths 1, 3, 15 and 45 (with corresponding point stabilizers $3 \times A_5$, A_5, A_4 and 2^2. Thus $f_{11} = 1$, $f_{12} = 3$, $f_{13} = 15$ and $f_{14} = 45$.
- m_{ij}'s are weights (attached to each class of $\overline{G}$) that will be used later in computing the Fischer matrices of $\overline{G}$. These weights are computed through the formula

$$m_{ij} = [N_{\overline{G}}(N\overline{g}_i) : C_{\overline{G}}(g_{ij})] = |N|\frac{|C_G(g_i)|}{|C_{\overline{G}}(g_{ij})|}, \qquad (1)$$

where N is the kernel of an extension $\overline{G}$ that is in consideration.

Table 1: The conjugacy classes of $\overline{G}$

$[g_i]_G$	k_i	f_{ij}	m_{ij}	$[g_{ij}]_{\overline{G}}$	$o(g_{ij})$	$\lvert[g_{ij}]_{\overline{G}}\rvert$	$\lvert C_{\overline{G}}(g_{ij})\rvert$
$g_1 = 1A$	$k_1 = 64$	$f_{11} = 1$	$m_{11} = 1$	g_{11}	1	1	11520
		$f_{12} = 3$	$m_{12} = 3$	g_{12}	2	3	3840
		$f_{13} = 15$	$m_{13} = 15$	g_{13}	2	15	768
		$f_{14} = 45$	$m_{14} = 45$	g_{14}	2	45	256
$g_2 = 2A$	$k_2 = 16$	$f_{21} = 1$	$m_{21} = 4$	g_{21}	2	60	192
		$f_{22} = 3$	$m_{22} = 12$	g_{22}	2	180	64
		$f_{23} = 6$	$m_{23} = 24$	g_{23}	4	360	32
		$f_{24} = 6$	$m_{24} = 24$	g_{24}	4	360	32
$g_3 = 3A$	$k_3 = 1$	$f_{31} = 1$	$m_{32} = 64$	g_{31}	3	64	180
$g_4 = 3B$	$k_4 = 1$	$f_{41} = 1$	$m_{42} = 64$	g_{41}	3	64	180
$g_5 = 3C$	$k_5 = 4$	$f_{51} = 1$	$m_{51} = 16$	g_{51}	3	320	36
		$f_{52} = 3$	$m_{52} = 48$	g_{52}	6	960	12
$g_6 = 3D$	$k_6 = 4$	$f_{61} = 1$	$m_{61} = 16$	g_{61}	3	320	36
		$f_{62} = 3$	$m_{62} = 48$	g_{62}	6	960	12
$g_7 = 3E$	$k_7 = 4$	$f_{71} = 1$	$m_{71} = 16$	g_{71}	3	320	36
		$f_{72} = 3$	$m_{72} = 48$	g_{72}	6	960	12
$g_8 = 5A$	$k_8 = 4$	$f_{81} = 1$	$m_{81} = 16$	g_{81}	5	192	60
		$f_{82} = 3$	$m_{82} = 48$	g_{82}	10	576	20
$g_9 = 5B$	$k_9 = 4$	$f_{91} = 1$	$m_{91} = 16$	g_{91}	5	192	60
		$f_{92} = 3$	$m_{92} = 48$	g_{92}	10	576	20
$g_{10} = 6A$	$k_{10} = 1$	$f_{10,1} = 1$	$m_{10,1} = 64$	$g_{10,1}$	6	960	12
$g_{11} = 6B$	$k_{11} = 1$	$f_{11,1} = 1$	$m_{11,1} = 64$	$g_{11,1}$	6	960	12
$g_{12} = 15A$	$k_{12} = 1$	$f_{12,1} = 1$	$m_{12,1} = 64$	$g_{12,1}$	15	768	15
$g_{13} = 15B$	$k_{13} = 1$	$f_{13,1} = 1$	$m_{13,1} = 64$	$g_{13,1}$	15	768	15
$g_{14} = 15C$	$k_{14} = 1$	$f_{14,1} = 1$	$m_{14,1} = 64$	$g_{14,1}$	15	768	15
$g_{15} = 15D$	$k_{15} = 1$	$f_{15,1} = 1$	$m_{15,1} = 64$	$g_{15,1}$	15	768	15

