AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 5(2) (2024) 111-116
https://doi.org/10.22060/AJMC.2022.21637.1093

Original Article

Approximate left ϕ-biprojectivity of θ-Lau product algebras

Salman Babayi[a], Mehdi Rostami[b], Mona Aj[c], Amir Sahami*[d]

[a] *Department of Mathematics, Faculty of Sciences, Urmia University, Urmia, Iran*
[b] *Department of Mathematics and Computer Science, Amirkabir University of Technology (Tehran Polytechnic), Iran*
[c] *Department of Mathematics, Farhangian University of Kermanshah, Kermanshah, Iran*
[d] *Department of Mathematics, Faculty of Basic Sciences Ilam University P.O. Box 69315-516 Ilam, Iran*

ABSTRACT: We continue [8] and we discuss approximately left ϕ-biprojectivity for θ-Lau product algebras. We give some Banach algebras among the category of θ-Lau product algebras which are not approximately left ϕ-biprojective. In fact, some class of matrix algebras under the notion of approximate left ϕ-biprojectivity is also discussed here.

Review History:

Received:28 July 2022
Revised:20 August 2022
Accepted:30 August 2022
Available Online:01 April 2024

Keywords:

Approximate left ϕ-biprojectivity
Approximate left ϕ-amenability
θ-Lau product

MSC (2020):

46M10; 43A07; 46H20

1. Introduction and Preliminaries

Helemskii studied the structure of Banach algebras by homological theory. There are two important notions in the homological theory, namely biflatness and biprojectivity. A Banach algebra A is called biprojective if there exists a bounded A-bimodule morphism $\rho : A \to A\widehat{\otimes}A$ such that $\pi_A \circ \rho(a) = a$, for all $a \in A$. Here $A\widehat{\otimes}A$ denotes the projective tensor product of A with A and $\pi_A : A\widehat{\otimes}A \to A$ is the product morphism which is given by $\pi_A(a\otimes b) = ab$ for all $a, b \in A$. For more information about homological Banach algebra's history see [6].

Zhang gave an approximate version of biprojectivity for Banach algebras. In fact A is approximately biprojective if there exists a net of A-bimodule morphism $\rho_\alpha : A \to A\widehat{\otimes}A$ such that $\pi_A \circ \rho_\alpha(a) \to a$ for all $a \in A$. He studied nilpotent ideals of Banach algebra using this notion, see [9].

Motivated by Zhang and Helemskii, Sahami and Pourabbas defined a notion of Banach homology with respect to a non-zero multiplicative linear functional. In fact for a non-zero multiplicative linear functional ϕ on A, the Banach algebras A is called approximate left ϕ-biprojective if there exists a net of bounded linear map $\rho_\alpha : A \longrightarrow A\widehat{\otimes}A$ such that

$$\rho_\alpha(ab) - a\cdot\rho_\alpha(b) \to 0, \quad \rho_\alpha(ab) - \phi(b)\rho_\alpha(a) \to 0 \quad \textit{and} \quad \phi\circ\pi_A\circ\rho_\alpha(a) - \phi(a) \to 0,$$

**Corresponding author.*
E-mail addresses: s.babayi@urmia.ac.ir, mross@aut.ac.ir, monaaj1373@yahoo.com, a.sahami@ilam.ac.ir

for all $a, b \in A$. They studied approximately left ϕ-biprojectivity of group algebras, Segal algebras and measure algebras over a locally compact group.

In this paper, We continue [8] and we discuss approximately left ϕ-biprojectivity for θ-Lau product algebras. The relations with its subalgebras also studied here. We give some Banach algebras among the category of θ-Lau product algebras which are not approximately left ϕ-biprojective.

