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Original Article

Normal supercharacter theory of the dihedral groups

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ABSTRACT: Diaconis and Isaacs defined the supercharacter theory for finite groups as a natural generalization of the classical ordinary character theory of finite groups. Supercharacter theory of many finite groups such as the cyclic groups, the Frobenius groups, etc. are well studied and well-known. In this paper we find the normal and automorphic supercharacter theories of the dihedral groups in special cases.

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(*Dedicated to Professor Jamshid Moori*)

1. Introduction

Let $Irr(G)$ denote the set of all the irreducible complex characters of a finite group G, and let $Con(G)$ denote the set of all the conjugacy classes of G. The identity element of G is denoted by 1 and the trivial character is denoted by 1_G. By definition a supercharacter theory for G is a pair $(\mathcal{X}, \mathcal{K})$ where $\mathcal{X}$ and $\mathcal{K}$ are partitions of $Irr(G)$ and G respectively, $|\mathcal{X}| = |\mathcal{K}|$, $\{1\} \in \mathcal{K}$, and for each $X \in \mathcal{X}$ there is a character σ_X such that $\sigma_X(x) = \sigma_X(y)$ for all $x, y \in K$, $K \in \mathcal{K}$. We call σ_X supercharacter and each member of $\mathcal{K}$ a superclass. We write $Sup(G)$ for the set of all the supercharacter theories of G.

Supercharacter theory of a finite group were defined by Diaconis and Isaacs [5] as a general case of the ordinary character theory. In fact, in a supercharacter theory characters play the role of irreducible ordinary characters and union of conjugacy classes play the role of conjugacy classes. In [5] it is shown that $\{1_G\} \in \mathcal{X}$ and if $X \in \mathcal{X}$ then σ_X is a constant multiple of $\sum_{\chi \in X} \chi(1)\chi$, and that we may assume that

$$\sigma_X = \sum_{\chi \in X} \chi(1)\chi.$$

We call σ_X supercharacter and each member of $\mathcal{K}$ a superclass. We write $Sup(G)$ for the set of all the supercharacter theories of G.

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Any non-trivial finite group G has two trivial supercharacter theories which are denoted by $m(G)$ and $M(G)$ as follows:

$$M(G) = (\{\{1_G\}, (\rho_G - 1_G)\}, \{\{1\}, G - \{1\}\})$$
$$m(G) = (\bigcup_{\chi \in Irr(G)} \{\chi\}, \bigcup_{x \in G} \{x\})$$

In $M(G)$ the supercharacters are 1_G and $\rho_G - 1$, where ρ_G is the regular character of G, and in $m(G)$ all the irreducible characters are supercharacters.

The set of all the supercharacter theories of a finite group forms a lattice in the natural way [8]. In general, if S is a set, then the set of all partitions of S which is denoted by $P(S)$ forms a lattice, in which:

$\mathcal{K} \leqslant \mathcal{L} \Leftrightarrow$ every part of $\mathcal{K}$ is a subset of some part of $\mathcal{L}$ where $\mathcal{K}, \mathcal{L} \in P(S)$ obviously $Sup(G)$ is a poset in the natural way by defining

$$(\mathcal{X}, \mathcal{K}) \preceq (\mathcal{Y}, \mathcal{L}) \Leftrightarrow \mathcal{X} \leq \mathcal{Y} \text{ (or } \mathcal{K} \leq \mathcal{L})$$

This makes $Sup(G)$ to a lattice in which the minimal element is $m(G)$ and the maximal element is $M(G)$.

Among construction of supercharacter theories of a finite group G the following is of great importance which is a lemma by Brauer on character tables of groups. Let A be a subgroup of $\mathbb{A}ut(G)$ and

$$Irr(G) = \{\chi_1 = 1_G, \ldots, \chi_h\}$$
$$Con(G) = \{\mathcal{C}_1 = \{1\}, \ldots, \mathcal{C}_h\}.$$

Suppose for each $\alpha \in A$, ${\mathcal{C}_i}^\alpha = \mathcal{C}_j$, $1 \leq i \leq h$, and ${\chi_i}^\alpha(g) = \chi_i(g^\alpha)$ for all $g \in G$, then the number of conjugacy classes fixed by α equals the number of irreducible characters fixed by α, and moreover the number of orbits of A on $Con(G)$ equals the number of orbits of A on $Irr(G)$, [6]. It is easy to see that the orbits of A on $Irr(G)$ and $Con(G)$ yield a supercharacter theory for G. This supercharacter theory of G is called automorphic. In [7] it is shown that all the supercharacter theories of the cyclic group of order p, p prime, are automorphic.