3. The inertia factor groups of $\overline{G}$

We recall that knowledge of the appropriate character tables of inertia factor groups is crucial in calculating the full character table of any group extension. Since in our extension $\overline{G}$, the normal subgroup 2^6 is abelian and the extension splits, it follows by applications of Mackey's Theorem (see for example Theorem 3.3.4 of Whitley [20]), that every character of 2^6 is extendible to an ordinary character of its respective inertia group $\overline{H}_k$. Thus all the character tables of the inertia factor groups that we will use to construct the character tables of $\overline{G}$ are the ordinary ones. Next we determine the structures of the inertia factor groups.

We have seen from Section 2 that the action of $\overline{G} = 2^6{:}(3 \times A_5)$ (which can be reduced to the action of $3 \times A_5$) on the classes of $N = 2^6$ yielded four orbits of lengths 1, 3, 15 and 45 (and the corresponding point stabilizers were $3 \times A_5$, A_5, A_4 and 2^2). By a theorem of Brauer (see for example [3]), it follows that the action of $\overline{G}$ on $\mathrm{Irr}(N)$ will also produce four orbits. The orbits' lengths on the two actions may not be the same. Indeed in our case, we used Programme C of [19] to determine the lengths of the orbits of $\overline{G}$ or just $3 \times A_5$ on $\mathrm{Irr}(N)$. We found that the action of $3 \times A_5$ on $\mathrm{Irr}(N)$ produces four orbits of lengths 1, 15, 18 and 30. Let H_1, H_2, H_3 and H_4 be the respective inertia factor groups of the representatives of characters from the orbits with previous lengths. We notice that these inertia factors have indices 1, 15, 18 and 30 respectively in $3 \times A_5$. Since A_5 has 3 maximal subgroups (see the Atlas), it is easy to see that the group $3 \times A_5$ will have 4 maximal subgroups, namely A_5 itself together with the direct product of each maximal subgroup of A_5 by $\mathbb{Z}_3$. That is the maximal subgroups of $3 \times A_5$ are A_5, $3 \times A_4$, $3 \times D_{10}$ and $3 \times S_3$ with respective indices 3, 5, 6 and 10 in $3 \times A_5$. In Table 2, we give few information on these maximal subgroups, where we used T_i to denote a representative of a conjugacy class of maximal subgroups of $3 \times A_5$.

Now the first inertia factor group H_1 of $3 \times A_5$ has an index 1 and thus $H_1 = 3 \times A_5$ itself. Since we have the character table of A_5 (see the Atlas) we can easily construct the character table of $3 \times A_5$, which we supply below as Table 3.

The second inertia factor group H_2 has index 15 in $3 \times A_5$. From Table 2 we can see that the only index of a maximal subgroup that divides 15 is either 3 or 5. It follows that H_2 is either an index 5 subgroup of A_5 or an index 3 subgroup of $3 \times A_4$. In either case, it is clear that H_2 is isomorphic to the group A_4. The character table

Table 2: The maximal subgroups of $3 \times A_5$

T_i	$\|T_i\|$	$[3 \times A_5 : T_i]$
A_5	60	3
$3 \times A_4$	36	5
$3 \times D_{10}$	30	6
$3 \times S_3$	18	10