We remind some definitions and notations which we need in this paper. For an arbitrary Banach algebra A, the character space is denoted by $\sigma(A)$ consists of all non-zero multiplicative linear functionals on A and any element of $\sigma(A)$ is called a character. The θ-Lau product was first introduced by Lau [4] for F-algebras. Monfared [5] introduced and investigated θ-Lau product space $A \times_\theta B$, for Banach algebras in general. Indeed for two Banach algebras A and B such that $\sigma(B) \neq \emptyset$ and θ be a non-zero character on B, the Cartesian product $A \times B$ by following multiplication and norm

$$(a, b)(a', b') = (aa' + \theta(b')a + \theta(b)a', bb'),$$

$$\|(a, b)\| = \|a\|_A + \|b\|_B,$$

is a Banach algebra, for all $a, a' \in A$ and $b, b' \in B$. The Cartesian product $A \times B$ with the above properties called the $\theta-$Lau product of A and B which is denoted by $A \times_\theta B$. From [5] we identify $A \times \{0\}$ with A, and $\{0\} \times B$ with B. Thus, it is clear that A is a closed two-sided ideal while B is a closed subalgebra of $A \times_\theta B$, and $(A \times_\theta B)/A$ is isometrically isomorphic to B. If $\theta = 0$, then we obtain the usual direct product of A and B. Since direct products often exhibit different properties, we have excluded the possibility that $\theta = 0$. Moreover, if $B = \mathbb{C}$, the complex numbers, and θ is the identity map on $\mathbb{C}$, then $A \times_\theta B$ is the unitization $A^\sharp$ of A. Note that, by [5, Proposition 2.4], the character space $\sigma(A \times_\theta B)$ of $A \times_\theta B$ is equal to

$$\{(\phi, \theta) : \ \phi \in \sigma(A)\} \bigcup \{(0, \psi) : \ \psi \in \sigma(B)\}.$$

Also, the dual space $(A \times_\theta B)^*$ of $A \times_\theta B$ is identified with $A^* \times B^*$ such that for each $(a, b) \in A \times_\theta B$, $\phi \in \sigma(A)$ and $\psi \in \sigma(B)$ we have

$$\langle (\phi, \psi), (a, b) \rangle = \phi(a) + \psi(b).$$

Now, suppose that A^{**}, B^{**} and $(A \times_\theta B)^{**}$ are equipped with their first Arens products. Then $(A \times_\theta B)^{**}$ is isometrically isomorphic with $A^{**} \times_\theta B^{**}$. Also, for all $(m, n), (p, q) \in (A \times_\theta B)^{**}$ the first Arens product is defined by

$$(m, n)\Box(p, q) = (m\Box p + n(\theta)p + q(\theta)m, n\Box q);$$

see [5, Proposition 2.12]. Note that every $\phi \in \sigma(A)$ has a unique extension to a character on A^{**} is given by $\tilde{\phi}$ where $\tilde{\phi}(m) = m(\phi)$, for all $m \in A^{**}$.

Note that A and B are closed two-sided ideal and closed subalgebra of $L := A \times_\theta B$, respectively. So, we can write $a = (a, 0)$ and $b = (0, b)$ for all $a \in A$ and $b \in B$. Therefore, $L = A \times_\theta B$ is a Banach A-bimodule and also is a Banach B-bimodule.

We recall that if X is a Banach A-bimodule, then with the following actions X^* is also a Banach A-bimodule:

$$a \cdot f(x) = f(x \cdot a), \quad f \cdot a(x) = f(a \cdot x) \qquad (a \in A, x \in X, f \in X^*).$$

The projective tensor product of A with A is denoted by $A \widehat{\otimes} A$. The Banach algebra $A \widehat{\otimes} A$ is a Banach A-bimodule with the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

Let $\phi \in \sigma(A)$. Then ϕ has a unique extension on A^{**} denoted by $\tilde{\phi}$ and defined by $\tilde{\phi}(F) = F(\phi)$ for every $F \in A^{**}$. Clearly this extension remains to be a character on A^{**}.

