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Original Article

Computation of μ-symmetry and μ-conservation law for the Camassa-Holm and Hunter-Saxton equations

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ABSTRACT: This work is intended to compute the μ-symmetry and μ-conservation laws for the Cammasa-Holm (CH) equation and the Hunter-Saxton (HS) equation. In other words, μ-symmetry, μ-symmetry reduction, variational problem, and μ-conservation laws for the CH equation and the HS equation are provided. Since the CH equation and the HS equation are of odd order, they do not admit a variational problem. First we obtain μ-conservation laws for both of them in potential form because they admit a variational problem and then using them, we obtain μ-conservation laws for the CH equation and the HS equation.

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1. Introduction

Olver [11] introduced Lie symmetry analysis of differential equations and provided a powerful and fundamental framework for the exploitation of systematic procedures that goes to the integration by quadrature of ordinary differential equations (ODEs).

A new class of symmetries is introduced in [7, 6]. These symmetries are called λ-symmetries, which are vector fields depending on a function λ. Recently, these symmetries have gained increasing importance. The exponential terms are replaced by a new method of prolonging vector fields, in [7, 6]. This is identified as λ-prolongation, which leads to the notion of λ-symmetries. If a system does not have a Lie point symmetry, we explain several of the processes of reduction of order, by demonstrating in the invariance of the equation under λ-symmetries. Hence, we present the notion of a λ-symmetry by the new technique of λ-prolongations and by certain conditions of invariance. Consequently, we gain a new method of reduction for ODEs.

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In [3], by horizontal one-form $\mu = \lambda_i dx^i$ on first order jet space $\pi\colon J^{(1)} \to M$, where μ is a compatible, i.e. $D_i\lambda_j - D_j\lambda_i = 0$, this approach is applied PDE frame with p independent variables $x = (x^1, \ldots, x^p)$ and q dependent variables $u = (u^1, \ldots, u^q)$, which leads to μ-symmetries. Based on λ-symmetries, the notion of the variational problem and conservation law, and adapted formulation of the Noether's theorem for the λ-symmetry of ODEs are presented in [8]. G. Cicogna and G. Gaeta [2] assert the result of λ-symmetries case to the case of μ-symmetries. The corresponding conservation law for the μ-symmetry of the Lagrangian is called μ-conservation law.

The Camassa-Holm (CH) equation [1] is $m_t + 2mu_x + m_x u = 0$, where $m = u - u_{xx}$ is equivalent to

$$\Delta^c : u_t - u_{xxt} + 3uu_x - 2u_x u_{xx} - uu_{xxx} = 0, \tag{1}$$

A nonlinear dispersive wave equation waves over a flat bed, beside the water waves moving over an underlying shear flow. It models the propagation of unidirectional irrotational shallow water [1]. The CH equation (1) can use in regarding some non-Newtonian fluids and models finite length, small amplitude radial deformation waves in cylindrical hyperelastic rods. Due to [1], the CH equation (1) has a bi-Hamiltonian structure, which is completely integrable. By substituting $m = -u_{xx}$ in Eq. (1), the short wave limit of the CH equation (1) becomes the Hunter-Saxton (HS) equation as follows

$$\Delta^s : u_{xxt} + 2u_x u_{xx} + uu_{xxx} = 0. \tag{2}$$

The Hunter-Saxton equation represents the propagation of waves in a massive director field of a nematic liquid crystal [4]. The field of unit vectors $(\cos u(x,t), \sin u(x,t))$ explains the orientation of the molecules, where x is the space variable in a reference frame moving with the linearized wave velocity, and t is a 'slow time variable'.

The liquid crystal state is a distinct phase of matter observed between the solid and liquid states. A nematic liquid crystal is characterized by long rigid molecules that do not have any positional order but tend to point in the same direction (along with the director). In Eq. (2), $u(x,t)$ is a measure of the average orientation of the medium locally around x at time t. Eq.(2) is a bi-variational, completely integrable system with a bi-Hamiltonian structure, which indicates the existence of an infinite family of commuting Hamiltonian flows in addition to an infinite sequence of conservation laws [5].

The outline of this paper is as follows. In Section 2, we express the Lie symmetry analysis, the optimal system of 1-dimensional subalgebras, Lie invariants, and reduction of the CH equation (1) and the HS equation (2). Nadjafikhah and Shirvani [10] attain the symmetry of the CH equation, and Nadjafikhah and Ahangari [9] attain the symmetry of the HS equation. We state these results in section 2.

In Section 3, we compute the μ-symmetry and order reduction of Eq. (1) and Eq. (2). Finally, in Section 4, we obtain the Lagrangian of equations (1) and (2) in potential form, and by using it, we compute μ-conservation laws of equations (1) and (2).

2. Lie Symmetry method for the CH equation and the HS equation

Suppose $\Delta(x, u^n) = 0$ is a PDE, which involves p independent variables $x = (x^1, \ldots, x^p)$ and q dependent variables $u = (u^1, \ldots, u^q)$, defined over the total space $M = X \times U$, whose coordinates represent the independent and dependent variables and the space $M^{(n)} = X \times U \times U^1 \times U^2 \times \ldots \times U^n$, the derivatives of dependent variables up to order n is called the n-th order jet space on the underlying space $X \times U$. Suppose $f(x) : X \longrightarrow U$ is a smooth real-valued function which contains p independent variables, then the n-jet or n-th prolongation of f is $\mathrm{Pr}^{(n)} f : X \longrightarrow U^{(n)}$, every point of $U^{(n)}$ is shown by $u^{(n)}$. The graph of $\mathrm{Pr}^{(n)} f(x)$ lies in the n-jet space $M^{(n)}$ and a smooth solution of $\Delta(x, u^n) = 0$ is a smooth function $u = f(x)$.

