



Original Article

Approximate left ϕ -biprojectivity of θ -Lau product algebras

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ABSTRACT: We continue [8] and we discuss approximately left ϕ -biprojectivity for θ -Lau product algebras. We give some Banach algebras among the category of θ -Lau product algebras which are not approximately left ϕ -biprojective. In fact, some class of matrix algebras under the notion of approximate left ϕ -biprojectivity is also discussed here.

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1. Introduction and Preliminaries

Helemskii studied the structure of Banach algebras by homological theory. There are two important notions in the homological theory, namely biflatness and biprojectivity. A Banach algebra A is called biprojective if there exists a bounded A -bimodule morphism $\rho : A \rightarrow A \widehat{\otimes} A$ such that $\pi_A \circ \rho(a) = a$, for all $a \in A$. Here $A \widehat{\otimes} A$ denotes the projective tensor product of A with A and $\pi_A : A \widehat{\otimes} A \rightarrow A$ is the product morphism which is given by $\pi_A(a \otimes b) = ab$ for all $a, b \in A$. For more information about homological Banach algebra's history see [6].

Zhang gave an approximate version of biprojectivity for Banach algebras. In fact A is approximately biprojective if there exists a net of A -bimodule morphism $\rho_\alpha : A \rightarrow A \widehat{\otimes} A$ such that $\pi_A \circ \rho_\alpha(a) \rightarrow a$ for all $a \in A$. He studied nilpotent ideals of Banach algebra using this notion, see [9].

Motivated by Zhang and Helemskii, Sahami and Pourabbas defined a notion of Banach homology with respect to a non-zero multiplicative linear functional. In fact for a non-zero multiplicative linear functional ϕ on A , the Banach algebras A is called approximate left ϕ -biprojective if there exists a net of bounded linear map $\rho_\alpha : A \rightarrow A \widehat{\otimes} A$ such that

$$\rho_\alpha(ab) - a \cdot \rho_\alpha(b) \rightarrow 0, \quad \rho_\alpha(ab) - \phi(b)\rho_\alpha(a) \rightarrow 0 \quad \text{and} \quad \phi \circ \pi_A \circ \rho_\alpha(a) - \phi(a) \rightarrow 0,$$

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for all $a, b \in A$. They studied approximately left ϕ -biprojectivity of group algebras, Segal algebras and measure algebras over a locally compact group.

In this paper, We continue [8] and we discuss approximately left ϕ -biprojectivity for θ -Lau product algebras. The relations with its subalgebras also studied here. We give some Banach algebras among the category of θ -Lau product algebras which are not approximately left ϕ -biprojective.

We remind some definitions and notations which we need in this paper. For an arbitrary Banach algebra A , the character space is denoted by $\sigma(A)$ consists of all non-zero multiplicative linear functionals on A and any element of $\sigma(A)$ is called a character. The θ -Lau product was first introduced by Lau [4] for F-algebras. Monfared [5] introduced and investigated θ -Lau product space $A \times_{\theta} B$, for Banach algebras in general. Indeed for two Banach algebras A and B such that $\sigma(B) \neq \emptyset$ and θ be a non-zero character on B , the Cartesian product $A \times B$ by following multiplication and norm

$$(a, b)(a', b') = (aa' + \theta(b')a + \theta(b)a', bb'),$$

$$\|(a, b)\| = \|a\|_A + \|b\|_B,$$

is a Banach algebra, for all $a, a' \in A$ and $b, b' \in B$. The Cartesian product $A \times B$ with the above properties called the θ -Lau product of A and B which is denoted by $A \times_{\theta} B$. From [5] we identify $A \times \{0\}$ with A , and $\{0\} \times B$ with B . Thus, it is clear that A is a closed two-sided ideal while B is a closed subalgebra of $A \times_{\theta} B$, and $(A \times_{\theta} B)/A$ is isometrically isomorphic to B . If $\theta = 0$, then we obtain the usual direct product of A and B . Since direct products often exhibit different properties, we have excluded the possibility that $\theta = 0$. Moreover, if $B = \mathbb{C}$, the complex numbers, and θ is the identity map on \mathbb{C} , then $A \times_{\theta} B$ is the unitization $A^{\#}$ of A . Note that, by [5, Proposition 2.4], the character space $\sigma(A \times_{\theta} B)$ of $A \times_{\theta} B$ is equal to

$$\{(\phi, \theta) : \phi \in \sigma(A)\} \cup \{(0, \psi) : \psi \in \sigma(B)\}.$$