2. Approximate left ϕ-biprojectivity

Here $p_A : L \longrightarrow A$ and $p_B : L \longrightarrow B$ denote the usual projections defined by $p_A(a, b) = a$ and $p_B(a, b) = b$ for all $a \in A$ and $b \in B$. Let $q_A : A \longrightarrow L$ and $q_B : B \longrightarrow L$ be injections given by $q_A(a) = (a, 0)$ and $q_B(b) = (0, b)$. Thus for q_A and p_B we define

$$q_A \otimes q_A : A \widehat{\otimes} A \longrightarrow L \widehat{\otimes} L$$

and

$$p_B \otimes p_B : L \widehat{\otimes} L \longrightarrow B \widehat{\otimes} B$$

with

$$(q_A \otimes q_A)(a \otimes c) = (a, 0) \otimes (c, 0)$$

and

$$(p_B \otimes p_B)((a,b)\otimes(c,d)) = b\otimes d,$$

for all $a,c\in A$ and $b,d\in B$, respectively. One can show that q_A and $q_A\otimes q_A$ are A-bimodule morphisms and also p_B, q_B and $p_B\otimes p_B$ are B-bimodule morphisms.

For a unital Banach algebra A with unit e. Set $r_A : L \longrightarrow A$ and $S_B : B \longrightarrow L$ with $r_A(a,b) = a+\theta(b)e$ and $S_B(b) = (-\theta(b)e, b)$, respectively, for every $a\in A, b\in B$. Now

$$r_A\otimes r_A : L\widehat{\otimes}L \longrightarrow A\widehat{\otimes}A$$

and

$$S_B\otimes S_B : B\widehat{\otimes}B \longrightarrow L\widehat{\otimes}L$$

follows that

$$(r_A\otimes r_A)((a,b)\otimes(c,d)) = (a+\theta(b)e)\otimes(c+\theta(d)e)$$

and

$$(S_B\otimes S_B)(b\otimes d) = (-\theta(b)e,b)\otimes(-\theta(d)e,d),$$

respectively. Clearly r_A and $r_A\otimes r_A$ are A-bimodule morphism and S_B is a B-bimodule morphism.

Proposition 2.1. *Suppose that A and B are Banach algebras. Let A has a unit e. Also let $\phi\in\sigma(A)$ and $\theta\in\sigma(B)$. If L is approximately left (ϕ,θ)-biprojective. Then A is approximately left ϕ-biprojective.*

Proof. Let L be left (ϕ,θ)-biprojective. Then there exists a net of bounded linear maps $\rho_\alpha : L\longrightarrow L\widehat{\otimes}L$ such that

$$\rho_\alpha(xy) - x\cdot\rho_\alpha(y)\to 0 \quad \rho_\alpha(xy)-\phi(y)\rho_\alpha(x)\to 0,\quad \phi\circ\pi_L\circ\rho_\alpha(x)-\phi(x)\to 0,\quad (x,y\in L).$$

It is easy to see that

$$r_A\circ\pi_L = \pi_A\circ(r_A\otimes r_A),\qquad \phi\circ r_A = (\phi,\theta).$$

Now define $\eta_\alpha : A\longrightarrow A\widehat{\otimes}A$ by $\eta_\alpha = (r_A\otimes r_A)\circ\rho_\alpha\circ q_A$. Consider

$$\begin{aligned}\eta_\alpha(a_1a_2) - a_1\eta_\alpha(a_2) &= (r_A\otimes r_A)\circ\rho_\alpha\circ q_A(a_1a_2) - a_1\cdot(r_A\otimes r_A)\circ\rho_\alpha\circ q_A(a_2)\\ &= (r_A\otimes r_A)\circ\rho_\alpha\circ q_A(a_1a_2) - (r_A\otimes r_A)\circ\rho_\alpha(a_1\cdot q_A(a_2))\to 0\end{aligned}$$

and

$$\begin{aligned}\eta_\alpha(a_1a_2)-\phi(a_2)\eta_\alpha(a_1) &= (r_A\otimes r_A)\circ\rho_\alpha\circ q_A(a_1a_2) - \phi(a_2)(r_A\otimes r_A)\circ\rho_\alpha\circ q_A(a_1)\\ &= (r_A\otimes r_A)\circ\rho_\alpha(q_A(a_1)\cdot a_2) - \phi(a_2)(r_A\otimes r_A)\circ\rho_\alpha\circ q_A(a_1)\\ &= (r_A\otimes r_A)\circ\rho_\alpha(q_A(a_1)\cdot a_2) - (r_A\otimes r_A)(\phi(a_2)\rho_\alpha(q_A(a_1)))\to 0\end{aligned}$$

