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Original Article

A Bi-level formulation for a sequential stochastic attacker-defender game via conditional value at risk

Nader Biranvand*[a], Mohammad Moghaddas[a], Morteza Maleknia[a]

[a]*Imam Ali University, Faculty of sciences, Department of mathematics, Tehran, Iran*

ABSTRACT: In this study, we present a bi-level formulation for a sequential stochastic attacker-defender game with multiple targets. In this game, the vulnerability of targets is a stochastic parameter, and the attacker has only one attack type. The defender's aim is to find the optimal allocation of the budget to minimize the conditional value at risk of damage. In response to the defender's decisions, the attacker seeks an optimal allocation of its budget to maximize the expected damage. By using Karush-Kuhn-Tucker transformations, we reduce the proposed bi-level formulation to a single-level one. We also explore some important relationships between the solutions of the single-level and bi-level problems. Finally, by means of numerical experiments, we apply our formulation to several stochastic attacker-defender games to show the efficiency of our formulation in practice.

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1. Introduction

This paper aims to formulate a sequential stochastic attacker-defender game by a bi-level optimization problem. Defending a country, organization or system against strategic threats is a difficult problem. In particular, in the theory of risk management, one challenging issue is to find an optimal allocation of the defensive resources to minimize the damage caused by a strategic attacker. In this respect, a lot of effort has gone into the problem of attacker-defender. The key role of game theory in formulating attacker-defender problems is shown in [13]. Powell [11] studies the influence of the defense cost effectiveness on optimal allocation of the budget against non-strategic and strategic attackers. The positive consequence of predicting the attacker's strategies against multiple targets is investigated in [21] . In [17], different targets are simulated by using the network theory. A multi-period attacker-defender game with multiple targets is investigated in [14]. Zhang and Zhuang [19] formulates an attacker-defender game with multiple attack types. A game in which the defender protects multiple targets against a group of attackers has been simulated in [6]. Phillips [10] formulates an attacker-defender game by the techniques that are commonly used in portfolio optimization. In [11], natural disasters, as non-strategic components, are added to an

**Corresponding author.*
E-mail addresses: nabiranvand@gmail.com, mo_moghaddas@yahoo.com, m.maleknia@aut.ac.ir

attacker-defender game. A discrete simultaneous game to protect crucial infrastructures against strategic threats is presented in [18]. Zhang et al. [20] presents a game to secure the borders between two neighboring countries. To reduce the vulnerability of the power systems, a tri-level game is provided in [9]. A zero-sum pursuit-evasion differential game can be found in [15]. Some Heuristic algorithms for solving a game which is formulated by a bi-level optimal control problem is studied in [16].

This study aims to present a bi-level formulation for a sequential attacker-defender game. In this game, both the defender and the attacker are strategic. In the proposed model, there are multiple targets and the attacker has only one attack type. Moreover, the vulnerability of a specific target is a stochastic parameter. This feature makes this study different from the most existing research that has appeared in the literature. The main aim of the defender is to find an optimal allocation of the limited budget to minimize the risk of damage. For this purpose, the defender employs the **C**onditional **V**alue at **R**isk (**CVaR**) to measure the risk of damage. On the other hand, the main goal of the attacker is to maximize the damage. To this end, based on the defender's decisions, the attacker seeks an allocation of its limited budget which maximizes the expected damage.

To find a solution for the proposed bi-level optimization problem, we employ **KKT** (**K**arush-**K**uhn-**T**ucker) transformations to reduce the bi-level problem to a single-level one [5]. Furthermore, we explore the relations between the solutions of the bi-level and single-level problems. More precisely, we show that every global solution to the single-level problem is also a global solution to the bi-level optimization problem. We also discover a relationship between local solutions of bi-level and single-level problems. Using the **KKT** transformations, we obtain a single-level optimization problem containing complementarity constraints. To deal with these constraints, we employ a perturbed variant of the Fischer-Burmeister function. Through numerical experiments, we apply the proposed formulation to some examples of stochastic attacker-defender games and report the most important results.

