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Original Article

Biamenability of Banach algebras and its applications

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ABSTRACT: In this paper, we introduce the concept of biamenability of Banach algebras and we show that despite the apparent similarities between amenability and biamenability of Banach algebras, they lead to very different, and somewhat opposed, theories. In this regard, we show that commutative Banach algebras and the group algebra $L^1(G)$, for each locally compact group G, tend to lack biamenability, while they may be amenable and highly non-commutative Banach algebras such as $B(H)$ for an infinite-dimensional Hilbert space H tend to be biamenable, while they are not amenable. Also, we show that although the unconditional unitization of an amenable Banach algebra is amenable but in general unconditional unitization of a Banach algebra is not biamenable. This concept may be applied for studying the character space of some Banach algebras and also for studying some spansion or density problems.

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1. Introduction

A derivation from a Banach algebra A to a Banach A-bimodule X with the continuous module operations is a bounded linear mapping $d : A \to X$ such that

$$d(ab) = d(a)b + ad(b), \quad (a, b \in A).$$

For each $x \in X$ the mapping $\delta_x : a \to ax - xa$, $(a \in A)$ is a bounded derivation, called an inner derivation.
Let X be a Banach A-bimodule. Then X^* is a dual Banach A-bimodule, by defining $a.f$ and $f.a$, for each $a \in A$ and $f \in X^*$ by

$$a.f(x) = f(xa), \quad f.a(x) = f(ax) \qquad (x \in X).$$

Similarly, the higher duals $X^{(n)}$ can be made into Banach A-bimodules in a natural fashion.

A Banach algebra A is called amenable if for each Banach A-bimodule X, the only derivations from A to X^* are inner derivations. For more details about this notion see [10].

Let A be a Banach algebra and X be a Banach A-bimodule, a bounded bilinear mapping $D : A \times A \to X$ is called a biderivation if D is a derivation with respect to both arguments. That is the mappings ${}_aD : A \to X$ and

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$D_b : A \to X$ are derivations. Where

$$_aD(b) = D(a, b) = D_b(a) \qquad\qquad (a, b \in A).$$

We denote the space of such biderivations by $BZ^1(A, X)$.

Consider the subspace $Z(A, X) = \{x \in X : ax = xa \quad \forall a \in A\}$ of X. Then for each $x \in Z(A, X)$, the mapping $\Delta_x : A \times A \to X$ defined by

$$\Delta_x(a, b) = x[a, b] = x(ab - ba) \qquad (a, b \in A)$$

is a basic example of a biderivation and called an inner biderivation. We denote the space of such inner biderivations by $BN^1(A, X)$. For more applications of biderivations, see [5, Section 3]. Some algebraic aspects of biderivations on certain algebras have been studied by many authors; see for example [3, 7], which the structures of biderivations on triangular algebras and generalized matrix algebras have been investigated, and particularly the question of whether biderivations on these algebras are inner, has been studied.

We define the first bicohomology group $BH^1(A, X)$ as follows,

$$BH^1(A, X) = \frac{BZ^1(A, X)}{BN^1(A, X)}.$$

Obviously $BH^1(A, X) = 0$ if and only if every biderivation from $A \times A$ to X is an inner biderivation. Now we are motivated to define the concept of biamenability of Banach algebras as follows.
A Banach algebra A is biamenable if for each Banach A-bimodule X we have $BH^1(A, X^*) = 0$.

Although one might expect that biderivations and biamenability must run parallel to derivations and amenability of Banach algebras what is true is that although there are some external similarities between them they lead to very different, and somewhat opposed, theories. Indeed we show that commutative Banach algebras tend to lack biamenability, while highly non-commutative Banach algebras tend to be biamenable. Thus, for instance, the ground algebras $\mathbb{C}$ and $\mathbb{R}$ are not biamenable (while they are trivially amenable) and $B(H)$, the algebra of all bounded operators on an infinite dimensional Hilbert space H, is biamenable, but not amenable. Moreover, if H is finite-dimensional, it turns out that $B(H)$ is amenable, but it fails to be biamenable. We show also, the unconditional unitization of a Banach algebra A is not biamenable. Although it may be amenable when A is amenable[10].