Also, the dual space $(A \times_{\theta} B)^*$ of $A \times_{\theta} B$ is identified with $A^* \times B^*$ such that for each $(a, b) \in A \times_{\theta} B$, $\phi \in \sigma(A)$ and $\psi \in \sigma(B)$ we have

$$\langle (\phi, \psi), (a, b) \rangle = \phi(a) + \psi(b).$$

Now, suppose that A^{**} , B^{**} and $(A \times_{\theta} B)^{**}$ are equipped with their first Arens products. Then $(A \times_{\theta} B)^{**}$ is isometrically isomorphic with $A^{**} \times_{\theta} B^{**}$. Also, for all $(m, n), (p, q) \in (A \times_{\theta} B)^{**}$ the first Arens product is defined by

$$(m, n) \square (p, q) = (m \square p + n(\theta)p + q(\theta)m, n \square q);$$

see [5, Proposition 2.12]. Note that every $\phi \in \sigma(A)$ has a unique extension to a character on A^{**} is given by $\tilde{\phi}$ where $\tilde{\phi}(m) = m(\phi)$, for all $m \in A^{**}$.

Note that A and B are closed two-sided ideal and closed subalgebra of $L := A \times_{\theta} B$, respectively. So, we can write $a = (a, 0)$ and $b = (0, b)$ for all $a \in A$ and $b \in B$. Therefore, $L = A \times_{\theta} B$ is a Banach A -bimodule and also is a Banach B -bimodule.

We recall that if X is a Banach A -bimodule, then with the following actions X^* is also a Banach A -bimodule:

$$a \cdot f(x) = f(x \cdot a), \quad f \cdot a(x) = f(a \cdot x) \quad (a \in A, x \in X, f \in X^*).$$

The projective tensor product of A with A is denoted by $A \widehat{\otimes} A$. The Banach algebra $A \widehat{\otimes} A$ is a Banach A -bimodule with the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

Let $\phi \in \sigma(A)$. Then ϕ has a unique extension on A^{**} denoted by $\tilde{\phi}$ and defined by $\tilde{\phi}(F) = F(\phi)$ for every $F \in A^{**}$. Clearly this extension remains to be a character on A^{**} .

2. Approximate left ϕ -biprojectivity

Here $p_A : L \rightarrow A$ and $p_B : L \rightarrow B$ denote the usual projections defined by $p_A(a, b) = a$ and $p_B(a, b) = b$ for all $a \in A$ and $b \in B$. Let $q_A : A \rightarrow L$ and $q_B : B \rightarrow L$ be injections given by $q_A(a) = (a, 0)$ and $q_B(b) = (0, b)$. Thus for q_A and p_B we define

$$q_A \otimes q_A : A \widehat{\otimes} A \rightarrow L \widehat{\otimes} L$$

and

$$p_B \otimes p_B : L \widehat{\otimes} L \rightarrow B \widehat{\otimes} B$$

with

$$(q_A \otimes q_A)(a \otimes c) = (a, 0) \otimes (c, 0)$$

and

$$(p_B \otimes p_B)((a, b) \otimes (c, d)) = b \otimes d,$$

for all $a, c \in A$ and $b, d \in B$, respectively. One can show that q_A and $q_A \otimes q_A$ are A -bimodule morphisms and also p_B, q_B and $p_B \otimes p_B$ are B -bimodule morphisms.

For a unital Banach algebra A with unit e . Set $r_A : L \rightarrow A$ and $S_B : B \rightarrow L$ with $r_A(a, b) = a + \theta(b)e$ and $S_B(b) = (-\theta(b)e, b)$, respectively, for every $a \in A, b \in B$. Now

$$r_A \otimes r_A : L \widehat{\otimes} L \rightarrow A \widehat{\otimes} A$$

and

$$S_B \otimes S_B : B \widehat{\otimes} B \rightarrow L \widehat{\otimes} L$$

follows that

$$(r_A \otimes r_A)((a, b) \otimes (c, d)) = (a + \theta(b)e) \otimes (c + \theta(d)e)$$

and

$$(S_B \otimes S_B)(b \otimes d) = (-\theta(b)e, b) \otimes (-\theta(d)e, d),$$

respectively. Clearly r_A and $r_A \otimes r_A$ are A -bimodule morphism and S_B is a B -bimodule morphism.