for every a_1 and a_2 in A. Also we have

$$\begin{aligned}\phi\circ\pi_A\circ\eta_\alpha(a)-\phi(a) &= \phi\circ\pi_A\circ(r_A\otimes r_A)\circ\rho_\alpha\circ q_A(a)-\phi(a)\\ &= (\phi\circ r_A\circ\pi_L\circ\rho_\alpha)(a,0)-\phi(a)\\ &= ((\phi,\theta)\circ\pi_L\circ\rho_\alpha)(a,0)-\phi(a)\to 0\end{aligned}$$

for all $a\in A$. So A is approximately left ϕ-biprojective. □

Proposition 2.2. *Suppose that A and B are Banach algebras and $\psi\in\sigma(B)$. If L is approximtely left $(0,\psi)$-biprojective, then B is approximaately left ψ-biprojective. Converse holds whenever A is unital.*

Proof. Since L is approximately left $(0,\psi)$-biprojective, there exists a net of bounded linear maps $\rho_L^\alpha : L\longrightarrow L\widehat{\otimes}L$ such that $(0,\psi)\circ\pi_L\circ\rho_L^\alpha - (0,\psi)\to 0$ and

$$\rho_L^\alpha(l_1l_2) - l_1\cdot\rho_L^\alpha(l_2)\to 0,\quad \rho_L^\alpha(l_1l_2)-(0,\psi)(l_2)\cdot\rho_L^\alpha(l_1)\to 0,\qquad (l_1,l_2\in L).$$

Set $\rho_B^\alpha : B\longrightarrow B\widehat{\otimes}B$ which is given by $\rho_B^\alpha = (p_B\otimes p_B)\circ\rho_L^\alpha\circ q_B$. It is easy to see that

$$\pi_B\circ(p_B\otimes p_B) = p_B\circ\pi_L,\qquad \psi\circ p_B = (0,\psi).$$

Now consider

$$\begin{aligned}\rho_B^\alpha(b_1b_2)-\psi(b_2)\rho_B^\alpha(a_1) &= (p_B\otimes p_B)\circ\rho_L^\alpha\circ q_B(b_1b_2) - \psi(b_2)(p_B\otimes p_B)\circ\rho_L^\alpha\circ q_B(a_1)\\ &= (p_B\otimes p_B)\circ\rho_L^\alpha(q_B(b_1)\cdot b_2) - \psi(b_2)(p_B\otimes p_B)\circ\rho_L^\alpha(q_B(b_1))\to 0\end{aligned}$$

and

$$\begin{aligned}\rho_B^\alpha(b_1b_2) - b_1\cdot\rho_B^\alpha(b_2) &= (p_{_B}\otimes p_{_B})\circ\rho_L^\alpha\circ q_{_B}(b_1b_2) - b_1\cdot(p_{_B}\otimes p_{_B})\circ\rho_L^\alpha\circ q_{_B}(b_2)\\ &= (p_{_B}\otimes p_{_B})\circ\rho_L^\alpha\circ q_{_B}(b_1b_2) - (p_{_B}\otimes p_{_B})\circ\rho_L^\alpha(b_1\cdot q_{_B}(b_2)) \to 0\end{aligned}$$

for every b_1 and b_2 in B. Also we have

$$\begin{aligned}\Big(\psi\circ\pi_B\circ\rho_B^\alpha\Big)(b) - \psi(b) &= \Big(\psi\circ\pi_B\circ(p_{_B}\otimes p_{_B})\rho_L^\alpha\circ q_{_B}\Big)(b) - \psi(b)\\ &= \Big(\psi\circ p_{_B}\circ\pi_L\circ\rho_L^\alpha\Big)(0,b) - \psi(b)\\ &= \Big((0,\psi)\circ\pi_L\circ\rho_L^\alpha\Big)(0,b) - \psi(b)\to 0,\end{aligned}$$

for each $b\in B$.