We may apply this concept for studying the character space of some Banach algebras and also for studying some Spansion or density problems. For example, we show that the character space of $B(H)$ and some module extension Banach algebras constructing with $B(H)$ is empty and for some Banach algebras such as $A = L^1(G)$, for some locally compact group G, $Span[A, A] = Span\{ab - ba; a, b \in A\} \neq A$. Also we study some density problems in section 3.

2. biamenable Banach algebras

For an example of a biamenable Banach algebra we commence with the next lemmas. The following lemma is similar to Corollary 2.4 of [4], where it is introduced for a biderivation $D : A \times A \to A$.

Lemma 2.1. *For each Banach A-bimodule X and each biderivation $D : A \times A \to X$,*

$$D(a, b)[c, d] = [a, b]D(c, d) \quad (a, b, c, d \in A).$$

In Proposition 2.1.3 of [10] it is shown that if A has a bounded approximate identity, and one of module actions is trivial, then the only derivations from A to X^* are inner derivations. The following lemma introduces a condition that not only implies the innerness of biderivations but it also forces them to be zero.

Lemma 2.2. *If a Banach algebra A has a bounded left approximate identity and $span\{ab - ba : a, b \in A\}$ is dense in A, then for every Banach A-bimodule X such that $XA = 0$ we have $BZ^1(A, X) = 0$.*

Proof. Let (e_α) be a left approximate identity of A. Lemma 2.1 says that for each $D \in BZ^1(A, X)$,

$$[a, b]D(c, d) = 0 \quad (a, b, c, d \in A).$$

So by density we have $aD(b, c) = 0$, for each $a, b, c \in A$. Now since $XA = 0$ we have

$$\begin{aligned} D(a, b) &= \lim_\alpha D(e_\alpha a, b) \\ &= \lim_\alpha [e_\alpha D(a, b) + D(e_\alpha, b)a] \\ &= 0. \end{aligned}$$

That is $BZ^1(A, X) = 0$. □

A very similar proof may be applied if A has a right approximate identity and the left module action is trivial.
The following lemma introduces a condition that under which some biderivations are inner. This result will lead to a condition implying biamenability of a Banach algebra. We shall see that $B(H)$, for each infinite dimensional Hilbert space H, satisfies this condition.

Lemma 2.3. *If A is unital and $A = span\{ab - ba : a, b \in A\}$, then for every unital Banach A-bimodule X, every biderivation $D : A \times A \to X$ is an inner biderivation.*

Proof. Let D be a biderivation and e be the identity of A. Since $A = span\{ab - ba : a, b \in A\}$, there exist a_i and b_i in A such that $e = \sum_i [a_i, b_i]$. So by Lemma 2.1, for every $a, b \in A$, we have

$$\begin{aligned} D(a,b) &= D(a,b)e \\ &= D(a,b)\sum_i [a_i,b_i] \\ &= \sum_i D(a,b)[a_i,b_i] \\ &= \sum_i [a,b]D(a_i,b_i) \\ &= [a,b]\lambda. \end{aligned}$$

Where $\lambda = \sum_i D(a_i, b_i)$. Similarly we have $D(a, b) = \lambda[a, b]$, and so

$$\lambda[a,b] = [a,b]\lambda \qquad (a \in A, b \in B).$$

Now since $A = span\{ab - ba : a, b \in A\}$, $\lambda \in Z(A, X)$. So $D = \Delta_\lambda$ is an inner biderivation. □

Lemma 2.4. *If a Banach algebra A has a bounded approximate identity and $span\{ab - ba : a, b \in A\}$ is dense in A, then the following statements are equivalent.*

(i) A is biamenable.

(ii) $BH^1(A, X^*) = 0$, for every left approximately unital Banach A-bimodule X.

(iii) $BH^1(A, X^*) = 0$, for every right approximately unital Banach A-bimodule X.

(iv) $BH^1(A, X^*) = 0$, for every approximately unital Banach A-bimodule X.