Suppose G is a local group transformation which acts on M, then a 1-parameter Lie group $G : I \times M \longrightarrow M$ is as follows:

$$(\varepsilon, (x, u)) \longmapsto \varphi(\varepsilon) = \Big(x^1 + \varepsilon\xi^1(x,u) + O(\varepsilon^2), \ldots, u^1 + \varepsilon\eta^1(x,u) + \ldots\Big),$$

where $I \subseteq \mathbb{R}$, and C-curve is the graph of G on M, that in each its point the tangent vector

$$\mathbf{v} = \dot{\varphi}(\varepsilon) = \frac{d\varphi}{d\varepsilon}\Big|_{\varepsilon=0} = \sum_{i=1}^{p} \xi^i(x,u)\partial_{x^i} + \sum_{\alpha=1}^{q} \eta^\alpha(x,u)\partial_{u^\alpha},$$

is an infinitesimal transformation of G in $\mathfrak{g}$ acts on $X \times U \times U^{(1)}$. G is a symmetry group of $\Delta(x, u^n) = 0$, transforming solutions of PDE to other solutions of PDE. Here, we want to determine the symmetry group via the classical infinitesimal symmetry condition. By acting n-th prolongation of $\mathbf{v}$; i.e., $\mathrm{Pr}^{(n)}\mathbf{v}$, on $\Delta(x, u^{(n)})$, we obtain

$$\mathrm{Pr}^{(n)}\mathbf{v}\big[\Delta(x, u^{(n)})\big] \equiv 0 \mod \Delta(x, u^{(n)}) = 0,$$

where $\mathrm{Pr}^{(n)}\mathbf{v}$ is

$$\mathrm{Pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q}\sum_{J}\Big[\mathrm{D}_J(\varphi_\alpha - \sum_{i=1}^{p}\xi^i u_i^\alpha) + \sum_{i=1}^{p}\xi^i u_{J,i}^\alpha\Big]\partial_{u_J^\alpha}, \tag{3}$$

and $J = (j_1, \ldots, j_k)$ with $1 \leq k \leq p$, and each D_i is the total derivative with respect to x^i [11]. If we solve this system, we find the symmetry group of $\Delta(x, u^{(n)})$. The equation (3) is equivalent to $\mathrm{Pr}^{(n)}\mathbf{v} = \mathbf{v}+\sum_{\alpha=1}^{q}\sum_J[D_i\varphi_\alpha^J - \sum_{m=1}^{p} u_{J,m}D_i\xi^m]\partial_{u_J^\alpha}$, where $\varphi^0 = \varphi$.

Suppose $\mathrm{v} = \xi(x,t,u)\partial_x + \tau(x,t,u)\partial_t + \varphi(x,t,u)\partial_u$ is an infinitesimal generator of the classical Lie point symmetry groups for the CH equation, then its third prolongation is

$$\mathrm{Pr}^{(3)}\mathrm{v} = \mathrm{v} + \varphi^x\partial_{u_x} + \varphi^t\partial_{u_t} + \varphi^{xx}\partial_{u_{xx}} + \cdots + \varphi^{ttt}\partial_{u_{ttt}},$$

where coefficients $\mathrm{Pr}^{(3)}\mathrm{v}$ are given by

$$\begin{aligned} \varphi^x &= D_x\varphi - u_xD_x\xi - u_tD_x\tau, & \varphi^t &= D_t\varphi - u_xD_t\xi - u_tD_t\tau, \\ \varphi^{xx} &= D_x\varphi^x - u_{xx}D_x\xi - u_{xt}D_x\tau, & \varphi^{ttt} &= D_t\varphi^{tt} - u_{ttx}D_t\xi - u_{ttt}D_t\tau, & \ldots. \end{aligned}$$

2.1. Lie Symmetry method for the CH equation

To compute the symmetry group of the CH equation, we substitute the $Pr^{(3)}\mathrm{v}$ on the Eq. (1), then we obtain $Pr^{(3)}\mathrm{v}[\Delta_u] = 0$. Next, by substituting $u_{xxt} - 3uu_x + 2u_xu_{xx} + uu_{xxx}$ to u_t, the remaining is a polynomial equation including several derivatives of $u(x,t)$ whose coefficients are certain derivatives of ξ, τ, and φ. Suppose any coefficients is equivalent to zero, leads to $\xi = c_3$, $\tau = c_1t + c_2$, and $\varphi = -c_1u$, where c_1, c_2 and c_3 are arbitrary constants.

Corollary 2.1. *The Lie algebra of infinitesimal projectable symmetries of the CH equation is spanned by the vector fields* $\mathbf{v}_1 = \partial_x$, $\mathbf{v}_2 = \partial_t$, $\mathbf{v}_3 = t\partial_t - u\partial_u$. □

The vector fields $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ generate the 1−parameter groups G_i,

$$G_1(x,t,u) = (x+\varepsilon, t, u), \quad G_2(x,t,u) = (x, t+\varepsilon, u), \quad G_3(x,t,u) = (x, te^{\varepsilon}, ue^{-\varepsilon}).$$

The entries give the transformed point $\exp(\varepsilon\mathbf{v}_i)(x,t,u) = (\tilde{x}, \tilde{t}, \tilde{u})$. Due to the fact that every G_i is a symmetry group, if $u = f(x,t)$ be a solution of the Eq. (1), it implies that the functions $u_1 = f(x-\epsilon, t)$, $u_2 = f(x, t-\epsilon)$ and $u_3 = f(x, te^{-\varepsilon})e^{-\varepsilon}$ are the solutions of the Eq. (1).

Lie groups are essential in finding the exact solutions of PDEs, and any transformation in the symmetry group will take a solution to the other solution. Every 1-parameter subgroup of the symmetry group of a PDE will be correspond to a family of solutions, such solutions called, invariant solutions. Finding an optimal system of subgroups equals to finding an optimal system of subalgebras. Classification problem 1-dimensional subalgebras is the same as the classification of the orbits of the adjoint representations. Hence by taking a general component of the Lie algebra and subjecting it to different adjoint transformations, one can simplify it as much as possible.

Table 1 shows commutation table of Lie algebra $\mathfrak{g}$ for the CH equation within vector fields $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$, where i-th row and j-th column is defined as $[\mathbf{v}_i, \mathbf{v}_j] = \mathbf{v}_i\mathbf{v}_j - \mathbf{v}_j\mathbf{v}_i$. Note that the Lie algebra $\mathfrak{g}$ is solvable. Table 1 also shows the adjoint representation of the CH equation, where the adjoint action is a Lie series as $\mathrm{Ad}(\exp(\varepsilon\mathbf{v}_i))\mathbf{v}_j = \mathbf{v}_j - \varepsilon[\mathbf{v}_i, \mathbf{v}_j] + \varepsilon^2[\mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j]]/2 - \ldots$.