Table 3: The character table of $H_1 = G = 3 \times A_5$

	$1A$	$2A$	$3A$	$3B$	$3C$	$3D$	$3E$	$5A$	$5B$	$6A$	$6B$	$15A$	$15B$	$15C$	$15D$
$\|C_{H_1}(h)\|$	180	12	180	180	9	9	9	15	15	12	12	15	15	15	15
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	A	$\overline{A}$	A	$\overline{A}$	1	1	1	A	$\overline{A}$	A	$\overline{A}$	A	$\overline{A}$
χ_3	1	1	$\overline{A}$	A	$\overline{A}$	A	1	1	1	$\overline{A}$	A	$\overline{A}$	A	$\overline{A}$	A
χ_4	3	-1	3	3	0	0	0	B	B^*	-1	-1	B^*	B^*	B	B
χ_5	3	-1	3	3	0	0	0	B^*	B	-1	-1	B	B	B^*	B^*
χ_6	3	-1	C	$\overline{C}$	0	0	0	B	B^*	$-A$	$-\overline{A}$	D	$\overline{D}$	$\overline{E}$	E
χ_7	3	-1	$\overline{C}$	C	0	0	0	B	B^*	$-\overline{A}$	$-A$	$\overline{D}$	D	E	$\overline{E}$
χ_8	3	-1	$\overline{C}$	C	0	0	0	B^*	B	$-\overline{A}$	$-A$	E	$\overline{E}$	$\overline{D}$	D
χ_9	3	-1	C	$\overline{C}$	0	0	0	B^*	B	$-A$	$-\overline{A}$	$\overline{E}$	E	D	$\overline{D}$
χ_{10}	4	0	4	4	1	1	1	-1	-1	0	0	-1	-1	-1	-1
χ_{11}	4	0	F	$\overline{F}$	A	$\overline{A}$	1	-1	-1	0	0	$-A$	$-\overline{A}$	$-A$	$-\overline{A}$
χ_{12}	4	0	$\overline{F}$	F	$\overline{A}$	A	1	-1	-1	0	0	$-\overline{A}$	$-A$	$-\overline{A}$	$-A$
χ_{13}	5	1	5	5	-1	-1	-1	0	0	1	1	0	0	0	0
χ_{14}	5	1	G	$\overline{G}$	$-A$	$-\overline{A}$	-1	0	0	A	$\overline{A}$	0	0	0	0
χ_{15}	5	1	$\overline{G}$	G	$-\overline{A}$	$-A$	-1	0	0	$\overline{A}$	A	0	0	0	0

where in Table 3, $A = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, $B = \frac{1}{2} - \frac{\sqrt{5}}{2}$, $B^* = \frac{1}{2} + \frac{\sqrt{5}}{2}$, $C = -\frac{3}{2} - i\frac{3\sqrt{3}}{2}$, $D = -E(15) - E(15)^4$, $E = -E(15)^2 - E(15)^8$, $F = -2 - i2\sqrt{3}$ and $G = -\frac{5}{2} - i\frac{5\sqrt{3}}{2}$.

of A_4 can be obtained easily using GAP and for convenience we supply it here as Table 4. The character table of $H_2 = A_4$ is given below as Table 4.

Table 4: The character table of $H_2 = A_4$

	$1a$	$2a$	$3a$	$3b$
$\|C_{H_2}(h)\|$	12	4	3	3
χ_1	1	1	1	1
χ_2	1	1	$\frac{-1-i\sqrt{3}}{2}$	$\frac{-1+i\sqrt{3}}{2}$
χ_3	1	1	$\frac{-1+i\sqrt{3}}{2}$	$\frac{-1-i\sqrt{3}}{2}$
χ_4	3	-1	0	0

Turning to the third inertia factor group H_3, which has index 18 in $G \cong 3 \times A_5$, we can see from Table 2 that H_3 is either an index 6 of A_5 or an index 3 subgroup of $3 \times D_{10}$. In either case, it turns out that H_3 is isomorphic to the group D_{10}. Again the character table of D_{10} can be constructed easily using GAP and for convenience we supply it here as Table 5.