Proof. We only prove (i) is equivalent to (ii). The equivalence of (i) and (iii) is similar and then the equivalence of (i) and (iv) is obvious.
Clearly if A is biamenable then (ii) is true. Now let $BH^1(A, Y^*) = 0$, for every left approximately unital Banach A-bimodule Y and let X be a Banach A-bimodule. Then Corollary 2.9.26 of [6] implies that $X_0 = AX$ is a left approximately unital closed submodule of X. Also $A(\frac{X}{X_0}) = 0$ and so $(\frac{X}{X_0})^* A = 0$. Therefore Lemma 2.2 says that $BZ^1(A, X_0^\perp) = BZ^1(A, (\frac{X}{X_0})^*) = 0$.
Let $D \in BZ^1(A, X^*)$ and $J : X_0 \to X$ be the inclusion mapping. Then $J^* \circ D \in BZ^1(A, X_0^*)$ and by assumption $J^* \circ D = \Delta_{\phi_0}$, for some $\phi_0 \in Z(A, X_0^*)$. Now the equation $X^* = X_0^* \oplus X_0^\perp$, which is implied from Theorem 4.9 of [9], shows that there exists an extension ϕ of ϕ_0 such that $\phi \in Z(A, X^*)$.
Define $D_0 = D - \Delta_\phi$. Then $D_0 \in BZ^1(A, X_0^\perp) = 0$ and so $D = \Delta_\phi$. □

A similar result of the previous lemma in the area of amenability is given in Proposition 2.1.5 of [10].
Now combination of the Lemmas 2.3 and 2.4 gives the following theorem that leads to a condition for biamenability of Banach algebras and then we can find some examples of biamenable Banach algebras which are not amenable.

Theorem 2.5. *Each unital Banach algebra A with $A = span\{ab - ba : a, b \in A\}$, is biamenable.*

Corollary 2.6. *If $A = span\{ab - ba : a, b \in A\}$ and A has an identity, then the only biderivation $D : A \times A \to A^*$ is zero.*

Proof. Let $D : A \times A \to A^*$ be a biderivation. By Theorem 2.5 A is biamenable, so D is inner. That is there exists an $f \in Z(A, A^*)$ such that for every $a, b \in A$, $D(a, b) = f[a, b]$. Now

$$\langle f, [a,b] \rangle = \langle fa - af, b \rangle = 0.$$

Hence our assumption implies that $f = 0$ and so $D = 0$. □

Every bounded bilinear mapping $f : X \times Y \to Z$ on normed spaces X, Y and Z, has two natural extensions f^{***} and f^{t***t} from $X^{**} \times Y^{**}$ to Z^{**} as follows.
We define the adjoint $f^* : Z^* \times X \to Y^*$ of f by

$$\langle f^*(z^*, x), y\rangle = \langle z^*, f(x, y)\rangle,$$

where $x \in X, y \in Y$ and $z^* \in Z^*$. We then define $f^{**} = (f^*)^*$ and $f^{***} = (f^{**})^*$. Let $f^t : Y \times X \longrightarrow Z$ be the flip map of f which is defined by $f^t(y, x) = f(x, y)$ $(x \in X, y \in Y)$. If we continue the latter process with f^t instead of f, we come to the bounded bilinear mapping $f^{t***t} : X^{**} \times Y^{**} \to Z^{**}$.
Where π is the multiplication of a Banach algebra A, π^{***} and π^{t***t} are actually the first and second Arens products, which are denoted by $\Box$ and $\Diamond$, respectively. For more detailes see [2] and [1].

Now we give some examples of biamenable Banach algebras.

Example 2.1. *According to Lemma 5.8 of [11], since for each infinite dimensional Hilbert space H and every integer $n \geq 0$,*

$$B(H)^{(2n)} = span\{au - ua : a \in B(H), u \in B(H)^{(2n)}\}.$$

So Theorem 2.5 help us to find some biamenable Banach algebras such as the Banach algebra $B(H)^{(2n)}$ and the module extension Banach algebra $B(H) \oplus B(H)^{(2n)}$ with the product and norm as follows.

$$(a, u)(b, v) = (ab, av + ub), \quad \|(a, u)\| = \|a\| + \|u\|, \quad (a, b \in B(H), u, v \in B(H)^{(2n)}).$$

Although $B(H)$ is not amenable in general. Note that since $\{au - ua : a \in B(H), u \in B(H)^{(2n)}\}$ is a subset of $\{uv - vu : u, v \in B(H)^{(2n)}\}$ and $\{[a, v] - [b, u] : a, b \in B(H), u, v \in B(H)^{(2n)}\}$. Therefore the commutators span the whole of $B(H)^{(2n)}$ and $B(H) \oplus B(H)^{(2n)}$.
Also similar to last corollary we can show that the only biderivation from $B(H) \times B(H)$ to $B(H)^{(2n+1)}$ is zero.