Table 1: The commutator table and adjoint representation table of Eq.1

$[\mathbf{v}_i, \mathbf{v}_j]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	0	0	0
$\mathbf{v}_2$	0	0	$\mathbf{v}_2$
$\mathbf{v}_3$	0	$-\mathbf{v}_2$	0

$\mathrm{Ad}(\exp(\varepsilon\mathbf{v}_i)\mathbf{v}_j)$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3 - \varepsilon\mathbf{v}_2$
$\mathbf{v}_3$	$\mathbf{v}_1$	$\mathbf{v}_2 + \varepsilon\mathbf{v}_2$	$\mathbf{v}_3$

Theorem 2.2. *An optimal system of 1-dimensional Lie algebras of Eq. (1) is provided by* $a_1\mathbf{v}_1 + \mathbf{v}_3$, $a_1\mathbf{v}_1 + \mathbf{v}_2$ *and* $\mathbf{v}_1$.

Proof. Suppose the vector fields $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ span the symmetry algebra $\mathfrak{g}$ of the CH equation. Let $F_i^\varepsilon : \mathfrak{g} \longrightarrow \mathfrak{g}$ be a linear map which is defined by $\mathbf{v} \mapsto \mathrm{Ad}(\exp(\varepsilon \mathbf{v}_i))\mathbf{v}$. The matrices M_i^ε of F_i^ε, $i = 1, 2, 3$, with respect to basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are

$$M_1^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_2^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \varepsilon \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_3^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-\varepsilon} & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A nonzero vector $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$ determines a 1-dimensional subalgebra of $\mathfrak{g}$. Here a_i are arbitrary constants. By acting these matrices on $\mathbf{v}$, we construct coefficients a_i as simple as possible. We obtain 3 cases:

Case 1. Let $a_3 \neq 0$, then suppose that $a_3 = 1$, acting M_1^ε and M_2^ε on $\mathbf{v}$, the coefficient of $\mathbf{v}_2$ vanishes and the coefficient of $\mathbf{v}_1$ doesn't change, then $\mathbf{v}$ reduces to $\mathbf{v} = a_1\mathbf{v}_1 + \mathbf{v}_3$.

Case 2. Let $a_3 = 0$ and $a_2 \neq 0$, then we can suppose that $a_2 = 1$, acting M_i^ε on $\mathbf{v}$, the coefficient of $\mathbf{v}_1$ doesn't change and $\mathbf{v}$ reduces to $\mathbf{v} = a_1\mathbf{v}_1 + \mathbf{v}_2$.

Case 3. Let $a_3 = 0$ and $a_2 = 0$, acting M_i^ε on $\mathbf{v}$, the coefficient of $\mathbf{v}_1$ doesn't change, in this case $\mathbf{v} = \mathbf{v}_1$. □

If we integrate the characteristic equations, we can obtain the invariants associated with the symmetry operators. For instance, the characteristic equation of the operator $\alpha \mathrm{v}_1 = \alpha\partial_x$ is $dx/\alpha = dt/0 = du/0$, and its corresponding invariants are $y = t$, $w = u$. The derivatives of u are given in terms of y and $w(y)$ as $u_t = w_y$, $u_{txx} = u_x = u_{xx} = u_{xxx} = 0$. By substituting them into the Eq. (1), we obtain the ordinary differential equation $w_y = 0$. Table 2 and Table 3 show the results.

Table 2: Invariant of Eq.(1)

operator	y	w	u
$\alpha\mathbf{v}_1$	t	u	$w(y)$
$\alpha\mathbf{v}_1 + \mathbf{v}_2$	$x - \alpha t$	u	$w(y)$
$\alpha\mathbf{v}_1 + \mathbf{v}_3$	$te^{-x/\alpha}$	$ue^{x/\alpha}$	$w(y)e^{-x/\alpha}$

Table 3: Reduction of Eq.(1)

operator	reduced equations
$\alpha\mathbf{v}_1$	$w_y = 0$
$\alpha\mathbf{v}_1 + \mathbf{v}_2$	$-\alpha w_y + \alpha w_{yyy} - 3ww_y + ww_{yy} = 0$
$\alpha\mathbf{v}_1 + \mathbf{v}_3$	$-\frac{1}{\alpha^3}(y^3 + wy^4)w_{yyy} + (\frac{2}{\alpha^3}wy^2 - \frac{5}{\alpha}y^2 - \frac{6}{\alpha^3}wy^3)w_{yy} + (\frac{\alpha^2-4}{\alpha^2}y - \frac{3}{\alpha^3})wy^2w_y + (\frac{2}{\alpha^3} - \frac{3}{\alpha})w^2y = 0$

2.2. Lie Symmetry method for the HS equation

Here we want to compute the symmetry group of the HS equation. First, let the $Pr^{(3)}\mathrm{v}$ on the Eq. (2) i.e. $Pr^{(3)}\mathrm{v}[\Delta^s] = 0$, then by substituting $-2u_xu_{xx} - uu_{xxx}$ to u_{xxt}, the remaining is a polynomial equation involving the different derivatives of $u(x,t)$ whose coefficients are certain derivatives of ξ, τ, and φ. Now, suppose every coefficient equals to zero. By solving these equations, we have $\xi = c_1x + c_3$, $\tau = c_2$, $\varphi = c_1u$, where c_1, c_2 and c_3 are arbitrary constant. The following vector fields span the Lie algebra of infinitesimal projectable symmetries of the HS equation

$$\mathbf{v}_1 = \partial_x, \qquad \mathbf{v}_2 = \partial_t, \qquad \mathbf{v}_3 = x\partial_x + u\partial_u. \tag{4}$$

The vector fields $\mathbf{v}_1, \mathbf{v}_2$, and $\mathbf{v}_3$ generate the 1-parameter groups G_i,

$$G_1(x,t,u) = (x+\varepsilon, t, u), \quad G_2(x,t,u) = (x, t+\varepsilon, u), \quad G_3(x,t,u) = (xe^\varepsilon, t, ue^\varepsilon),$$

that the entries give the transformed point $\exp(\varepsilon\mathbf{v}_i)(x,t,u) = (\tilde{x}, \tilde{t}, \tilde{u})$. Since every G_i is a symmetry group, by putting $u = f(x,t)$ as a solution of the Eq. (2), we have the functions $u_1 = f(x-\epsilon, t)$, $u_2 = f(x, t-\epsilon)$ and $u_3 = e^\varepsilon f(xe^{-\varepsilon}, t)$ are solutions of the Eq. (2).