Proposition 2.1. *Suppose that A and B are Banach algebras. Let A has a unit e . Also let $\phi \in \sigma(A)$ and $\theta \in \sigma(B)$. If L is approximately left (ϕ, θ) -biprojective. Then A is approximately left ϕ -biprojective.*

Proof. Let L be left (ϕ, θ) -biprojective. Then there exists a net of bounded linear maps $\rho_\alpha : L \rightarrow L \widehat{\otimes} L$ such that

$$\rho_\alpha(xy) - x \cdot \rho_\alpha(y) \rightarrow 0 \quad \rho_\alpha(xy) - \phi(y)\rho_\alpha(x) \rightarrow 0, \quad \phi \circ \pi_L \circ \rho_\alpha(x) - \phi(x) \rightarrow 0, \quad (x, y \in L).$$

It is easy to see that

$$r_A \circ \pi_L = \pi_A \circ (r_A \otimes r_A), \quad \phi \circ r_A = (\phi, \theta).$$

Now define $\eta_\alpha : A \rightarrow A \widehat{\otimes} A$ by $\eta_\alpha = (r_A \otimes r_A) \circ \rho_\alpha \circ q_A$. Consider

$$\begin{aligned} \eta_\alpha(a_1 a_2) - a_1 \eta_\alpha(a_2) &= (r_A \otimes r_A) \circ \rho_\alpha \circ q_A(a_1 a_2) - a_1 \cdot (r_A \otimes r_A) \circ \rho_\alpha \circ q_A(a_2) \\ &= (r_A \otimes r_A) \circ \rho_\alpha \circ q_A(a_1 a_2) - (r_A \otimes r_A) \circ \rho_\alpha(a_1 \cdot q_A(a_2)) \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} \eta_\alpha(a_1 a_2) - \phi(a_2) \eta_\alpha(a_1) &= (r_A \otimes r_A) \circ \rho_\alpha \circ q_A(a_1 a_2) - \phi(a_2)(r_A \otimes r_A) \circ \rho_\alpha \circ q_A(a_1) \\ &= (r_A \otimes r_A) \circ \rho_\alpha(q_A(a_1) \cdot a_2) - \phi(a_2)(r_A \otimes r_A) \circ \rho_\alpha \circ q_A(a_1) \\ &= (r_A \otimes r_A) \circ \rho_\alpha(q_A(a_1) \cdot a_2) - (r_A \otimes r_A)(\phi(a_2)\rho_\alpha(q_A(a_1))) \rightarrow 0 \end{aligned}$$

for every a_1 and a_2 in A . Also we have

$$\begin{aligned} \phi \circ \pi_A \circ \eta_\alpha(a) - \phi(a) &= \phi \circ \pi_A \circ (r_A \otimes r_A) \circ \rho_\alpha \circ q_A(a) - \phi(a) \\ &= (\phi \circ r_A \circ \pi_L \circ \rho_\alpha)(a, 0) - \phi(a) \\ &= ((\phi, \theta) \circ \pi_L \circ \rho_\alpha)(a, 0) - \phi(a) \rightarrow 0 \end{aligned}$$

for all $a \in A$. So A is approximately left ϕ -biprojective. □

Proposition 2.2. *Suppose that A and B are Banach algebras and $\psi \in \sigma(B)$. If L is approximately left $(0, \psi)$ -biprojective, then B is approximately left ψ -biprojective. Converse holds whenever A is unital.*

Proof. Since L is approximately left $(0, \psi)$ -biprojective, there exists a net of bounded linear maps $\rho_L^\alpha : L \rightarrow L \widehat{\otimes} L$ such that $(0, \psi) \circ \pi_L \circ \rho_L^\alpha - (0, \psi) \rightarrow 0$ and

$$\rho_L^\alpha(l_1 l_2) - l_1 \cdot \rho_L^\alpha(l_2) \rightarrow 0, \quad \rho_L^\alpha(l_1 l_2) - (0, \psi)(l_2) \cdot \rho_L^\alpha(l_1) \rightarrow 0, \quad (l_1, l_2 \in L).$$