A similar method as Lemma 5.7 of [11], can show that for the Banach algebra $K(H)$ of compact operators on H,

$$K(H) = span\{ku - uk : k \in K(H) : u \in B(H)\}.$$

So similarly $B(H) \oplus K(H)$ is biamenable. Although Remark 5.10 of [11] says that it is not amenable.

Let G be a locally compact group. $m \in L^\infty(G)^*$ is a mean on $L^\infty(G)$ if $m(1) = \|m\| = 1$. A mean m on $L^\infty(G)$ is called a left-invariant mean if for each $x \in G$ and $g \in L^\infty(G)$, $m(\delta_x * g) = m(g)$. G is called amenable if there is a left-invariant mean on $L^\infty(G)$.
Consider $L^\infty(G)$ as an $L^1(G)$-bimodule with the left and right module actions

$$\pi_\ell : L^1(G) \times L^\infty(G) \to L^\infty(G) \quad \pi_r : L^\infty(G) \times L^1(G) \to L^\infty(G),$$

defined by $\pi_\ell(f, g) = f * g$ and $\pi_r(g, f) = (\int_G f)g$. Then we have the following proposition.

Proposition 2.7. *Let G be a locally compact group such that $Z(L^1(G), L^\infty(G)^*)$ contains an element n such that $n(1) \neq 0$. Then $L^1(G)$ is amenable.*

Proof. Define $|n|$ by $|n|(\phi) = |n(g)|$, for each $g \in L^\infty(G)$. Then $m = \frac{|n|}{|n(1)|}$ is a positive element of $Z(L^1(G), L^\infty(G)^*)$ such that $m(1) = 1$. Therefore by [10, Proposition 1.1.2], m is a mean on $L^\infty(G)$.
Now we have

$$\begin{aligned}
\langle m, f * g\rangle &= \langle m, \pi_\ell(f, g)\rangle \\
&= \langle \pi_\ell^*(m, f), g\rangle \\
&= \langle \pi_r^{t*t}(f, m), g\rangle \\
&= \langle m, \pi_r(g, f)\rangle \\
&= \langle m, g\rangle \int_G f \\
&= \langle m, g\rangle \qquad (f \in P(G), g \in L^\infty(G)).
\end{aligned}$$

Where $P(G) = \{f \in L^1(G); \|f\|_1 = 1, f \geq 0\}$. So [10, Lemma 1.1.7] and Johnson's theorem imply that $L^1(G)$ is amenable. □

As a direct result of the last proposition, we conclude that for every abelian group G, $L^1(G)$ is amenable. Of course, this is a known result by applying Johnson's theorem and the Example 1.1.5 of [10]. However, in the following example, we show that a big class of Banach algebras, such as $L^1(G)$, are not biamenable, although they may be amenable.

Example 2.2. *(i) If there is a Banach A-bimodule X such that $Z(A, X) = \{0\}$ and there is a non zero biderivation from $A \times A$ into X^*, then A is not biamenable. Since in this case $Z(A, X^*) = \{0\}$ and so the only inner biderivation $D : A \times A \to X^*$ is zero.*

(ii) For every locally compact group G, $L^1(G)$ is not biamenable. Since if we consider $L^\infty(G)$ as an $L^1(G)$-module with the zero right module action and the left module action π_ℓ defined by $\pi_\ell(f, g) = f * g$, for every $f \in L^1(G)$ and $g \in L^\infty(G)$, then for each nonzero $g \in L^\infty(G)$, the bilinear mapping $D : L^1(G) \times L^1(G) \to L^\infty(G)$ defined by $D(f, h) = f * h * g$ is a nonzero biderivation. But since $Z(L^1(G), L^\infty(G)) = \{0\}$, the only inner biderivation from : $L^1(G) \times L^1(G)$ to $L^\infty(G)$ is zero.