Table 4 shows the commutation table of Lie algebra $\mathfrak{g}$ for the HS equation between vector fields $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$, where i-th row and j-th column is defined as $[\mathbf{v}_i, \mathbf{v}_j] = \mathbf{v}_i\mathbf{v}_j - \mathbf{v}_j\mathbf{v}_i$. Note that the Lie algebra $\mathfrak{g}$ is solvable. Table 4 also shows the adjoint representation of the HS equation, where the adjoint action is a Lie series as $\mathrm{Ad}(\exp(\varepsilon\mathbf{v}_i))\mathbf{v}_j = \mathbf{v}_j - \varepsilon[\mathbf{v}_i, \mathbf{v}_j] + \varepsilon^2[\mathbf{v}_i, [\mathbf{v}_i, \mathbf{v}_j]]/2 - \ldots$.

Table 4: The commutator table and adjoint representation table of Eq. (2)

$[\mathbf{v}_i, \mathbf{v}_j]$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	0	0	$\mathbf{v}_1$
$\mathbf{v}_2$	0	0	0
$\mathbf{v}_3$	$-\mathbf{v}_1$	0	0

$\mathrm{Ad}(\exp\varepsilon\mathbf{v}_i)\mathbf{v}_j)$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_1$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3 - \varepsilon\mathbf{v}_1$
$\mathbf{v}_2$	$\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$
$\mathbf{v}_3$	$\mathbf{v}_1 + \varepsilon\mathbf{v}_1$	$\mathbf{v}_2$	$\mathbf{v}_3$

Theorem 2.3. *An optimal system of one-dimensional Lie algebras of Eq. (2) is provided by $a_2\mathbf{v}_2 + \mathbf{v}_3$, $a_1\mathbf{v}_1 + \mathbf{v}_2$ and $\mathbf{v}_1$.*

Proof. Suppose the vector fields $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ span the symmetry algebra $\mathfrak{g}$ of the HS equation. Let $F_i^\varepsilon : \mathfrak{g} \longrightarrow \mathfrak{g}$ be a linear map which is defined by $\mathbf{v} \mapsto \mathrm{Ad}(\exp(\varepsilon\mathbf{v}_i))\mathbf{v}$. The matrices M_i^ε of F_i^ε, $i = 1, 2, 3$, with respect to basis $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are

$$M_1^\varepsilon = \begin{pmatrix} 1 & 0 & \varepsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_2^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad M_3^\varepsilon = \begin{pmatrix} e^{-\varepsilon} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

A nonzero vector $\mathbf{v} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3$ determines a 1-dimensional subalgebra of $\mathfrak{g}$. Here a_i are arbitrary constants. By acting these matrices on $\mathbf{v}$, we construct coefficients a_i as simple as possible. We obtain 3 cases:

Case 1. Let $a_3 \neq 0$, then suppose $a_3 = 1$, which M_1^ε and M_2^ε act on $\mathbf{v}$, the coefficient of $\mathbf{v}_1$ vanishes and the coefficient of $\mathbf{v}_2$ doesn't change, then $\mathbf{v}$ reduces to $\mathbf{v} = a_2\mathbf{v}_2 + \mathbf{v}_3$.

Case 2. Let $a_3 = 0$ and $a_2 \neq 0$, then we can assume that $a_2 = 1$, acting M_i^ε on $\mathbf{v}$, the coefficient of $\mathbf{v}_1$ doesn't change and $\mathbf{v}$ reduces to $\mathbf{v} = a_1\mathbf{v}_1 + \mathbf{v}_2$.

Case 3. Let $a_3 = 0$ and $a_2 = 0$, acting M_i^ε on $\mathbf{v}$, the coefficient of $\mathbf{v}_1$ doesn't change. In this case, $\mathbf{v} = \mathbf{v}_1$. □

Table 5 shows the calculation of the invariants associated with the symmetry operators and reduction of the Eq. (2).

Table 5: Invariants and reduced form of Eq.(2)

operator	y, w, u	Reduced equation
$\alpha\mathbf{v}_1$	$t, u, w(y)$	$w_y = 0$
$\alpha\mathbf{v}_1 + \mathbf{v}_2$	$x - \alpha t, u, w(y)$	$-\alpha w_{yyy} + 2w_y w_{yy} + w w_{yyy} = 0$
$\alpha\mathbf{v}_2 + \mathbf{v}_3$	$xe^{-t/\alpha}, ue^{-t/\alpha}, w(y)e^{t/\alpha}$	$(\alpha yw - y^2)w_{yyy} + (3\alpha w - 2y + \alpha y)w_{yy} + (3\alpha + \alpha w/y)wy = 0$

3. μ-symmetry method for the CH equation and the HS equation

Suppose $\mu = \lambda_i dx^i$ is a horizontal 1-form on first order jet space $\pi : J^{(1)} \to M$ which is compatible, i.e. $D_i\lambda_j - D_j\lambda_i = 0$, where each D_i is the total derivative with respect to x^i and $\lambda_i : J^{(1)}M \longrightarrow \mathbb{R}$ [3]. Muriel and Romero [6] introduced a new technique to order reduction of ODEs. This method is called λ-symmetry method to order reduction of ODEs. In 2004, Gaeta and Morando extended this method of ODEs to μ-symmetries method of PDEs [3].

Suppose $\Delta = \Delta(x, u^{(n)}) = 0$ is a scalar PDE of order n, which includes p independent variables $x = (x^1, \ldots, x^p)$ and one dependent variable $u = u(x^1, \ldots, x^p)$. Suppose $X = \sum_{i=1}^p \xi^i(x, u)\partial_{x^i} + \varphi(x, u)\partial_u$ is a vector field on M. The vector field $Y = X + \sum_{J=1}^n \Psi^J\, \partial_{u_J}$, is μ-prolongation of X on n-th order jet space $J^n M$, if its coefficients satisfy the μ-prolongation formula $\Psi^{J,i} = (D_i + \lambda_i)\Psi^J - u_{J,m}(D_i + \lambda_i)\xi^m$, where $\Psi^0 = \varphi$. Suppose $\mathcal{S} \subset J^{(n)}M$ is the solution manifold for Δ. If $Y : \mathcal{S} \longrightarrow T\mathcal{S}$, then X is called a μ-symmetry for Δ. In general, if $\mu = 0$, ordinary prolongation and ordinary symmetry will arise.