Set $\rho_B^\alpha : B \rightarrow B \widehat{\otimes} B$ which is given by $\rho_B^\alpha = (p_B \otimes p_B) \circ \rho_L^\alpha \circ q_B$. It is easy to see that

$$\pi_B \circ (p_B \otimes p_B) = p_B \circ \pi_L, \quad \psi \circ p_B = (0, \psi).$$

Now consider

$$\begin{aligned} \rho_B^\alpha(b_1 b_2) - \psi(b_2) \rho_B^\alpha(a_1) &= (p_B \otimes p_B) \circ \rho_L^\alpha \circ q_B(b_1 b_2) - \psi(b_2)(p_B \otimes p_B) \circ \rho_L^\alpha \circ q_B(a_1) \\ &= (p_B \otimes p_B) \circ \rho_L^\alpha(q_B(b_1) \cdot b_2) - \psi(b_2)(p_B \otimes p_B) \circ \rho_L^\alpha(q_B(b_1)) \rightarrow 0 \end{aligned}$$

and

$$\begin{aligned} \rho_B^\alpha(b_1b_2) - b_1 \cdot \rho_B^\alpha(b_2) &= (p_B \otimes p_B) \circ \rho_L^\alpha \circ q_B(b_1b_2) - b_1 \cdot (p_B \otimes p_B) \circ \rho_L^\alpha \circ q_B(b_2) \\ &= (p_B \otimes p_B) \circ \rho_L^\alpha \circ q_B(b_1b_2) - (p_B \otimes p_B) \circ \rho_L^\alpha(b_1 \cdot q_B(b_2)) \rightarrow 0 \end{aligned}$$

for every b_1 and b_2 in B . Also we have

$$\begin{aligned} (\psi \circ \pi_B \circ \rho_B^\alpha)(b) - \psi(b) &= (\psi \circ \pi_B \circ (p_B \otimes p_B) \rho_L^\alpha \circ q_B)(b) - \psi(b) \\ &= (\psi \circ p_B \circ \pi_L \circ \rho_L^\alpha)(0, b) - \psi(b) \\ &= ((0, \psi) \circ \pi_L \circ \rho_L^\alpha)(0, b) - \psi(b) \rightarrow 0, \end{aligned}$$

for each $b \in B$.

For converse, suppose that B is approximately left ψ -biprojective. Then there exists a net of bounded linear maps $\rho_B : B \rightarrow B \widehat{\otimes} B$ such that

$$\rho_B^\alpha(ab) - a \cdot \rho_B^\alpha(b) \rightarrow 0, \quad \rho_B^\alpha(ab) - \psi(b)\rho_B^\alpha(a) \rightarrow 0$$

and

$$\psi \circ \pi_B \circ \rho_B^\alpha(b) - \psi(b) \rightarrow 0$$

for each $a, b \in B$. Define $\rho_L^\alpha : L \rightarrow L \widehat{\otimes} L$ by

$$\rho_L^\alpha(a, b) := (S_B \otimes S_B) \circ \rho_B^\alpha(b),$$

for all $a \in A$ and $b \in B$. It is easy to see that

$$\pi_L \circ (S_B \otimes S_B) = S_B \circ \pi_B, \quad (0, \psi) \circ S_B = \psi, \quad ((S_B \otimes S_B) \circ \lambda_B(b)) \cdot x = 0,$$

for all $b \in B$ and $x \in A$. By these facts we can show that ρ_L^α is a net of bounded linear maps such that

$$\rho_L^\alpha(l_1l_2) - (0, \psi)(l_2)\rho_L^\alpha(l_1) \rightarrow 0, \quad \rho_L^\alpha(l_1l_2) - l_1 \cdot \rho_L^\alpha(l_2) \rightarrow 0$$

for all $l_1, l_2 \in L$. Also we have

$$(0, \psi) \circ \pi_L \circ \rho_L^\alpha(l) - (0, \psi)(l) \rightarrow 0,$$

for each $l \in L$. It follows that L is approximately left $(0, \psi)$ -biprojective. □