(iii) Let $\sigma(A)$ be the spectrum of A. If $\sigma(A) \neq \emptyset$, then A is not biamenable. Since if f is an element in $\sigma(A)$ and X is a non zero Banach A-bimodule with module actions

$$ax = 0, \quad xa = f(a)x \qquad (a \in A, x \in X).$$

Then $Z(A, X) = \{0\}$. But for a non-zero element $h \in X^*$

$$\begin{array}{rcl} D : A \times A & \to & X^* \\ (a, b) & \mapsto & f(a)(bh - hb) \end{array}$$

is a non-zero biderivation.
In particular, by Theorem 1.3.3 of [8], we conclude that every unital commutative Banach algebra is not biamenable. For example, $\mathbb{C}$, $\mathbb{R}$, $C(\Omega)$, for each Hausdorff space Ω and the group algebra $M(G)$, for each locally compact abelian group G are not biamenable. In the next section, we extend this result to arbitrary commutative Banach algebras.
Also, let A be a Banach space and θ be a non zero element of A^*. Then A is a Banach algebra with the multiplication

$$ab = \theta(a)b, \qquad (a, b \in A).$$

Now since $\theta \in \sigma(A)$, A with this multiplication is not biamenable.

(iv) If there exists a non-zero derivation $d : A \to A^{**}$ on a commutative Banach algebra A such that for some $a, b \in A$, $d(a)\Box d(b) \neq 0$, then A is not biamenable. Since the map

$$\begin{array}{rcl} D : A \times A & \longrightarrow & A^{**} \\ (a, b) & \mapsto & d(a)\Box d(b) \end{array}$$

where $\Box$ denotes the first Arens product of A^{**}, defines a biderivation which is not inner.

As an application, a combination of Lemma 2.2, Example 2.1 and Example 2.2 implies the following results.

Corollary 2.8. *(i) For every locally compact group G, $Span[L^1(G), L^1(G)] \neq L^1(G)$.*

(ii) For each integer $n \geq 0$ and each infinite dimensional Hilbert space H, $\sigma(B(H)^{(2n)}) = \emptyset$ and $\sigma(B(H) \oplus B(H)^{(2n)}) = \emptyset$.

3. Some properties

In this section we study some properties of biamenable Banach algebras and we tend to some another examples of non biamenable Banach algebras.

Theorem 3.1. *Let A be a Banach algebra and consider $\mathbb{C}$ as a Banach A-bimodule. If there is a nonzero derivation $d : A \to \mathbb{C}$, then biamenability of A implies amenability of A.*

Proof. Let X be a Banach A-bimodule and $d' : A \to X^*$ be a bounded derivation. Then

$$\begin{array}{rcl} D : A \times A & \to & X^* \\ (a, b) & \mapsto & d(a)d'(b) \end{array}$$

is a bounded biderivation and so there is $f \in Z(A, X^*)$ such that

$$d(a)d'(b) = D(a, b) = f[a, b] \quad (a, b \in A).$$

Therefore for every $b \in A$ and for some $a \in A$ such that $d(a) \neq 0$ we have $d'(b) = \delta_{-\frac{fa}{d(a)}}(b)$. □

Example 3.1. *Let $\mathbb{D} = \{z \in \mathbb{C}; |z| \leq 1\}$ be the unit disc, and $A(\mathbb{D})$ be the disc algebra. We can consider $\mathbb{C}$ as an $A(\mathbb{D})$-bimodule with module actions $\alpha f = \alpha f(0) = f\alpha$ and*

$$\begin{array}{rcl} d : A(\mathbb{D}) & \to & \mathbb{C} \\ f & \mapsto & f'(0) \end{array}$$

is a nonzero derivation. Therefore since $A(\mathbb{D})$ is not amenable so it is not biamenable.

We know that every amenable Banach algebra has an approximate identity (See Proposition 2.2.1 of [10]). A similar result is given in the following.

Proposition 3.2. *If $A = span\{ab-ba : a, b \in A\}$ and A is biamenable, then A has a bounded approximate identity.*

Proof. If A is biamenable, then for the biderivation

$$\begin{array}{rcl} D : A \times A & \to & A^{**} \\ (a, b) & \mapsto & [a, b] \end{array}$$

there is $E \in Z(A, A^{**})$ such that for each $a, b \in A$, $E[a, b] = [a, b]$. Now let (e_α) be a bounded net in A which is w^*-convergent to E. Then we have

$$\begin{array}{rcl} \lim_\alpha e_\alpha[a, b] & = & w - \lim_\alpha e_\alpha[a, b] \\ & = & E[a, b] \\ & = & [a, b] \\ & = & [a, b]E \\ & = & w - \lim_\alpha [a, b]e_\alpha \\ & = & \lim_\alpha [a, b]e_\alpha, \end{array}$$

and by assumption, A has an approximate identity (e_α). □

Note that the converse of this proposition is not true in general. For example, in the sequel, we see that every commutative Banach algebra is not biamenable. Although it may be unital or approximately unital.