The computation of μ-symmetries of a given equation $\Delta = 0$ like the ordinary symmetries. Let X be a vector field which acts in M then its μ-prolongation Y of order n acts in $J^{(n)}M$. Later proceed it to apply Y to Δ, restricts the obtained expression to the solution manifold $\mathcal{S}_\Delta \subset J^{(n)}M$. The result is Δ^* up to ξ, τ, φ, and λ_i. Let λ_i be functions on $J^{(k)}M$, then the dependences on u_J are precise, and one obtains a system of determining equation. This system complemented with the compatibility conditions within λ_i. If we define a priori form of μ, we have a system of linear equations of ξ, τ, φ. Similarly, if we fix a vector field X and try to find the μ to have a μ-symmetry of Δ, we have a system of quasilinear equation for the λ_i [3].

Let X be a vector field on M and $V = \exp\big(\int \mu\big)X$ be an exponential vector field. Then V is a general symmetry of Δ if and only if X be a μ-symmetry of Δ.

Theorem 3.1. *[3] (order reduction of PDEs under μ-symmetry method) Let Δ be a scalar PDE of order n for $u = u(x^1, \dots, x^p)$. Let $X = \xi^i(x,u)\partial_{x^i} + \varphi(x,u)\partial_u$ be a vector field on M, with characteristic $Q = \varphi - u_i\xi^i$, and let Y be the μ-prolong of order n of X. If X is a μ-symmetry for Δ, then $Y : \mathcal{S}_X \longrightarrow T\mathcal{S}_X$, where $\mathcal{S}_X \subset J^{(n)}M$ is the solution manifold for the system Δ_X made of Δ and of $E_J := D_JQ = 0$ for all J with $\mid J \mid= 0, 1, \dots, n-1$.* □

Here, we want to compute μ-symmetry of the Eq. (1). Suppose $\mu = \lambda_1 dx + \lambda_2 dt$ is a horizontal 1-form with the compatibility condition $D_t\lambda_1 = D_x\lambda_2$, whenever $\Delta_u = 0$. Suppose $X = \xi\partial_x + \tau\partial_t + \varphi\partial_u$ is a vector field on M. In order to calculate μ-prolongation Y of order 3 of X,we can use of (4), then we have $Y = X + \Psi^x\partial_{u_x} + \Psi^t\partial_{u_t} + \Psi^{xx}\partial_{u_{xx}} + \cdots + \Psi^{ttt}\partial_{u_{ttt}}$. In this case, the coefficients of Y are given by

$$\begin{aligned}
\Psi^x &= (D_x + \lambda_1)\varphi - u_x(D_x + \lambda_1)\xi - u_t(D_x + \lambda_1)\tau, \\
\Psi^t &= (D_t + \lambda_2)\varphi - u_x(D_t + \lambda_2)\xi - u_t(D_t + \lambda_2)\tau, \\
&\vdots \\
\Psi^{ttt} &= (D_t + \lambda_2)\Psi^{tt} - u_{ttx}(D_t + \lambda_2)\xi - u_{ttt}(D_t + \lambda_2)\tau.
\end{aligned} \tag{5}$$

To achieve μ-symmetry method of the CH equation, by applying Y to Eq. (1) and substituting $u_t + 3uu_x - 2u_xu_{xx} - uu_{xxx}$ in u_{xxt}, we reach to the following system

$$-3\xi_u u = 0, \qquad 2\tau_u = 0, \qquad 4\tau_{uu} = 0, \qquad -u\tau_u + \xi_u = 0, \qquad \dots. \tag{6}$$

Suppose λ_1 and λ_2 are any choice of the type

$$\lambda_1 = D_x[f(x,t)] + g(x), \qquad \lambda_2 = D_t[f(x,t)] + h(t).$$

where satisfy to the compatibility condition, i.e. $D_t\lambda_1 = D_x\lambda_2$, and $f(x,t)$, $g(x)$ and $h(t)$ are arbitrary functions. For simplicity in calculating μ-symmetry of the Eq. (1), suppose $g(x) = 0, h(t) = 0$ and $f(x,t) = -\ln(F(x,t))$ in λ_1 and λ_2. Let $F := F(x,t)$ be an arbitrary positive function, then by substituting $\lambda_1 = -F_x/F$ and $\lambda_2 = -F_t/F$ into the system of (6) and solving them, we obtain $\xi = F$, $\tau = 0$, $\varphi = 0$. Consequently, the vector field $X = F\partial_x$ is μ-symmetry of Eq. (1) and the vector field $V = \exp\big(\int \lambda_1 dx + \lambda_2 dt\big)X = \exp\big(\int -\frac{F_x}{F}dx - \frac{F_t}{F}dt\big)X$ is a general symmetry of exponential type corresponds to X. In this case, by using Theorem 3.1, the order reduction of Eq. (1) is $Q = \varphi - \xi u_x - \tau u_t = -Fu_x$. These three cases are shown in Table 6 where $f(x,t) = -\ln(F(x,t))$.

By applying Y to Eq. (2) and substituting $-2u_xu_{xx} - uu_{xxx}$ to u_{xxt} , we get the system

$$-3\xi_u u = 0, \qquad -2\tau_u = 0, \qquad -4\tau_{uu} = 0, \qquad \dots. \tag{7}$$

Suppose $\lambda_1 = D_x[f(x,t)] + g(x)$ and $\lambda_2 = D_t[f(x,t)] + h(t)$ satisfy the compatibility condition, i.e. $D_t\lambda_1 = D_x\lambda_2$. Similar to μ-symmetry method for the CH equation, we consider these three cases in Tables 7, where $f(x,t) = -\ln(F(x,t))$.