This paper is organized as follows. In Section 2, we provide some mathematical preliminaries. Section 3 presents our bi-level formulation for the stochastic attacker-defender game. In Section 4, we reduce the bi-level problem to a single-level one, and some relations between local and global solutions are provided. Numerical experiments are presented in Section 5, and Section 6 concludes the paper.

2. Mathematical Preliminaries

In this paper, the n-dimensional Euclidean space is denoted by $\mathcal{R}^n$ and, for $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$, the inner product is given by $\mathbf{x}^T\mathbf{y} := \sum_{i=1}^n x_i y_i$. Moreover, $\|\mathbf{x}\| := \sqrt{\mathbf{x}^T\mathbf{x}}$. Vector $\mathbf{1} \in \mathcal{R}^n$ is the vector of ones, i.e., $\mathbf{1}^T := (1, 1, \dots, 1)$. For any $i \in \{1, \dots, n\}$, $\mathbf{e}_i \in \mathbb{R}^n$ is a vector with a 1 in the i-th coordinate and zeros elsewhere. Furthermore, the element-wise product of vectors $\mathbf{x}, \mathbf{y} \in \mathcal{R}^n$ is denoted by

$$\mathbf{x} \odot \mathbf{y} := (x_1y_1, x_2y_2, \dots, x_ny_n)^T.$$

In our model formulation, the vulnerability of each target is a random parameter. Thus, we need to employ some risk measures to provide a bi-level formulation for the considered stochastic attacker-defender game. In particular, in this paper, mathematical expectation and **C**onditional **V**alue at **R**isk (**CVaR**) are of especial importance.

The mathematical expectation or mean value of the random variable $\mathbf{X}$ is denoted by $\mathbf{E}(\mathbf{X})$. In case the random variable $\mathbf{X}$ is continuous, $\mathbf{E}(\mathbf{X})$ is given by [7]

$$\mathbf{E}(\mathbf{X}) := \int_{-\infty}^{\infty} x\mathbf{F}(x)dx,$$

and for the discrete case

$$\mathbf{E}(\mathbf{X}) := \sum_x x\mathbf{P}(\mathbf{X} = x),$$

in which $\mathbf{F}(x)$ and $\mathbf{P}(\mathbf{X} = x)$ are the **P**robability **D**ensity **F**unction (**PDF**) and **P**robability **M**ass **F**unction (**PMF**) of $\mathbf{X}$, respectively.

The conditional value at risk with confidence level $1 - \alpha$ of the random variable $\mathbf{X}$ is defined as follows [8]

$$\mathrm{CVaR}_{1-\alpha}(\mathbf{X}) := \inf_{t \in \mathcal{R}} \left\{ t + \frac{1}{\alpha}\mathbf{E}(\mathbf{X} - t)^+ \right\},$$

in which $\alpha \in (0, 1]$ and $(\mathbf{X} - t)^+ := \max\{0, \mathbf{X} - t\}$. It is recalled that the mathematical expectation and conditional value at risk are coherent risk measures [8].

Next, we concisely review the general framework of a bi-level optimization problem. A bi-level program is a mathematical program which contains an optimization problem as a constraint. The main problem is known as the upper-level problem (leader) and the nested one is known as the lower-level problem (follower). Each level has its own decision vector, namely upper-level and lower-level decision vectors. For any given leader's decision vector, the follower provides an optimal response. In other words, the lower-level optimization problem is a parametric one. Let $\mathbf{x}_u \in \mathcal{R}^n$ and $\mathbf{x}_l \in \mathcal{R}^m$ be the decision vectors of the upper-level and lower-level problems, respectively. Then, if we denote the upper-level objective function by $F_u : \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}$ and the lower-level objective function by $F_l : \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}$, one can formulate a bi-level program as follows