For each integer $n \geq 0$ put

$$AA^{(2n)} + A^{(2n)}A = \{aa^{(2n)} + b^{(2n)}b : a, b \in A, a^{(2n)}, b^{(2n)} \in A^{(2n)}\}.$$

Then we have the following proposition.

Proposition 3.3. *If A is biamenable, then for each integer $n \geq 0$, $span(AA^{(2n)} + A^{(2n)}A)$ is dense in $A^{(2n)}$.*

Proof. If $span(AA^{(2n)} + A^{(2n)}A)$ is not dense in $A^{(2n)}$, then there exists a non-zero linear functional $f \in A^{(2n+1)}$ such that it is zero on $span(AA^{2n} + A^{(2n)}A)$. Now the bilinear mapping

$$\begin{array}{rcl} D : A \times A & \to & A^{(2n+1)} \\ (a, b) & \mapsto & f(a)f(b)f \end{array}$$

is a biderivation which is not inner. So A is not biamenable, which is a contradiction. □

Let A be a Banach algebra and

$$A^n = span\{a_1...a_n : a_i \in A\} \qquad (n \in \mathbb{N}).$$

As a corollary of the latter proposition we have:

Corollary 3.4. *If A is biamenable then for each $n \in \mathbb{N}$, A^n is dense in A.*

Proof. By Proposition 3.3 A^2 is dense in A. Now by applying the density of A^2 in A we can prove that A^3 is dense in A and also by an inductive method we can prove that for each n, A^n is dense in A. □

For a Banach algebra A, put

$$[A, A] = \{[a, b] : a, b \in A\} \quad \text{and} \quad [A, A]A = \{[a, b]c : a, b, c \in A\}.$$

The following proposition gives a big class of non-biamenable Banach algebras.

Proposition 3.5. *Let A be a biamenable Banach algebra. Then $span([A,A]\cup[A,A]A)$ is dense in A.*

Proof. Suppose $S = span([A,A]\cup[A,A]A)$ is not dense in A. Then there exists a nonzero element $f\in A^*$ such that $f|_S = 0$. In particular for each $a,b,c\in A$, we have $f(ab)=f(ba)$ and $c.f(ab)=c.f(ba)$. Consider $X=\overline{f.A}$ as an A-bimodule with module actions

$$(f.a).b = f.ab, \quad \text{and} \quad b.(f.a)=0 \qquad (a,b\in A).$$

Then $Z(A,X^*)=\{0\}$ and so the only inner biderivation from $A\times A$ to X^* is zero. Now by Corollary 3.4 the bilinear mapping $D: A\times A\to X^*$ defined by

$$\langle D(a,b), f.c\rangle = f(abc), \qquad (a,b,c\in A)$$

is nonzero. Also for each $a,b,c,d\in A$ we have

$$\begin{aligned}\langle D(ab,c), f.d\rangle &= f(abcd)\\ &= f(bcda)\\ &= \langle D(b,c), f.da\rangle\\ &= \langle D(b,c), (f.d).a)\rangle\\ &= \langle aD(b,c), f.d\rangle\\ &= \langle aD(b,c)+D(a,c)b, f.d\rangle,\end{aligned}$$

and similarly

$$\begin{aligned}\langle D(a,bc), f.d\rangle &= f(abcd)\\ &= f(cdab)\\ &= (b.f)(cda)\\ &= (b.f)(acd)\\ &= f(acdb)\\ &= \langle D(a,c), f.db\rangle\\ &= \langle bD(a,c)+D(a,b)c, f.d\rangle.\end{aligned}$$

So D is a nonzero biderivation and so it is not inner. That is a contradiction. □

Note that if a biamenable Banach algebra A has a right approximate identity, then $[A,A]\subseteq\overline{[A,A]A}$ and therefore $span([A,A]A)$ is dense in A. This may be compared with the converse of Proposition 3.2, for each biamenable Banach algebra.