From another point of view, the use of the supercharacter theory of finite groups for computation of the conjugacy classes and irreducible characters of $U_n(F)$ is still open. The group $U_n(F)$ is the group of $n \times n$ unimodular upper triangular matrices over the Galois field $GF(p^m)$, p prime. In [2] the author has developed an applicable supercharacter theory for $U_n(F)$. This result is reviewed in [5].

Recently a new method is presented in [1] for constructing supercharacter theories for a finite group called normal supercharacter theory. Motivated by [1] we consider the dihedral group and find its normal and automorphic supercharacter theories in special cases.

2. Supercharacter table

At this point we define the supercharacter table. Let $(\mathcal{X}, \mathcal{K})$ be a supercharacter theory for a finite group G. Suppose

$$\mathcal{X} = \{X_1, X_2, \ldots, X_h\}$$

is a partition of $Irr(G)$ with corresponding supercharacter $\sigma_i = \sum_{\chi \in X_i} \chi(1)\chi$. Let

$$\mathcal{K} = \{K_1, K_2, \ldots, K_h\}$$

be the partition of G into superclasses. In fact, $X_1 = \{1_G\}$, $K_1 = \{1\}$ and K_i's are union of conjugacy classes of G. The supercharacter table of G corresponding to $(\mathcal{X}, \mathcal{K})$ is Table 1.

Table 1: Supercharacter table

	K_1	K_2	$\ldots$	K_j	$\ldots$	K_h
σ_1	$\sigma_1(K_1)$	$\sigma_1(K_2)$	$\ldots$	$\sigma_1(K_j)$	$\ldots$	$\sigma_1(K_h)$
σ_2	$\sigma_2(K_1)$	$\sigma_2(K_2)$	$\ldots$	$\sigma_2(K_j)$	$\ldots$	$\sigma_2(K_h)$
$\vdots$	$\vdots$	$\vdots$				$\vdots$
σ_i	$\sigma_i(K_1)$	$\sigma_i(K_2)$	$\ldots$	$\sigma_i(K_j)$	$\ldots$	$\sigma_i(K_h)$
$\vdots$	$\vdots$	$\vdots$				$\vdots$
σ_h	$\sigma_h(K_1)$	$\sigma_h(K_2)$	$\ldots$	$\sigma_h(K_j)$	$\ldots$	$\sigma_h(K_h)$

Let us set $S = (\sigma_i(K_j))_{i,j=1}^h$, and call it the supercharacter table of G.

Recall that a class function on G is a function $f : G \longrightarrow \mathbb{C}$ which is constant on conjugacy classes of G. The set of all the class functions on G, $Cf(G)$ has the structure of a vector space over $\mathbb{C}$ with an orthonormal basis $Irr(G)$ with respect the inner product

$$\langle f, g\rangle = \frac{1}{|G|}\sum_{x\in G} f(x)\overline{g(x)}.$$

Since supercharacters are constant on superclasses, it is natural to call them superclass functions. We have:

$$\langle \sigma_i, \sigma_j\rangle = \frac{1}{|G|}\sum_{k=1}^{h}|K_k|\sigma_i(K_k)\overline{\sigma_j(K_k)}$$

But using the orthogonality of $Irr(G)$ we also can write:

$$\langle \sigma_i, \sigma_j\rangle = \langle \sum_{\chi\in X_i}\chi(1)\chi, \sum_{\varphi\in X_j}\varphi(1)\varphi\rangle = \delta_{ij}\sum_{\chi\in X_i}\chi(1)^2$$