Finally the fourth inertia factor group H_4 has index 30 in $G \cong 3 \times A_5$. From Table 2 we can see that H_4 is

- an index 10 subgroup of A_5,
- an index 6 subgroup of $3 \times A_4$,
- an index 5 subgroup of $3 \times D_{10}$ or
- an index 3 subgroup of $3 \times S_3$.

It is clear that if H_4 is an index 10 subgroup of A_5 or an index 3 subgroup of $3 \times S_3$, then H_4 will be isomorphic to the group S_3. If H_4 is an index 6 subgroup of $3 \times A_4$, we note that the group $3 \times A_4$ has 5 conjugacy classes of

Table 5: The character table of $H_3 = D_{10}$

	$1a$	$2a$	$5a$	$5b$
$\|C_{H_3}(h)\|$	10	2	5	5
χ_1	1	1	1	1
χ_2	1	-1	1	1
χ_3	2	0	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1-\sqrt{5}}{2}$
χ_4	2	0	$\frac{-1-\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$

maximal subgroups represented by A_4 (3 isomorphic non-conjugate copies), $\mathbb{Z}_6 \times \mathbb{Z}_2$ and $\mathbb{Z}_3 \times \mathbb{Z}_3$. We know that H_4 can not be a subgroup of A_4 as A_4 has no subgroup of order 6. Therefore if H_4 is an index 6 subgroup of $3 \times A_4$, then the only possibility is that $H_4 < \mathbb{Z}_6 \times \mathbb{Z}_2$ and it is clear that it will be isomorphic to $\mathbb{Z}_6$. If H_4 is an index 5 subgroup of $3 \times D_{10}$, then it must be maximal there. Checking the maximal subgroups of $3 \times D_{10}$, we can see that there are three conjugacy classes of maximal subgroups represented by $\mathbb{Z}_{15}$, D_{10} and $\mathbb{Z}_6$. Therefore if H_4 is an index 5 subgroup of $3 \times D_{10}$, then it will be isomorphic to $\mathbb{Z}_6$. It follows that $H_4 \in \{S_3, \mathbb{Z}_6\}$. In the following proposition we determine the structure of the fourth inertia factor group.

Proposition 3.1. *$H_4 \cong S_3$.*

Proof. Recall from Section 2 that the group $\overline{G} \cong 2^6{:}(3 \times A_5)$ has 26 conjugacy classes and therefore $|\mathrm{Irr}(\overline{G})| = 26$. Also since the extension $\overline{G}$ splits and the kernel of the extension is an elementary abelian group, it follows that all the character tables of the inertia factor groups that we will use to construct the character table of $\overline{G}$ will be the ordinary ones. We know that $|\mathrm{Irr}(\overline{G})| = \sum_{i=1}^{4} |\mathrm{Irr}(H_i)|$. From Tables 3, 4 and 5 we have that $|\mathrm{Irr}(H_1)| = 15$, $|\mathrm{Irr}(H_2)| = 4$ and $|\mathrm{Irr}(H_3)| = 4$. It follows that $|\mathrm{Irr}(H_4)| = |\mathrm{Irr}(\overline{G})| - (|\mathrm{Irr}(H_1)| + |\mathrm{Irr}(H_2)| + |\mathrm{Irr}(H_3)|) = 26 - (15+4+4) = 26 - 23 = 3$. We also know that $|\mathrm{Irr}(\mathbb{Z}_6)| = 6$ and $|\mathrm{Irr}(S_3)| = 3$. This simple statement shows that $H_4 \cong S_3$ as claimed. □

Below we supply the character table of $H_4 \cong S_3$.

Table 6: The character table of $H_4 = S_3$

	$1a$	$2a$	$3a$
$\|C_{H_4}(h)\|$	6	2	3
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Next we turn to determine the fusions of classes of the inertia factor groups of the extension into the classes of $G = 3 \times A_5$. We have used the permutation characters of G on the inertia factor groups and the centralizer sizes to determine the fusions of these inertia factors into G. We list these fusions in Table 7.