Corollary 3.6. *Every non zero commutative Banach algebra is not biamenable.*

Theorem 3.7. *Let A be a Banach algebra, X be a Banach A-bimodule and I be a closed ideal of it such that $Z(A,X^*)=Z(I,X^*)$. Then if $BH^1(I,X^*)=\{0\}$ and $\frac{A}{I}$ is biamenable, then $BH^1(A,X^*)=\{0\}$.*

Proof. Let $D: A\times A\to X^*$ be a biderivation. Then $D_0 = D|_{I\times I}\in BZ^1(I,X^*)$ and so $D_0=\Delta_E$, for some $E\in Z(A,X^*)$. Put $\tilde{D}=D-\Delta_E$. Then $\tilde{D}(I\times I)=0$ and so $\mathfrak{D}:\frac{A}{I}\times\frac{A}{I}\to X^*$, defined by $\mathfrak{D}((a+I,b+I))=\tilde{D}((a,b))$ is a well defined map.
Put $X_0 = IX+XI$. Then

$$(\frac{X}{X_0})^* = X_0^\perp = \{\phi\in X^*; \phi i = 0 = i\phi, \text{ for all } \ i\in I\}.$$

and so we can consider $X_0^\perp$ as an $\frac{A}{I}$-bimodule with the module actions $(a+I).\phi = a.\phi$ and $\phi.(a+I)=\phi.a$, for each $a\in A$ and $\phi\in X_0^\perp$. On the other hand we have

$$\begin{aligned}\tilde{D}(a,b)ij &= (\tilde{D}(ai,b)-a\tilde{D}(i,b))j\\ &= \tilde{D}(ai,bj)-b\tilde{D}(ai,j)-a\tilde{D}(i,bj)+ab\tilde{D}(i,j)\\ &= 0, \quad (i,j\in I, a,b\in A).\end{aligned}$$

Similarly we can show that $ij\tilde{D}(a,b)=0$. Therefore by density of I^2 in I (Proposition 3.3) we conclude that $\mathfrak{D}(\frac{A}{I}\times\frac{A}{I})\subseteq X_0^\perp$ and then we can coclude that $\mathfrak{D}:\frac{A}{I}\times\frac{A}{I}\to X_0^\perp$ is a biderivation. So there is $\psi\in Z(\frac{A}{I},X_0^\perp)\subseteq Z(A,X^*)$ such that $D-\Delta_E=\tilde{D}=\Delta_\psi$. Hence $D=\Delta_{E+\psi}$ and $E+\psi\in Z(A,X^*)$. □

We know that a Banach algebra A is amenable if and only if the unconditional unitization $A^\flat$ of A (see [6, Definition 1.3.3]) is amenable [10, Corollary 2.3.11]. But it is not true for biamenability of Banach algebras. Indeed we show that the unconditional unitization of each Banach algebra is not biamenable.

Lemma 3.8. *If $\theta : A \to B$ is a continuous homomorphism of Banach algebras with dense range and A is biamenable, then so is B.*

Proof. Let X be a Banach B-bimodule. Consider X as an A-bimodule with module actions $ax = \theta(a)x$ and $xa = x\theta(a)$. Now for each $D \in BZ^1(B, X^*)$, $D \circ (\theta \times \theta) \in BZ^1(A, X^*)$ and biamenability of A implies that $D \circ (\theta \times \theta) = \Delta_\phi$ for some $\phi \in Z(A, X^*)$. Now by density we conclude that $\phi \in Z(B, X^*)$ and $D = \Delta_\phi$. □

Corollary 3.9. *If A is biamenable then for each closed ideal I of A, $\frac{A}{I}$ is biamenable.*

For analoges of the above two results in the area of amenability see Proposition 2.3.1 and Corollary 2.3.2 of [10].

Now we have the following theorem for the unconditional unitization $A^\flat$ of a Banach algebra A.

Theorem 3.10. *For each Banach algebra A, the unconditional unitization $A^\flat$ is not biamenable.*

Proof. If $A^\flat$ is biamenable then $\mathbb{C} = \frac{A^\flat}{A}$ is biamenable by Corollary 3.9 (recall that A is a closed ideal in $A^\flat$). A contradiction. □

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