Therefore,

$$\frac{1}{|G|}\sum_{k=1}^{h}|K_k|\sigma_i(K_k)\overline{\sigma_j(K_k)} = \delta_{ij}\sum_{\chi\in X_i}\chi(1)^2.$$

If we set the matrix

$$U = \frac{1}{\sqrt{|G|}}\left[\frac{\sigma_i(K_j)\sqrt{|K_j|}}{\sqrt{\sum_{\chi\in X_i}\chi(1)^2}}\right]_{i,j=1}^{h}.$$

We see that U is a unitary matrix with the following properties, which are proved in [3]: We have $U = U^t$, $U^2 = P$ where P is a permutation matrix and $U^4 = I$.

In the course of studying the supercharacter theory of a group G finding the supercharacter table of G and the matrix U is of great importance. In this paper, we will do this task for certain groups acting on certain sets.

3. Normal supercharacter theories for the dihedral group

Normal supercharacter theory for a finite group G was defined in [1]. Let G be a finite group, then $Norm(G)$ is denoted the set of all the normal subgroups of G. This set has the structure of a semigroup because the product of any two normal subgroups of G is again a normal subgroup of G. A certain subset of $Norm(G)$ forms a Lattice as defined below:

Definition 1. *Let $S \subseteq Norm(G)$. The smallest subsemigroup of $Norm(G)$ generated by S is denoted by $A(S)$ and has the following properties:*

(a) $\{1\}$, $G \in A(S)$;
(b) $S \subseteq A(S)$;
(c) $A(S)$ is closed under intersection.

It is clear that if $S = \{\{1\}, G\}$, then $A(S) = \{\{1\}, G\}$ satisfies (a), (b) and (c). Also $S = A(S) = Norm(G)$ satisfies (a), (b) and (c).

For $N \in A(S)$, we define

$$N^0 = N \setminus \bigcup_{H\subset N, H\in A(S)} H$$

It follows that $\{1\}^0 = \{1\}$.

For each $N \in A(S)$, we set

$$\mathcal{X}^N = \{\varphi \in Irr(G) \mid N \leq \ker\varphi\}$$

and $\chi^N = \sum_{\varphi\in\mathcal{X}^N}\varphi(1)\varphi$. We also set

$$\mathcal{X}^{N^*} = \mathcal{X}^N \setminus \bigcup_{N\subset K, K\in A(S)} \mathcal{X}^K.$$

Theorem 2. *[1] For an orbitrary $S \subseteq Norm(G)$,*

$$(\{\mathcal{X}^{N^*} \neq \emptyset \mid N \in A(S)\}, \{N^0 \neq \emptyset \mid N \in A(S)\})$$

is a supercharacter theory for G.

The supercharacter theory in above theorem is called the normal supercharacter theory generated by S. It is easy to check that if we choose $A(S) = \{\{1\}, G\}$. Then $\{1\}^0 = \{1\}$, $G^0 = G - \{1\}$ and we obtain the supercharacter theory

$$M(G) = (\{\{1_G\}, \rho_G - 1_G\}, \{\{1\}, G - \{1\}\})$$

where ρ_G denotes the regular character of G.

By [1] every sublattice of $Norm(G)$ containing $\{1\}$ and G yield a normal supercharacter theory of G.

The dihedral group is defined by generators and relations as follows:

$D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1}\rangle$. We have $|D_{2n}| = 2n$, $\langle a\rangle \trianglelefteq D_{2n}$, and every subgroup of $\langle a\rangle$ is normal in D_{2n}.

Also following lemma about normal subgroup of dihedral group is stated in Theorem 3.8 of [4], but we prefer this notations:

Lemma 3. *In D_{2n}, every subgroup of $\langle a\rangle$ is a normal subgroup. This describes all proper normal subgroups of D_{2n}, when n is odd, and the only additional proper normal subgroups when n is even are $\langle a^2, b\rangle$ and $\langle a^2, ab\rangle$, which are isomorphic to D_n.*

The character table of D_{2n} is well-known and is computed in [9].