For converse, suppose that B is approximately left ψ-biprojective. Then there exists a net of bounded linear maps $\rho_B : B\longrightarrow B\widehat{\otimes}B$ such that

$$\rho_B^\alpha(ab) - a\cdot\rho_B^\alpha(b)\to 0, \rho_B^\alpha(ab) - \psi(b)\rho_B^\alpha(a)\to 0$$

and

$$\psi\circ\pi_B\circ\rho_B^\alpha(b) - \psi(b)\to 0$$

for each $a,b\in B$. Define $\rho_L^\alpha : L\longrightarrow L\widehat{\otimes}L$ by

$$\rho_L^\alpha(a,b) := (S_{_B}\otimes S_{_B})\circ\rho_B^\alpha(b),$$

for all $a\in A$ and $b\in B$. It is easy to see that

$$\pi_L\circ(S_{_B}\otimes S_{_B}) = S_{_B}\circ\pi_B,\qquad (0,\psi)\circ S_{_B} = \psi,\qquad ((S_{_B}\otimes S_{_B})\circ\lambda_B(b))\cdot x = 0,$$

for all $b\in B$ and $x\in A$. By these facts we can show that ρ_L^α is a net of bounded linear maps such that

$$\rho_L^\alpha(l_1l_2) - (0,\psi)(l_2)\rho_L^\alpha(l_1)\to 0,\quad \rho_L^\alpha(l_1l_2) - l_1\cdot\rho_L^\alpha(l_2)\to 0$$

for all $l_1,l_2\in L$. Also we have

$$(0,\psi)\circ\pi_L\circ\rho_L^\alpha(l) - (0,\psi)(l)\to 0,$$

for each $l\in L$. It follows that L is approximately left $(0,\psi)$-biprojective. □

Remark 2.3. *We show that approximately left (ϕ,θ)-biprojectivity of L implies that B is approximately left θ-biprojective. To see this, we know that there exists a net of bounded linear maps $\rho_L^\alpha : L\longrightarrow L\widehat{\otimes}L$ such that*

$$\rho_L^\alpha(ab) - a\cdot\rho_L^\alpha(b)\to 0,\quad \rho_L^\alpha(ab) - (\phi,\theta)(b)\rho_L^\alpha(a)\to 0$$

and

$$(\phi,\theta)\circ\pi_L\circ\rho_L(a) - (\phi,\theta)(a)\to 0,\qquad (a,b\in L).$$

Note that, we have

$$p_{_B}\circ\pi_L = \pi_B\circ(p_{_B}\otimes p_{_B}),\qquad r_A\circ\pi_L = \pi_A\circ(r_A\otimes r_A),\qquad \theta\circ p_{_B} = (0,\theta).$$

Define $\rho_B^\alpha : B\longrightarrow B\widehat{\otimes}B$ by $\rho_B^\alpha := (p_{_B}\otimes p_{_B})\circ\rho_L^\alpha\circ q_{_B}$. So by using

$$\Big((\phi,0)\circ\pi_L\circ\rho_L^\alpha\Big)(0,b)\to 0,$$

we have

$$\begin{aligned}\Big(\theta\circ\pi_B\circ\rho_B^\alpha\Big)(b) - \theta(b) &= \langle(\phi,\theta),(0,b)\rangle - \Big((\phi,0)\circ\pi_L\circ\rho_L^\alpha\Big)(0,b) - \theta(b)\\ &= \Big((\phi,0)\circ\pi_L\circ\rho_L^\alpha\Big)(0,b)\to 0,\end{aligned}$$

for every $b\in B$. Also we have

$$\rho_B^\alpha(b_1b_2) - b_1\cdot\rho_B^\alpha(b_2)\to 0,\quad \rho_B^\alpha(b_1b_2) - \theta(b_2)\rho_B(b_1)\to 0,\quad (b_1,b_2\in B).$$

It follows that B is approximately left θ-biprojective.

3. Applications and examples

Suppose that A is a Banach algebra and $\phi \in \sigma(A)$. We remind that a Banach algebra A is approximately left ϕ-amenable if there exists a net (m_α) in A such that $am_\alpha - \phi(a)m_\alpha \to 0$ and $\phi(m_\alpha) \to 1$ for all $a \in A$, see [1]. A Banach algebra A is called approximately left character amenable, if A is approximately left ϕ-amenable for all $\phi \in \sigma(A)$ and A posses a left approximate identity see [1].