4. μ-conservation laws of the CH equation and the HS equation

The concept of variational problem and conservation law in the case of symmetries to the case of λ-symmetries of ODEs has developed by Muriel, Romero, and Olver [8]. They suggested an adapted formulation of the Noether's Theorem for λ-symmetry of ODEs. The results of [8] are generalized by Cicogna and Gaeta [2]. They extended the case of λ-symmetries for ODEs to the case of μ-symmetries for PDEs and they also extended the Noether's

Table 6: μ-symmetry, symmetry of exponential type and order reduction of Eq.(1)

Case	$g(x)$	$h(t)$	λ_1	λ_2	ξ	τ	φ
1	0	0	$-\frac{F_x}{F}$	$-\frac{F_t}{F}$	F	0	0
2	0	$\frac{1}{t-c_1}$	$-\frac{F_x}{F}$	$-\frac{F_t}{F}+\frac{1}{t-c_1}$	0	F	$\frac{-u}{t-c_1}F$
3	0	$\frac{c_1}{c_1t+c_2}$	$-\frac{F_x}{F}$	$-\frac{F_t}{F}+\frac{c_1}{c_1t+c_2}$	$\frac{-F}{c_1t+c_2}$	F	$-\frac{c_1u}{c_1t+c_2}F$

μ-symmetry : X	symmetry of exponential type : V	order reduction
$X=F\partial_x$	$V=\exp\big(-\int\frac{F_x\,dx}{F}+\frac{F_t\,dt}{F}\big)X$	$-Fu_x=0$
$X=F\big(\partial_t-\frac{u}{t-c_1}\partial_u\big)$	$V=\exp\big(-\int\frac{F_x\,dx}{F}+(\frac{F_t}{F}-\frac{1}{t-c_1})dt\big)X$	$-(u_t+\frac{u}{t-c_1})F=0$
$X=F\big(-\frac{1}{c_1t+c_2}\partial_x+\partial_t-\frac{c_1u}{c_1t+c_2}\partial_u\big)$	$V=\exp\big(-\int\frac{F_x\,dx}{F}+(\frac{F_t}{F}+\frac{c_1}{c_1t+c_2})dt\big)X$	$-(u_t+\frac{c_1u}{(c_1t+c_2)}-\frac{1}{(c_1t+c_2)}u_x)F=0$

Table 7: μ-symmetry, symmetry of exponential type and order reduction of Eq.(2)

Case	$g(x)$	$h(t)$	λ_1	λ_2	ξ	τ	φ
1	0	$\frac{1}{t-c_1}$	$-\frac{F_x}{F}$	$-\frac{F_t}{F}+\frac{1}{t-c_1}$	0	F	$\frac{-uF}{t-c_1}$
2	0	$h(t)$	$-\frac{F_x}{F}$	$-\frac{F_t}{F}+h(t)$	F	0	$h(t)F$
3	$\frac{1}{x-k(t)}$	$-\frac{k'(t)}{x-k(t)}$	$-\frac{F_x}{F}+\frac{1}{x-k(t)}$	$-\frac{F_t}{F}-\frac{k'(t)}{x-k(t)}$	F	0	$\frac{u-k'(t)}{x-k(t)}$

μ-symmetry : X	symmetry of exponential type : V	order reduction
$X=F\big(\partial_t-\frac{u}{t-c_1}\,\partial_u\big)$	$V=\exp\big(\int-\frac{F_x}{F}dx-(\frac{F_t}{F}+\frac{1}{t-c_1})dt\big)X$	$-F(u_t+\frac{u}{t-c_1})=0$
$X=F\big(\partial_x+h(t)\,\partial_u\big)$	$V=\exp\big(\int-\frac{F_x}{F}dx-(\frac{F_t}{F}+h(t))dt\big)X$	$(h(t)-u_x)F=0$
$X=F\big(\partial_x+\frac{u-k'(t)}{x-k(t)}\,\partial_u\big)$	$V=\exp\big(\int(-\frac{F_x}{F}+\frac{1}{x-k(t)})dx-(\frac{F_t}{F}-\frac{k'(t)}{x-k(t)})dt\big)X$	$(\frac{u-k'(t)}{x-k(t)}-u_x)F=0$

Theorem for λ-symmetry of ODEs to the Noether's Theorem for μ-symmetry of PDEs [2]. Also, the conservation law called μ-conservation law in the case of μ-symmetry of the Lagrangian.

A conservation law of PED is a divergence expression $\operatorname{Div}\mathbf{P}:=D_iP^i=0$, where $\mathbf{P}=(P^1,\ldots,P^p)$ is a p-dimensional vector. Suppose $\mu=\lambda_i dx^i$ is a horizontal 1-form with $D_i\lambda_j=D_j\lambda_i$. A μ-conservation law is a relation as $(D_i+\lambda_i)P^i=0$, where M-vector P^i is a (Matrix-valued) vector. This vector is called a μ-conserved vector.

Theorem 4.1. *[2] (Exist of M-vector) Consider the n-th order Lagrangian $\mathcal{L}=\mathcal{L}(x,u^{(n)})$, and vector field X, then X is a μ-symmetry for $\mathcal{L}$, i.e. $Y[\mathcal{L}]=0$ if and only if there exists $M-$vector P^i satisfying the μ-conservation law $(D_i+\lambda_i)P^i=0$.* □

Here, we calculate the M-vector P^i as [2]. To this aim suppose $\mathcal{L}=\mathcal{L}(x,u^{(2)})$ is a second order Lagrangian, and the vector field $X=\varphi\,\partial_u$ is a μ-symmetry for $\mathcal{L}$. The M-vector P^i will be as follows

$$P^i:=\varphi\frac{\partial\mathcal{L}}{\partial u_i}+(D_j+\lambda_j)\varphi.\frac{\partial\mathcal{L}}{\partial u_{ij}}-\varphi D_j\frac{\partial\mathcal{L}}{\partial u_{ij}}. \tag{8}$$

A system admits a variational formulation if and only if its Frechet derivative is self-adjoint. Indeed, we have the following theorem.

Theorem 4.2. *[11] Let $\Delta=0$ be a system of differential equations. Then Δ is the Euler-Lagrange expression for some variational problem $\mathfrak{L}=\int L dx$, i.e. $\Delta=E(L)$, if and only if the Frechet derivative D_Δ is self-adjoint: $D^*_\Delta=D_\Delta$. In this case, a Lagrangian for Δ can be explicitly constructed using the homotopy formula $\mathcal{L}[u]=\int_0^1 u.\Delta[\lambda u]d\lambda$.* □

4.1. μ-conservation laws of the CH equation

The CH equation (Δ^c), Eq. (1), is of odd order, so it does not admit a variational problem. But the CH equation in potential form admits a variational problem.