Table 2: The character table of D_{2n}, n odd

	1	a^r $(1 \leq r \leq \frac{n-1}{2})$	b
χ_1	1	1	1
χ_2	1	1	-1
ψ_j $_{1\leq j\leq \frac{n-1}{2}}$	2	$\epsilon^{jr} + \epsilon^{-jr}$	0

Table 3: The character table of D_{2n}, n even

	1	$a^{\frac{n}{2}}$	a^r $(1 \leq r \leq \frac{n}{2} - 1)$	b	ab
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^{\frac{n}{2}}$	$(-1)^r$	1	-1
χ_4	1	$(-1)^{\frac{n}{2}}$	$(-1)^r$	-1	1
ψ_j $_{1\leq j\leq \frac{n}{2}-1}$	2	$2(-1)^j$	$\epsilon^{jr} + \epsilon^{-jr}$	0	0

$$\epsilon = e^{\frac{2\pi i}{n}}$$

To find normal supercharacters of D_{2n}, we need to know $Norm(D_{2n})$. If n is an odd prime p, then by Lemma 3 proper normal subgroups of D_{2n} are contained in $\langle a\rangle \cong \mathbb{Z}_p$. Therefore $A(S) = \{\{1\}, D_{2p}\}$ or $A(S) = \{\{1\}, \mathbb{Z}_p, D_{2p}\}$ are possible.

Proposition 4. *The group D_{2p} has only one non-trivial normal supercharacter theory.*

Proof. As we mentioned since by $A(S) = \{\{1\}, D_{2p}\}$ we obtain trivial supercharacter theory, then only

$$S = A(S) = \{\{1\}, \mathbb{Z}_p, D_{2p}\}$$

is possible. Using [1] we obtain:

$$\{1\}^0 = \{1\}, \mathbb{Z}_p^0 = \mathbb{Z}_p - \{1\}, D_{2p}^0 = D_{2p} - \mathbb{Z}_p,$$
$$\mathcal{X}^{D_{2p}} = \{\chi_1\} \Longrightarrow \mathcal{X}^{D_{2p}^*} = \{\chi_1\},$$
$$\mathcal{X}^{\mathbb{Z}_p} = \{\chi_1, \chi_2\} \Longrightarrow \mathcal{X}^{\mathbb{Z}_p^*} = \{\chi_2\},$$
$$\mathcal{X}^{\{1\}} = Irr(D_{2p}) \Longrightarrow \mathcal{X}^{\{1\}*} = Irr(D_{2p}) - \{\chi_1, \chi_2\}.$$
$$\sigma_1 = \chi_1, \sigma_2 = \chi_2, \sigma_3 = 2 \sum_{1 \le j \le \frac{p-1}{2}} \psi_j.$$

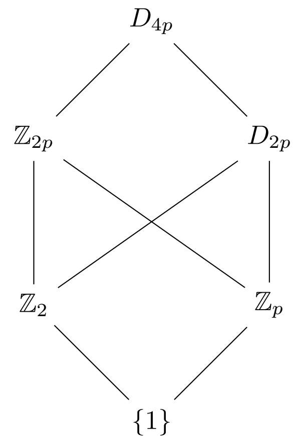
The corresponding supercharacter table is Table 4. □

Table 4: Table IV: Supercharacter table corresponding to S

	1	a	b
σ_1	1	1	1
σ_2	1	1	-1
σ_3	$2(p-1)$	-2	0

Proposition 5. *The dihedral group D_{4p}, p odd prime, has exactly 16 normal supercharacter theory.*

Proof. By Lemma 3 the lattice of normal subgroups of D_{4p} up to isomorphism is as follows:



Therefore we have the following possibilities for $S \subseteq Norm(D_{4p})$:

$S_1 = \{\{1\}, D_{4p}\}$ $\quad S_2 = \{\{1\}, D_{4p}, \mathbb{Z}_{2p}\}$
$S_3 = \{\{1\}, D_{4p}, \mathbb{Z}_p\}$ $\quad S_4 = \{\{1\}, D_{4p}, \mathbb{Z}_2\}$
$S_5 = \{\{1\}, D_{4p}, \mathbb{Z}_{2p}, \mathbb{Z}_p\}$ $\quad S_6 = \{\{1\}, D_{4p}, \mathbb{Z}_{2p}, \mathbb{Z}_2\}$
$S_7 = \{\{1\}, D_{4p}, \mathbb{Z}_{2p}, \mathbb{Z}_p, \mathbb{Z}_2\}$ $\quad S_8 = \{\{1\}, D_{4p}, \mathbb{Z}_p, \mathbb{Z}_2\}$
$S_9 = \{\{1\}, D_{4p}, D_{2p}\}$ $\quad S_{10} = \{\{1\}, D_{4p}, D_{2p}, \mathbb{Z}_2\}$
$S_{11} = \{\{1\}, D_{4p}, D_{2p}, \mathbb{Z}_p\}$ $\quad S_{12} = \{\{1\}, D_{4p}, D_{2p}, \mathbb{Z}_p, \mathbb{Z}_2\}$
$S_{13} = \{\{1\}, D_{4p}, \mathbb{Z}_{2p}, D_{2p}\}$ $\quad S_{14} = \{\{1\}, D_{4p}, D_{2p}, \mathbb{Z}_{2p}, \mathbb{Z}_2\}$
$S_{15} = \{\{1\}, D_{4p}, D_{2p}, \mathbb{Z}_{2p}, \mathbb{Z}_p\}$ $\quad S_{16} = \{\{1\}, D_{4p}, D_{2p}, \mathbb{Z}_{2p}, \mathbb{Z}_p, \mathbb{Z}_2\}$

Next we take one of the above sets, say S_5 and find the corresponding supercharacters and superclasses. In this case $A(S_5) = \{\{1\}, D_{4p}, \mathbb{Z}_{2p}, \mathbb{Z}_p\}$ where using the presentation of D_{4p} and table 3 we calculate:

$$\mathbb{Z}_{2p} = \langle a \rangle, \qquad \mathbb{Z}_p = \langle a^2 \rangle.$$
$$\{1\}^0 = \{1\}, \qquad \mathbb{Z}_p^0 = \mathbb{Z}_p - \{1\} = \{a^2, \ldots, a^{2p-2}\},$$
$$\mathbb{Z}_{2p}^0 = \mathbb{Z}_{2p} - (\mathbb{Z}_p \cup \{1\}) = \{a, a^3, \ldots, a^{2p-1}\}, \qquad D_{4p}^0 = \{a^i b \mid i = 0, 1, \ldots, 2p-1\}.$$

$$\begin{aligned}\mathcal{X}^{D_{4p}} &= \{\chi_1\} \Longrightarrow \mathcal{X}^{D_{4p}^*} = \{\chi_1\},\\ \mathcal{X}^{\mathbb{Z}_{2p}} &= \{\chi_1, \chi_2\} \Longrightarrow \mathcal{X}^{\mathbb{Z}_{2p}^*} = \{\chi_2\},\\ \mathcal{X}^{\mathbb{Z}_p} &= \{\chi_1, \chi_2, \chi_3, \chi_4\} \Longrightarrow \mathcal{X}^{\mathbb{Z}_p^*} = \{\chi_3, \chi_4\},\\ \mathcal{X}^{\{1\}} &= Irr(D_{4p}) \Longrightarrow \mathcal{X}^{\{1\}^*} = \{\psi_j \mid 1 \leq j \leq p-1\}.\end{aligned}$$

Hence: $\sigma_1 = \chi_1$, $\sigma_2 = \chi_2$, $\sigma_3 = \chi_3 + \chi_4$, $\sigma_4 = \sum_{j=1}^{p-1} 2\psi_j$. Therefore, the supercharacter table is Table 5. □

Table 5: The supercharacter table corresponding to S_5

	$\{1\}^0$	$\mathbb{Z}_p^0$	$\mathbb{Z}_{2p}^0$	D_{4p}^0
σ_1	1	1	1	1
σ_2	1	1	1	-1
σ_3	2	2	-2	0
σ_4	$4p-4$	-2	$-4p+4$	0