Example 3.1. *We give a Lau product Banach algebra which is not approximately left ϕ-biprojective. To see this, suppose that $C^1[0,1]$ is the set of all differentiable functions which its derivation is continuous. With the point-wise multiplication and the sup-norm, $C^1[0,1]$ becomes a Banach algebra. It is well-known that $\sigma(C^1[0,1]) = \{\phi_t : t \in [0,1]\}$, where $\phi_t(f) = f(t)$ for all $t \in [0,1]$. We assume conversely that $C^1[0,1] \times_\theta C^1[0,1]$ is approximately left (ϕ_t, θ)-biprojective or approximatley left $(0, \phi_t)$-biprojective, where $\phi_t(f) = f(t)$ for each $t \in [0,1]$. It is easy to see that function 1 is an identity for $C^1[0,1]$. Using Proposition 2.1 and Proposition 2.2 follows that $C^1[0,1]$ is approximatley left ϕ_t-biprojective. So there exists a net of bounded linear map $\rho^\alpha_{C^1[0,1]} : C^1[0,1] \longrightarrow C^1[0,1]\hat{\otimes}C^1[0,1]$ such that*

$$\rho^\alpha_{C^1[0,1]}(fg) - f \cdot \rho^\alpha_{C^1[0,1]}(g) \to 0, \quad \rho^\alpha_{C^1[0,1]}(fg) - \phi_t(g)\rho^\alpha_{C^1[0,1]}(f) \to 0$$

and

$$\tilde{\phi_t} \circ \pi_{C^1[0,1]} \circ \rho^\alpha_{C^1[0,1]}(f) - \phi_t(f) \to 0$$

for all $f, g \in C[0,1]$. Define $m_\alpha = \pi_{C_{[0,1]}} \circ \rho^\alpha_{C^1[0,1]}(1) \in A$. Then

$$\begin{aligned} f \cdot m_\alpha - \phi_t(f)m_\alpha &= f \cdot \pi_{C_{[0,1]}} \circ \rho^\alpha_{C^1[0,1]}(1) - \phi_t(f)\pi_{C_{[0,1]}} \circ \rho^\alpha_{C^1[0,1]}(1) \\ &= \pi_{C_{[0,1]}}(f \cdot \rho^\alpha_{C^1[0,1]}(1)) - \pi_{C_{[0,1]}}(\phi_t(f)\rho^\alpha_{C^1[0,1]}(1)) \to 0 \end{aligned}$$

and

$$\phi_t(m_\alpha) - 1 = \phi_t \circ \pi_{C_{[0,1]}} \circ \rho^\alpha_{C^1[0,1]}(1) - 1 \to \phi(1) - 1 = 0,$$

for all $f \in C^1[0,1]$. It follows that $C^1[0,1]$ is approximately left ϕ_t-amenable which is impossible by similar arguments as in [3, Example 2.5].

The Banach algebra A is called approximately left character biprojective if A is approximately left ϕ-biprojective for each $\phi \in \sigma(A)$, respectively, see [8].

Proposition 3.1. *Suppose that G is a locally compact group and also $M(G)$ is the measure algebra with respect to G. Let $\theta \in \sigma(M(G))$. Then $M(G) \times_\theta M(G)$ is approximately left character biprojective if and only if G is discrete and amenable.*

Proof. Suppose that $M(G) \times_\theta M(G)$ is approximtely left character biprojective. Since $M(G)$ has an identity, Proposition 2.1 implies that $M(G)$ is approximately left ϕ-biprojective for all $\phi \in \sigma(M(G))$. Following the arguments of previous Example, gives that $M(G)$ is approximately character amenable. Now by [1, Theorem 7.2], G is discrete and amenable.
For converse, suppose that G is discrete and amenable. Then we have $M(G) = \ell^1(G)$. Thus by Johnson Theorem $\ell^1(G)$ is amenable. So [2, Corollary 2.1] finishes the proof. □