The Frechet derivative of the Eq. (1) is

$$D_{\Delta^c}=3u_x-u_{xxx}+D_t+(3u-2u_{xx})D_x-2u_xD_x^2-uD_x^3-D_x^2D_t.$$

Clearly, the CH equation does not admit a variational problem since $D^*_{\Delta^c} \neq D_{\Delta^c}$. If we substitute $u = v_x$, then the related transformed the CH equation is $\Delta^c_v \equiv v_{xt} - v_{xxxt} + 3v_x v_{xx} - 2v_{xx}v_{xxx} - v_x v_{xxxx} = 0$. In this case, Δ^c_v is called "the CH equation in potential form", and the Frechet derivative of the Δ^c_v is

$$D_{\Delta^c_v} = (3v_{xx} - v_{xxxx})D_x + (3v_x - 2v_{xxx})D^2_x + D_x D_t - 2v_{xx}D^3_x - D^3_x D_t - v_x D^4_x,$$

which is self-adjoint, i. e., $D^*_{\Delta^c_v} = D_{\Delta^c_v}$. Using Theorem 4.2, the Δ^c_v has a Lagrangian of the form

$$\mathcal{L}[v] = \int_0^1 v.\Delta^c_v[\lambda v]d\lambda = -\frac{1}{2}\Big(v_x v_t + v_{xx}v_{xt} + v_x v^2_{xx} + v^3_x\Big) + \text{Div}\, P.$$

Therefore, Lagrangian of the Δ^c_v, up to Div-equivalence is

$$\mathcal{L}[v] = -\frac{1}{2}\Big(v_x v_t + v_{xx}v_{xt} + v_x v^2_{xx} + v^3_x\Big).$$

Now, we compute μ-conservation law of the $\Delta^c_v = E(\mathcal{L}[v])$. Suppose $X = \varphi\partial_v$ is a vector field for $\mathcal{L}[v]$ and $\mu = \lambda_1 dx + \lambda_2 dt$ is a horizontal 1−form with the condition $D_t\lambda_1 = D_x\lambda_2$ in case of $\Delta^c_v = 0$. With the aim of calculating μ-prolongation of order 2 of X. Therefore, $Y = \varphi\partial_v + \Psi^x\partial_{v_x} + \Psi^t\partial_{v_t} + \Psi^{xx}\partial_{v_{xx}} + \Psi^{xt}\partial_{v_{xt}} + \Psi^{tt}\partial_{v_{tt}}$, and the coefficients of Y are given by

$$\begin{aligned} & & \Psi^x = (D_x+\lambda_1)\varphi, \qquad & \Psi^t = (D_t+\lambda_2)\varphi,\\ \Psi^{xx} &= (D_x+\lambda_1)\Psi^x, \qquad & \Psi^{xt} = (D_t+\lambda_2)\Psi^x, \qquad & \Psi^{tt} = (D_t+\lambda_2)\Psi^t. \end{aligned} \tag{9}$$

By putting $-(v_x^{-1}v_{xx}v_{xt} + v^2_{xx} + v^2_x)$ instead of v_t and substituting it and (9) into Y and also admitting the μ-prolongation Y on the $\mathcal{L}[v]$, we have the following

$$(-1/2)\varphi_v = 0, \quad -(1/2)\varphi_{vv} = 0, \quad \lambda_1\varphi + \varphi_x = 0, \quad \dots . \tag{10}$$

Put $\varphi = F(x,t)$ into the system (10), where $F(x,t)$ is an arbitrary positive function satisfying $\mathcal{L}[v] = 0$. By solving the system, a special solution presented as

$$\lambda_1 = -\frac{F_x(x,t)}{F(x,t)}, \qquad \lambda_2 = -\frac{F_t(x,t)}{F(x,t)}, \tag{11}$$

In this case, λ_1 and λ_2 satisfy the condition $D_t\lambda_1 = D_x\lambda_2$. Consequently, $X = F(x,t)\partial_v$ is a μ-symmetry for $\mathcal{L}[v]$. With the help of Theorem 4.1, there exists $M-$vector P^i which satisfies in the μ-conservation law $(D_i + \lambda_i)P^i = 0$. By using (8), we achieve the $M-$vector P^i for $\mathcal{L}[v]$ as follows

$$P^1 = -\frac{F(x,t)}{2}\Big(v_t - 2v_{xxt} - v^2_{xx} + 3v^2_x - 2v_x v_{xxx}\Big), \quad P^2 = -\frac{F(x,t)}{2}v_x. \tag{12}$$

So, $(D_x + \lambda_1)P^1 + (D_t + \lambda_2)P^2 = 0$ is a μ-conservation law for second order Lagrangian $\mathcal{L}[v]$.

The results of what we expressed above, are as follows

Corollary 4.3. *μ-conservation law for the CH equation in potential form $\Delta^c_v = E(\mathcal{L}[v])$ is $D_x P^1 + D_t P^2 + \lambda_1 P^1 + \lambda_2 P^2 = 0$. In this case P^1 and P^2 are the $M-$vector P^i of (12).* □

We use the Noether's theorem for μ-symmetry which is given in [2] and so we have

Remark 4.4. *The CH equation in potential form satisfying to the Noether's Theorem for μ-symmetry and μ-conservation law, i.e.*

$$\begin{aligned} (D_i + \lambda_i)P^i &= (D_x + \lambda_1)P^1 + (D_t + \lambda_2)P^2\\ &= F(x,t)\big(v_{xt} - v_{xxxt} + 3v_x v_{xx} - 2v_{xx}v_{xxx} - v_x v_{xxxx}\big) = QE(\mathcal{L}[v]). \end{aligned}$$

By using the CH equation in potential form Δ^c_v, one can determine μ-conservation law of the CH equation Δ^c, (Eq.1). The Δ^c_v is agree with $D_x(2v_t - 2v_{txx} + 3v^2_x - v^2_{xx} - 2v_x v_{xxx}) = 0$, which equals to $2v_t - 2v_{txx} + 3v^2_x - v^2_{xx} - 2v_x v_{xxx} = g(t)$, where $g(t)$ is an arbitrary function. By substituting $2g(t) + 2v_{txx} - 3v^2_x + v^2_{xx} + 2v_x v_{xxx}$ instead of v_t and u instead of v_x into (12), one get $M-$vectors P^1 and P^2 as

$$P^1 = -\frac{F(x,t)}{4}\Big(2g(t) - 2u_{tx} + 3u^2 - u^2_x - 2uu_{xx}\Big), \quad P^2 = -\frac{F(x,t)}{2}u. \tag{13}$$

So we have the following results

Corollary 4.5. *The μ-conservation law of the CH equation, Eq. (1), is in the form $D_xP^1+D_tP^2+\lambda_1P^1+\lambda_2P^2=0$. Here, P^1 and P^2 are the $M-$vector P^i of (13).*

Remark 4.6. *The CH equation satisfies to the characteristic form, i.e.*

$$\begin{aligned}(D_i+\lambda_i)P^i &= (D_x+\lambda_1)P^1+(D_t+\lambda_2)P^2\\ &= F(x,t)(u_t-u_{xxt}+3uu_x-2u_xu_{xx}-uu_{xxx})=Q\Delta^c.\end{aligned}$$

4.2. μ-conservation laws of the HS equation

While the HS equation (Δ^s), Eq. (2), is of odd order, it does not admit a variational problem. But the HS equation in potential form admits a variational problem. The Frechet derivative of the Eq. (2) is $D_{\Delta^s}=u_{xxx}+2u_{xx}D_x+2u_xD_x^2+uD_x^3+D_x^2D_t$.