Table 7: The fusions of the classes of H_2, H_3 and H_4 into the classes of $G = H_1 = 3 \times A_5$

$[x]_{H_2}$	$\longrightarrow$	$[x]_G$	$[x]_{H_3}$	$\longrightarrow$	$[x]_G$	$[x]_{H_4}$	$\longrightarrow$	$[x]_G$
$1a$		$1A$	$1a$		$1A$	$1a$		$1A$
$2a$		$2A$	$2a$		$2A$	$2a$		$2A$
$3a$		$3C$	$5a$		$5A$	$3a$		$3E$
$3b$		$3D$	$5b$		$5B$			

4. Fischer matrices of $\overline{G}$

In this section, we use the arithmetical properties of Fischer matrices, given in Proposition 3.6 of [5], to calculate some of the entries of the Fischer matrices and also to build an algebraic system of equations. To build these systems of equations, we firstly recall that we label the top and bottom of the columns of the Fischer matrix $\mathcal{F}_i$, corresponding to g_i, by the sizes of the centralizers of g_{ij}, $1 \leq j \leq c(g_i)$ in $\overline{G}$ and m_{ij} respectively. In Table 1 we supplied $|C_{\overline{G}}(g_{ij})|$ and $m_{ij},\ 1 \leq i \leq 15,\ 1 \leq j \leq c(g_i)$. Also having obtained the fusions of the inertia factor groups into $3 \times A_5$, we are able to label the rows of the Fischer matrices as described in [3, 5].

Since the size of the Fischer matrix $\mathcal{F}_i$ is $c(g_i)$, it follows from Table 1 that the sizes of the Fischer matrices of $\overline{G}$ range between 1 and 4 for all $i \in \{1, 2, \cdots, 15\}$. Now with the help of the symbolic mathematical package Maxima [1], we were able to solve the systems of equations and hence we have computed all the Fischer matrices of $\overline{G}$, where we found that all these matrices are integer valued. Below we list these matrices.

$\mathcal{F}_1$

g_1		g_{11}	g_{12}	g_{13}	g_{14}
$o(g_{1j})$		1	2	2	2
$\lvert C_{\overline{G}}(g_{1j})\rvert$		11520	3840	762	256
(k, m)	$\lvert C_{H_k}(g_{1km})\rvert$				
$(1, 1)$	11520	1	1	1	1
$(2, 1)$	768	15	15	-1	-1
$(3, 1)$	640	18	-6	-6	2
$(4, 1)$	384	30	-10	6	-2
m_{1j}		1	3	15	45

$\mathcal{F}_2$

g_2		g_{21}	g_{22}	g_{23}	g_{24}
$o(g_{2j})$		2	2	4	4
$\lvert C_{\overline{G}}(g_{2j})\rvert$		192	64	32	32
(k, m)	$\lvert C_{H_k}(g_{2km})\rvert$				
$(1, 1)$	12	1	1	1	1
$(2, 1)$	4	3	3	-1	-1
$(3, 1)$	2	6	-2	-2	2
$(4, 1)$	2	6	-2	2	-2
m_{2j}		4	12	24	24

$\mathcal{F}_3$

g_3		g_{31}
$o(g_{3j})$		3
$\lvert C_{\overline{G}}(g_{3j})\rvert$		180
(k, m)	$\lvert C_{H_k}(g_{3km})\rvert$	
$(1, 1)$	180	1
m_{3j}		64

$\mathcal{F}_4$

g_4		g_{41}
$o(g_{3j})$		3
$\lvert C_{\overline{G}}(g_{4j})\rvert$		180
(k, m)	$\lvert C_{H_k}(g_{4km})\rvert$	
$(1, 1)$	180	1
m_{4j}		64

$\mathcal{F}_5$

g_5		g_{51}	g_{52}
$o(g_{5j})$		3	6
$\lvert C_{\overline{G}}(g_{5j})\rvert$		36	12
(k, m)	$\lvert C_{H_k}(g_{5km})\rvert$		
$(1, 1)$	9	1	1
$(2, 1)$	3	3	-1
m_{5j}		16	48