Remark 2.3. We show that approximately left (ϕ, θ) -biprojectivity of L implies that B is approximately left θ -biprojective. To see this, we know that there exists a net of bounded linear maps $\rho_L^\alpha : L \rightarrow L \widehat{\otimes} L$ such that

$$\rho_L^\alpha(ab) - a \cdot \rho_L^\alpha(b) \rightarrow 0, \quad \rho_L^\alpha(ab) - (\phi, \theta)(b)\rho_L^\alpha(a) \rightarrow 0$$

and

$$(\phi, \theta) \circ \pi_L \circ \rho_L^\alpha(a) - (\phi, \theta)(a) \rightarrow 0, \quad (a, b \in L).$$

Note that, we have

$$p_B \circ \pi_L = \pi_B \circ (p_B \otimes p_B), \quad r_A \circ \pi_L = \pi_A \circ (r_A \otimes r_A), \quad \theta \circ p_B = (0, \theta).$$

Define $\rho_B^\alpha : B \rightarrow B \widehat{\otimes} B$ by $\rho_B^\alpha := (p_B \otimes p_B) \circ \rho_L^\alpha \circ q_B$. So by using

$$((\phi, 0) \circ \pi_L \circ \rho_L^\alpha)(0, b) \rightarrow 0,$$

we have

$$\begin{aligned} (\theta \circ \pi_B \circ \rho_B^\alpha)(b) - \theta(b) &= \langle (\phi, \theta), (0, b) \rangle - ((\phi, 0) \circ \pi_L \circ \rho_L^\alpha)(0, b) - \theta(b) \\ &= ((\phi, 0) \circ \pi_L \circ \rho_L^\alpha)(0, b) \rightarrow 0, \end{aligned}$$

for every $b \in B$. Also we have

$$\rho_B^\alpha(b_1b_2) - b_1 \cdot \rho_B^\alpha(b_2) \rightarrow 0, \quad \rho_B^\alpha(b_1b_2) - \theta(b_2)\rho_B^\alpha(b_1) \rightarrow 0, \quad (b_1, b_2 \in B).$$

It follows that B is approximately left θ -biprojective.

3. Applications and examples

Suppose that A is a Banach algebra and $\phi \in \sigma(A)$. We remind that a Banach algebra A is approximately left ϕ -amenable if there exists a net (m_α) in A such that $am_\alpha - \phi(a)m_\alpha \rightarrow 0$ and $\phi(m_\alpha) \rightarrow 1$ for all $a \in A$, see [1]. A Banach algebra A is called approximately left character amenable, if A is approximately left ϕ -amenable for all $\phi \in \sigma(A)$ and A posses a left approximate identity see [1].

Example 3.1. We give a Lau product Banach algebra which is not approximately left ϕ -biprojective.

To see this, suppose that $C^1[0, 1]$ is the set of all differentiable functions which its derivation is continuous. With the point-wise multiplication and the sup-norm, $C^1[0, 1]$ becomes a Banach algebra. It is well-known that $\sigma(C^1[0, 1]) = \{\phi_t : t \in [0, 1]\}$, where $\phi_t(f) = f(t)$ for all $t \in [0, 1]$. We assume conversely that $C^1[0, 1] \times_\theta C^1[0, 1]$ is approximately left (ϕ_t, θ) -biprojective or approximatley left $(0, \phi_t)$ -biprojective, where $\phi_t(f) = f(t)$ for each $t \in [0, 1]$. It is easy to see that function 1 is an identity for $C^1[0, 1]$. Using Proposition 2.1 and Proposition 2.2 follows that $C^1[0, 1]$ is approximatley left ϕ_t -biprojective. So there exists a net of bounded linear map $\rho_{C^1[0,1]}^\alpha : C^1[0, 1] \rightarrow C^1[0, 1] \hat{\otimes} C^1[0, 1]$ such that

$$\rho_{C^1[0,1]}^\alpha(fg) - f \cdot \rho_{C^1[0,1]}^\alpha(g) \rightarrow 0, \quad \rho_{C^1[0,1]}^\alpha(fg) - \phi_t(g)\rho_{C^1[0,1]}^\alpha(f) \rightarrow 0$$

and

$$\tilde{\phi}_t \circ \pi_{C^1[0,1]} \circ \rho_{C^1[0,1]}^\alpha(f) - \phi_t(f) \rightarrow 0$$

for all $f, g \in C[0, 1]$. Define $m_\alpha = \pi_{C[0,1]} \circ \rho_{C^1[0,1]}^\alpha(1) \in A$. Then