Since $D^*_{\Delta^s}\neq D_{\Delta^s}$, the HS equation does not admit a variational problem. If we substitute $u=w_x$, then the related transformed the HS equation is $\Delta^s_w\equiv w_{xxxt}+2w_{xx}w_{xxx}+w_xw_{xxxx}=0$. In this case, Δ^s_w is called "the HS equation in potential form", and the Frechet derivative of the Δ^s_w is $D_{\Delta^s_w}=w_{xxxx}D_x+2w_{xxx}D_x^2+2w_{xx}D_x^3+D_x^3D_t+w_xD_x^4$, which is self-adjoint, i. e., $D^*_{\Delta^s_w}=D_{\Delta^s_w}$. Using Theorem 4.2, the Δ^s_w has a Lagrangian of the form

$$\mathcal{L}[w]=\int_0^1 w.\Delta^s_w[\lambda w]d\lambda=\frac{1}{2}\Big(w_{xx}w_{xt}+w_xw_{xx}^2\Big)+\mathrm{Div}\,P.$$

Therefore, Lagrangian of the Δ^s_w, up to Div-equivalence is

$$\mathcal{L}[w]=\frac{1}{2}\Big(w_{xx}w_{xt}+w_xw_{xx}^2\Big).$$

For calculating the μ-conservation law for the $\Delta^s_w=E(\mathcal{L}[w])$, suppose $X=\varphi\partial_w$ is a vector field for $\mathcal{L}[w]$, and $\mu=\lambda_1dx+\lambda_2dt$ is a horizontal 1-form with the condition $D_t\lambda_1=D_x\lambda_2$ in case of $\Delta^s_w=0$. With the purpose of calculating μ-prolongation of order 2 of X, by using $\Psi^{J,i}=(D_i+\lambda_i)\Psi^J-u_{J,m}(D_i+\lambda_i)\xi^m$, we obtain $Y=\varphi\partial_w+\Psi^x\partial_{w_x}+\Psi^t\partial_{w_t}+\Psi^{xx}\partial_{w_{xx}}+\Psi^{xt}\partial_{w_{xt}}+\Psi^{tt}\partial_{w_{tt}}$. In this case, the coefficients of Y are as (9). By putting these coefficients into Y and applying the μ-prolongation Y on the $\mathcal{L}[w]$, and also, substituting $-w_xw_{xx}$ instead of w_{xt} into it, we get the following

$$(1/2)\varphi_w=0,\quad (1/2)\varphi_{ww}=0,\quad (1/2)(\lambda_1\varphi+\varphi_x)=0,\qquad \ldots. \tag{14}$$

Put $\varphi=F(x,t)$ into the system (14), where $F(x,t)$ is an arbitrary positive function satisfying $\mathcal{L}[w]=0$. By solving this system, a special solution presented as

$$\lambda_1=-\frac{F_x(x,t)}{F(x,t)},\qquad \lambda_2=-\frac{F_t(x,t)}{F(x,t)}, \tag{15}$$

where λ_1 and λ_2 are satisfying to $D_t\lambda_1=D_x\lambda_2$. Hence, $X=F(x,t)\partial_w$ is a μ-symmetry for $\mathcal{L}[w]$. With the help of Theorem 4.1, there exists $M-$vector P^i satisfying in the μ-conservation law $(D_i+\lambda_i)P^i=0$. By using (8), we achieve the $M-$vector P^i for $\mathcal{L}[w]$ as follows

$$P^1=-\frac{F(x,t)}{2}\Big(2w_{xxt}+w_{xx}^2+2w_xw_{xxx}\Big),\qquad P^2=0. \tag{16}$$

So, $(D_x+\lambda_1)P^1+(D_t+\lambda_2)P^2=0$ is a μ-conservation law for second order Lagrangian $\mathcal{L}[w]$.

The results of what we expressed above, are as follows

Corollary 4.7. *μ-conservation law for the HS equation in potential form $\Delta^s_w=E(\mathcal{L}[w])$ is of $D_xP^1+D_tP^2+\lambda_1P^1+\lambda_2P^2=0$. In this case, P^1 and P^2 are the $M-$vector P^i of (16).* □

We use the Noether's theorem for μ-symmetry which is given in [2] and so we have

Remark 4.8. *The HS equation in potential form Δ^s_w satisfying to the Noether's Theorem fro μ-symmetry and μ-conservation law, i.e.*

$$\begin{aligned}(D_i+\lambda_i)P^i &= (D_x+\lambda_1)P^1+(D_t+\lambda_2)P^2\\ &= F(x,t)\big(w_{xxxt}+2w_{xx}w_{xxx}+w_xw_{xxxx}\big)=QE(\mathcal{L}[w]).\end{aligned}$$

For computing the μ-conservation law of the HS equation, Eq. (2), we substitute u instead of w_x into (16). Thus, $M-$vectors P^1 and P^2 are obtained as the following

$$P^1 = -\frac{F(x,t)}{2}\Big(2u_{tx} + u_x^2 + 2uu_{xx}\Big), \qquad P^2 = 0. \tag{17}$$

So we have the following results

Corollary 4.9. *The μ-conservation law of the HS equation, Eq. (2), is in the form $D_xP^1+D_tP^2+\lambda_1P^1+\lambda_2P^2 = 0$. In this case P^1 and P^2 are the $M-$vector P^i of (17).*

Remark 4.10. *The HS equation, Eq. (2), satisfies to the characteristic form, i.e.*

$$\begin{aligned} (D_i+\lambda_i)P^i &= (D_x+\lambda_1)P^1 + (D_t+\lambda_2)P^2 \\ &= F(x,t)(u_{xxt} + 2u_xu_{xx} + uu_{xxx}) = Q\Delta^s. \end{aligned}$$

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