$\mathcal{F}_6$

g_6		g_{61}	g_{62}
$o(g_{6j})$		3	6
$\lvert C_{\overline{G}}(g_{6j})\rvert$		36	12
(k, m)	$\lvert C_{H_k}(g_{6km})\rvert$		
$(1, 1)$	9	1	1
$(2, 1)$	3	3	-1
m_{6j}		16	48

$\mathcal{F}_7$

g_7		g_{71}	g_{72}
$o(g_{7j})$		3	6
$\lvert C_{\overline{G}}(g_{7j})\rvert$		36	12
(k, m)	$\lvert C_{H_k}(g_{7km})\rvert$		
$(1, 1)$	9	1	1
$(4, 1)$	3	3	-1
m_{7j}		16	48

$\mathcal{F}_8$

g_8		g_{81}	g_{82}
$o(g_{8j})$		5	10
$\lvert C_{\overline{G}}(g_{8j})\rvert$		60	20
(k, m)	$\lvert C_{H_k}(g_{8km})\rvert$		
$(1, 1)$	15	1	1
$(3, 1)$	5	3	-1
m_{8j}		16	48

$\mathcal{F}_9$

g_9		g_{91}	g_{92}
$o(g_{9j})$		5	10
$\lvert C_{\overline{G}}(g_{9j})\rvert$		60	20
(k, m)	$\lvert C_{H_k}(g_{9km})\rvert$		
$(1, 1)$	15	1	1
$(3, 1)$	5	3	-1
m_{9j}		16	48

$\mathcal{F}_{10}$

g_{10}		$g_{10,1}$
$o(g_{10j})$		6
$\lvert C_{\overline{G}}(g_{10j})\rvert$		12
(k, m)	$\lvert C_{H_k}(g_{10km})\rvert$	
$(1, 1)$	12	1
m_{10j}		64

$\mathcal{F}_{11}$

g_{11}		$g_{11,1}$
$o(g_{11j})$		6
$\lvert C_{\overline{G}}(g_{11j})\rvert$		12
(k, m)	$\lvert C_{H_k}(g_{11km})\rvert$	
$(1, 1)$	12	1
m_{11j}		64

$\mathcal{F}_{12}$

g_{12}		$g_{12,1}$
$o(g_{12j})$		15
$\lvert C_{\overline{G}}(g_{12j})\rvert$		15
(k, m)	$\lvert C_{H_k}(g_{12km})\rvert$	
$(1, 1)$	15	1
m_{12j}		64

$\mathcal{F}_{13}$

g_{13}		$g_{13,1}$
$o(g_{13j})$		15
$\lvert C_{\overline{G}}(g_{13j})\rvert$		15
(k, m)	$\lvert C_{H_k}(g_{13km})\rvert$	
$(1, 1)$	15	1
m_{13j}		64

$\mathcal{F}_{14}$

g_{14}		$g_{14,1}$
$o(g_{14j})$		15
$\lvert C_{\overline{G}}(g_{14j})\rvert$		15
(k, m)	$\lvert C_{H_k}(g_{14km})\rvert$	
$(1, 1)$	15	1
m_{14j}		64

$\mathcal{F}_{15}$

g_{15}		$g_{15,1}$
$o(g_{15j})$		15
$\lvert C_{\overline{G}}(g_{15j})\rvert$		15
(k, m)	$\lvert C_{H_k}(g_{15km})\rvert$	
$(1, 1)$	15	1
m_{15j}		64

5. The character table of $\overline{G}$

Throughout Sections 2, 3 and 4 we have found

- the conjugacy classes of $\overline{G}$ (Table 1),
- the inertia factor groups $H_1,\ H_2,\ H_3$ and H_4 of $\overline{G}$ and their character tables (Tables 3, 4, 5 and 6). Also we obtained the fusions of classes of the inertia factors H_2, H_3 and H_4 of $\overline{G}$ into the classes of $3 \times A_5$ (Table 7),
- the Fischer matrices of $\overline{G}$ (Section 4).