4. Automorphic supercharacter theory of the dihedral group

Also following lemma about automorphism group of D_{2n} is stated in Theorem 1.4 of [4], but we prefer this notations:

Lemma 6. *The automorphism group of D_{2n} is as follows:*

$$\mathbb{A}ut(D_{2n}) = \{f_{k,l} \mid f_{k,l}(a) = a^k, f_{k,l}(b) = a^l b, (k, n) = 1, 0 \leq l < n\}.$$

Therefore $\mathbb{A}ut(D_{2n}) \cong \mathbb{Z}_n \rtimes \cong es\mathbb{A}ut(\mathbb{Z}_n)$ is a group of order $n\varphi(n)$.

In this section we assume $n = p$ is an odd prime number and then use the Brauer theorem on character table to find some automorphic supercharacter theories for D_{2p}. Through this section we let $A = \mathbb{A}ut(D_{2p}) \cong \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$, $H = \{f_{1,l} \mid 0 \leq l < p\} \cong \mathbb{Z}_p$, $K = \{f_{k,0} \mid (k, p) = 1\} \cong \mathbb{Z}_{p-1}$.

Proposition 7. *D_{2p} has three A-invariant supercharacters and superclasses.*

Proof. By Table 2 we can see all classes of D_{2p} are invariant under A and has 3 orbits on $Con(D_{2p})$ and $Irr(D_{2p})$. □

Next we consider the dihedral group D_{4p}, where p is an odd prime. In this case

$$\begin{aligned}A = \mathbb{A}ut(D_{4p}) &\cong \mathbb{Z}_{2p} \rtimes \mathbb{Z}_{p-1}\\ &= \{f_{k,l} \mid f_{k,l}(a) = a^k, f_{k,l}(b) = a^l b, (k, 2p) = 1, 0 \leq l < 2p\},\\ H &= \{f_{1,l} \mid 0 \leq l < 2p\} \cong \mathbb{Z}_{2p},\\ K &= \{f_{k,0} \mid (k, 2p) = 1\} \cong \mathbb{Z}_{p-1}.\end{aligned}$$

Proposition 8. *D_{4p} has for A-variant supercharacters and superclasses.*

Proof. By table 3 and using elements of A we see that: Orbits of A on $Con(D_{4p})$ are $\{1\}$, $\{a^r \mid r \ \ odd\}$, $\{a^r \mid r \ \ even\}$, $\{a^i b \mid 1 \leq i < 2p\}$. Orbits of A on $Irr(D_{4p})$ are $\{\chi_1\}$, $\{\chi_2\}$, $\{\chi_3, \chi_4\}$, $\{\psi_j \mid 1 \leq j \leq p-1\}$. Therefore $\sigma_1 = \chi_1$, $\sigma_2 = \chi_2$, $\sigma_3 = \chi_3 + \chi_4$, $\sigma_4 = \sum_{j=1}^{p-1} 2\psi_j$. The corresponding supercharacter table is Table 6.

Table 6: Automorphic supercharacter table of D_{4p}

	$\{1\}$	$\{a^r \mid r \ \ odd\}$	$\{a^r \mid r \ \ even\}$	$\{a^i b \mid 1 \leq i < p\}$
σ_1	1	1	1	1
σ_2	1	1	1	-1
σ_3	2	-2	2	0
σ_4	$4p-4$	0	-1	0

□

Proposition 9. *H and K have $p+2$ and 8 orbits on $Con(D_{4p})$ respectively.*

Proof. Using table 3 and the action of H and K we have:
Orbits of H on $Con(D_{4p})$ are: $\{1\}$, $\{a^p\}$, $\{a^r \mid 1 \leq r \leq p-1\}$, $\{a^t b \mid 1 \leq t < 2p\}$.
Orbits of K on $Con(D_{4p})$ are: $\{1\}$, $\{a^p\}$, $\{b\}$, $\{a^p b\}$, $\{a^t \mid t \text{ is odd}\}$, $\{a^t \mid 0 \neq t \text{ is even}\}$, $\{a^t b \mid t \text{ is odd}\}$, $\{a^t b \mid 0 \neq t \text{ is even}\}$. □

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