$$\begin{aligned} f \cdot m_\alpha - \phi_t(f)m_\alpha &= f \cdot \pi_{C[0,1]} \circ \rho_{C^1[0,1]}^\alpha(1) - \phi_t(f)\pi_{C[0,1]} \circ \rho_{C^1[0,1]}^\alpha(1) \\ &= \pi_{C[0,1]}(f \cdot \rho_{C^1[0,1]}^\alpha(1)) - \pi_{C[0,1]}(\phi_t(f)\rho_{C^1[0,1]}^\alpha(1)) \rightarrow 0 \end{aligned}$$

and

$$\phi_t(m_\alpha) - 1 = \phi_t \circ \pi_{C[0,1]} \circ \rho_{C^1[0,1]}^\alpha(1) - 1 \rightarrow \phi(1) - 1 = 0,$$

for all $f \in C^1[0, 1]$. It follows that $C^1[0, 1]$ is approximately left ϕ_t -amenable which is impossible by similar arguments as in [3, Example 2.5].

The Banach algebra A is called approximately left character biprojective if A is approximately left ϕ -biprojective for each $\phi \in \sigma(A)$, respectively, see [8].

Proposition 3.1. Suppose that G is a locally compact group and also $M(G)$ is the measure algebra with respect to G . Let $\theta \in \sigma(M(G))$. Then $M(G) \times_\theta M(G)$ is approximately left character biprojective if and only if G is discrete and amenable.

Proof. Suppose that $M(G) \times_\theta M(G)$ is approximtely left character biprojective. Since $M(G)$ has an identity, Proposition 2.1 implies that $M(G)$ is approximately left ϕ -biprojective for all $\phi \in \sigma(M(G))$. Following the arguments of previous Example, gives that $M(G)$ is approximately character amenable. Now by [1, Theorem 7.2], G is discrete and amenable.

For converse, suppose that G is discrete and amenable. Then we have $M(G) = \ell^1(G)$. Thus by Johnson Theorem $\ell^1(G)$ is amenable. So [2, Corollary 2.1] finishes the proof. \square

Example 3.2. Let $A = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} : a, b, c \in \mathbb{C} \right\}$ be a matrix algebra. With matrix operation and ℓ^1 -norm A becomes a Banach algebra. Define $\phi : A \rightarrow \mathbb{C}$ by

$$\phi\left(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\right) = c.$$

It is easy to see that is a character on A . We claim that $A \times_\theta A$ is neither approximately (ϕ, θ) -biprojective nor is approximately left $(0, \phi)$ -biprojective, where $\theta \in \sigma(A)$. Suppose conversely that $A \times_\theta A$ is approximately left (ϕ, θ) -biprojective or approximately left $(0, \phi)$ -biprojective. Since A is unital, by Proposition 2.1 and Proposition 2.2 A is approximately left ϕ -biprojective. The existence of unit for A gives that A is approximately left ϕ -amenable. Define

$$J := \left\{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} : b, d \in \mathbb{C} \right\}$$

One can see that J is a closed ideal of A and $\phi|_J \neq 0$. Since A is left ϕ -amenable, by [3, Lemma 3.1] we have that J is $\phi|_J$ -amenable. Now [7, Proposition 5.1] follows that, there exists a net (u_α) in J such that $ju_\alpha - \phi(j)u_\alpha \rightarrow 0$ and $\phi(u_\alpha) \rightarrow 1$ for all $j \in J$. Set $j = \begin{pmatrix} 0 & j_1 \\ 0 & j_2 \end{pmatrix}$ and $u_\alpha = \begin{pmatrix} 0 & w_\alpha \\ 0 & v_\alpha \end{pmatrix}$, for some $j_1, j_2, w_\alpha, v_\alpha \in \mathbb{C}$. Thus,

$$ju_\alpha - \phi(j)u_\alpha = \begin{pmatrix} 0 & j_1w_\alpha \\ 0 & j_2v_\alpha \end{pmatrix} - \begin{pmatrix} 0 & j_2w_\alpha \\ 0 & j_2v_\alpha \end{pmatrix} \rightarrow 0.$$

It gives that $j_1v_\alpha - j_2w_\alpha \rightarrow 0$. If we put $j_1 = 1$ and $j_2 = 0$, then we have $v_\alpha \rightarrow 0$ which contradicts with $\phi(u_\alpha) = v_\alpha \rightarrow 1$.

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