It follows by [3, 5] that the full character table of $\overline{G}$ can be constructed easily in the format of Clifford-Fischer theory. This table will be partitioned into 60 parts corresponding to the 15 cosets and the four inertia factor groups. The full character table of $\overline{G}$ is 26×26 $\mathbb{C}$-valued matrix. In Table 8, we supply the character table of $\overline{G}$ in the format of Clifford-Fischer Theory. In this table we have also included the fusions of the conjugacy classes of $\overline{G}$ into the conjugacy classes of the Symplectic group $Sp(4,4)$, where the classes of $Sp(4,4)$ as in the Atlas. Finally we would like to remark that the accuracy of this character table has been tested using GAP.

Table 8: The character table of $\overline{G}$

$[g_i]_G$	$1A$				$2A$				$3A$	$3B$	$3C$		$3D$		$3E$		$5A$		$5B$		$6A$	$6B$	$15A$	$15B$	$15C$	$15D$
$[g_{ij}]_{\overline{G}}$	$1a$	$2a$	$2b$	$2c$	$2d$	$2e$	$4a$	$4b$	$3a$	$3b$	$3c$	$6a$	$3d$	$6b$	$3e$	$6c$	$5a$	$10a$	$5b$	$10b$	$6c$	$6d$	$15a$	$15b$	$15c$	$15d$
$\lvert C_{\overline{G}}(g_{ij})\rvert$	11520	3840	768	256	192	64	32	32	180	180	36	12	36	12	36	12	60	20	60	20	12	12	15	15	15	15
$\hookrightarrow Sp(4,4)$	$1A$	$2A$	$2B$	$2C$	$2A$	$2C$	$4A$	$4B$	$3B$	$3B$	$3A$	$6A$	$3A$	$6A$	$3B$	$6B$	$5D$	$10A$	$5C$	$10B$	$6B$	$6B$	$15C$	$15C$	$15D$	$15D$
χ_1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
χ_2	1	1	1	1	1	1	1	1	A	$\overline{A}$	A	A	$\overline{A}$	$\overline{A}$	1	1	1	1	1	1	A	$\overline{A}$	$\overline{A}$	A	A	$\overline{A}$
χ_3	1	1	1	1	1	1	1	1	$\overline{A}$	A	$\overline{A}$	$\overline{A}$	A	A	1	1	1	1	1	1	$\overline{A}$	A	A	$\overline{A}$	$\overline{A}$	A
χ_4	3	3	3	3	-1	-1	-1	-1	3	3	0	0	0	0	0	0	B	B	B^*	B^*	-1	-1	B	B	B^*	B^*
χ_5	3	3	3	3	-1	-1	-1	-1	3	3	0	0	0	0	0	0	B^*	B^*	B	B	-1	-1	B^*	B^*	B	B
χ_6	3	3	3	3	-1	-1	-1	-1	C	$\overline{C}$	0	0	0	0	0	0	B	B	B^*	B^*	$-A$	$-\overline{A}$	D	$\overline{D}$	$\overline{E}$	E
χ_7	3	3	3	3	-1	-1	-1	-1	C	$\overline{C}$	0	0	0	0	0	0	B^*	B^*	B	B	$-A$	$-\overline{A}$	E	$\overline{E}$	$\overline{D}$	D