Example 3.2. *Let $A = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : \ a, b, c \in \mathbb{C} \right\}$ be a matrix algebra. With matrix operation and ℓ^1-norm A becomes a Banach algebra. Define $\phi : A \longrightarrow \mathbb{C}$ by*

$$\phi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = c.$$

It is easy to see that is a character on A. We claim that $A \times_\theta A$ is neither approximately (ϕ, θ)-biprojective nor is approximately left $(0, \phi)$-biprojective, where $\theta \in \sigma(A)$. Suppose conversely that $A \times_\theta A$ is approximately left (ϕ, θ)-biprojective or approximately left $(0, \phi)$-biprojective. Since A is unital, by Proposition 2.1 and Proposition 2.2 A is approximately left ϕ-biprojective. The existence of unit for A gives that A is approximately left ϕ-amenable. Define

$$J := \left\{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} : \ b, d \in \mathbb{C} \right\}$$

One can see that J is a closed ideal of A and $\phi_{|_J} \neq 0$. Since A is left ϕ-amenable, by [3, Lemma 3.1] we have that J is $\phi_{|_J}$-amenable. Now [7, Proposition 5.1] follows that, there exists a net (u_α) in J such that $ju_\alpha - \phi(j)u_\alpha \longrightarrow 0$ and $\phi(u_\alpha) \to 1$ for all $j \in J$. Set $j = \begin{pmatrix} 0 & j_1 \\ 0 & j_2 \end{pmatrix}$ and $u_\alpha = \begin{pmatrix} 0 & w_\alpha \\ 0 & v_\alpha \end{pmatrix}$, for some $j_1, j_2, w_\alpha, v_\alpha \in \mathbb{C}$. Thus,

$$ju_\alpha - \phi(j)u_\alpha = \begin{pmatrix} 0 & j_1 w_\alpha \\ 0 & j_2 v_\alpha \end{pmatrix} - \begin{pmatrix} 0 & j_2 w_\alpha \\ 0 & j_2 v_\alpha \end{pmatrix} \longrightarrow 0.$$

It gives that $j_1 v_\alpha - j_2 w_\alpha \longrightarrow 0$. If we put $j_1 = 1$ and $j_2 = 0$, then we have $v_\alpha \to 0$ which contradicts with $\phi(u_\alpha) = v_\alpha \to 1$.

Acknowledgements

The authors are grateful to the referees for their useful comments which improved the manuscript. The corresponding author is thankful to Ilam university, for it's support.

References

[1] H. P. Aghababa, L. Y. Shi, and Y. J. Wu, *Generalized notions of character amenability*, Acta Math. Sin. (Engl. Ser.), 29 (2013), pp. 1329–1350.

[2] M. Askari-Sayah, A. Pourabbas, and A. Sahami, *Johnson pseudo-contractibility and pseudo-amenability of θ-Lau product*, Kragujevac J. Math., 44 (2020), pp. 593–601.

[3] E. Kaniuth, A. T. Lau, and J. Pym, *On ϕ-amenability of Banach algebras*, Math. Proc. Cambridge Philos. Soc., 144 (2008), pp. 85–96.

[4] A. T. M. Lau, *Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups and semigroups*, Fund. Math., 118 (1983), pp. 161–175.

[5] M. S. Monfared, *On certain products of Banach algebras with applications to harmonic analysis*, Studia Math., 178 (2007), pp. 277–294.

[6] V. Runde, *Lectures on amenability*, vol. 1774 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 2002.

[7] A. Sahami, *On biflatness and ϕ-biflatness of some Banach algebras*, Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys., 80 (2018), pp. 111–122.

[8] A. Sahami and A. Pourabbas, *On approximate left ϕ-biprojective Banach algebras*, Glas. Mat. Ser. III, 53(73) (2018), pp. 187–203.

[9] Y. Zhang, *Nilpotent ideals in a class of Banach algebras*, Proc. Amer. Math. Soc., 127 (1999), pp. 3237–3242.

Please cite this article using:

Salman Babayi, Mehdi Rostami, Mona Aj, Amir Sahami, Approximate left ϕ-biprojectivity of θ-Lau product algebras, AUT J. Math. Comput., 5(2) (2024) 111-116
https://doi.org/10.22060/AJMC.2022.21637.1093