χ_8	3	3	3	3	-1	-1	-1	-1	$\overline{C}$	C	0	0	0	0	0	0	B^*	B^*	B	B	$-\overline{A}$	$-A$	$\overline{E}$	E	D	$\overline{D}$
χ_9	3	3	3	3	-1	-1	-1	-1	$\overline{C}$	C	0	0	0	0	0	0	B	B	B^*	B^*	$-\overline{A}$	$-A$	$\overline{D}$	D	E	$\overline{E}$
χ_{10}	4	4	4	4	0	0	0	0	4	4	1	1	1	1	1	1	-1	-1	-1	-1	0	0	-1	-1	-1	-1
χ_{11}	4	4	4	4	0	0	0	0	F	$\overline{F}$	A	A	$\overline{A}$	$\overline{A}$	1	1	-1	-1	-1	-1	0	0	$-\overline{A}$	$-A$	$-A$	$-\overline{A}$
χ_{12}	4	4	4	4	0	0	0	0	$\overline{F}$	F	$\overline{A}$	$\overline{A}$	A	A	1	1	-1	-1	-1	-1	0	0	$-A$	$-\overline{A}$	$-\overline{A}$	$-A$
χ_{13}	5	5	5	5	1	1	1	1	5	5	-1	-1	-1	-1	-1	-1	0	0	0	0	1	1	0	0	0	0
χ_{14}	5	5	5	5	1	1	1	1	G	$\overline{G}$	$-A$	$-A$	$-\overline{A}$	$-\overline{A}$	-1	-1	0	0	0	0	A	$\overline{A}$	0	0	0	0
χ_{15}	5	5	5	5	1	1	1	1	$\overline{G}$	G	$-\overline{A}$	$-\overline{A}$	$-A$	$-A$	-1	-1	0	0	0	0	$\overline{A}$	A	0	0	0	0
χ_{16}	15	15	-1	-1	3	3	-1	-1	0	0	3	-1	3	-1	0	0	0	0	0	0	0	0	0	0	0	0
χ_{17}	15	15	-1	-1	3	3	-1	-1	0	0	$\overline{C}$	$-\overline{A}$	C	$-A$	0	0	0	0	0	0	0	0	0	0	0	0
χ_{18}	15	15	-1	-1	3	3	-1	-1	0	0	C	$-A$	$\overline{C}$	$-\overline{A}$	0	0	0	0	0	0	0	0	0	0	0	0
χ_{19}	45	45	-3	-3	-3	-3	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
χ_{20}	18	-6	-6	2	6	-2	-2	2	0	0	0	0	0	0	0	0	3	-1	3	-1	0	0	0	0	0	0
χ_{21}	18	-6	-6	2	-6	2	2	-2	0	0	0	0	0	0	0	0	3	-1	3	-1	0	0	0	0	0	0
χ_{22}	36	-12	-12	4	0	0	0	0	0	0	0	0	0	0	0	0	H	B^*	H^*	B	0	0	0	0	0	0
χ_{23}	36	-12	-12	4	0	0	0	0	0	0	0	0	0	0	0	0	H^*	B	H	B^*	0	0	0	0	0	0
χ_{24}	30	-10	6	-2	-6	2	-2	2	0	0	0	0	0	0	3	-1	0	0	0	0	0	0	0	0	0	0
χ_{25}	30	-10	6	-2	6	-2	2	-2	0	0	0	0	0	0	3	-1	0	0	0	0	0	0	0	0	0	0
χ_{26}	60	-20	12	-4	0	0	0	0	0	0	0	0	0	0	-3	1	0	0	0	0	0	0	0	0	0	0

where in Table 8, $A = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$, $B = \frac{1}{2} - \frac{\sqrt{5}}{2}$, $B^* = \frac{1}{2} + \frac{\sqrt{5}}{2}$, $C = -\frac{3}{2} - i\frac{3\sqrt{3}}{2}$, $D = -E(15)^2 - E(15)^8$, $E = -E(15)^{11} - E(15)^{14}$, $F = -2 - i2\sqrt{3}$ $G = -\frac{5}{2} - i\frac{5\sqrt{3}}{2}$, $H = -\frac{3}{2} - \frac{3\sqrt{5}}{2}$ and $H^* = -\frac{3}{2} + \frac{3\sqrt{5}}{2}$.

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