$$\begin{aligned} &\min_{(\mathbf{x}_u,\mathbf{x}_l)\in\mathcal{R}^n\times\mathrm{R}^m} \quad F_u(\mathbf{x}_u,\mathbf{x}_l) \\ &\text{s.t.} \;\; C_i(\mathbf{x}_u,\mathbf{x}_l) \le 0, \quad i=1,\dots,I, \\ &\mathbf{x}_l \in \arg\min_{\mathbf{x}_l} \{F_l(\mathbf{x}_u,\mathbf{x}_l) \;:\; c_k(\mathbf{x}_u,\mathbf{x}_l) \le 0, \quad k=1,\dots,K\}, \end{aligned} \tag{1}$$

in which $C_i : \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}, i = 1,\dots,I$ and $c_k : \mathcal{R}^n \times \mathcal{R}^m \to \mathcal{R}, k = 1,\dots,K$ denote the constraints of the upper-level and lower-level optimization problems, respectively. Let us denote the feasible set of the upper-level problem by

$$\mathbf{\Omega} := \{(\mathbf{x}_u,\mathbf{x}_l) \in \mathcal{R}^n \times \mathcal{R}^m \;:\; C_i(\mathbf{x}_u,\mathbf{x}_l) \le 0, \quad i=1,\dots,I\},$$

and define the set-valued map $\mathbf{S} : \mathcal{R}^n \rightrightarrows \mathcal{R}^m$ as

$$\mathbf{S}(\mathbf{x}_u) := \arg\min_{\mathbf{x}_l} \{F_l(\mathbf{x}_u,\mathbf{x}_l) \;:\; c_k(\mathbf{x}_u,\mathbf{x}_l) \le 0, \quad k=1,\dots,K\},$$

which maps any upper-level decision vector to the solution set of the lower-level problem. Now, the bi-level problem (1) is represented as follows

$$\begin{aligned} &\min_{(\mathbf{x}_u,\mathbf{x}_l)\in\mathcal{R}^n\times\mathcal{R}^m} \quad F_u(\mathbf{x}_u,\mathbf{x}_l) \\ &\text{s.t.} \;\; (\mathbf{x}_u,\mathbf{x}_l) \in \mathbf{\Omega}, \\ &\qquad \mathbf{x}_l \in \mathbf{S}(\mathbf{x}_u). \end{aligned} \tag{2}$$

Definition 2.1. *The feasible point $(\mathbf{x}_u^*,\mathbf{x}_l^*)$ is a local optimal solution for bi-level problem (2), if there is $\epsilon > 0$ such that for every $(\mathbf{x}_u,\mathbf{x}_l) \in \mathbf{\Omega} \cap \mathrm{graph}(\mathbf{S})$ with $\|(\mathbf{x}_u,\mathbf{x}_l) - (\mathbf{x}_u^*,\mathbf{x}_l^*)\| \le \epsilon$ one has*

$$F_u(\mathbf{x}_u^*,\mathbf{x}_l^*) \le F_u(\mathbf{x}_u,\mathbf{x}_l),$$

where $\mathrm{graph}(\mathbf{S}) := \{(\mathbf{x}_u,\mathbf{x}_l) \;:\; \mathbf{x}_l \in \mathbf{S}(\mathbf{x}_u)\}$ is the graph of the map $\mathbf{S}$. Also, $(\mathbf{x}_u^*,\mathbf{x}_l^*)$ is a global optimal solution, if ϵ can be chosen arbitrary large.

3. The Proposed Bi-level Formulation

In this section, a bi-level program for a stochastic attacker-defender game is provided. For this purpose, we first mention the assumptions and notations that are necessary to formulate the game.

In this attacker-defender game, we have $n \in \mathbb{N}$ targets which are indexed by $j \in \{1,\dots,n\}$. The vulnerability of the target j is a discrete random parameter which is denoted by V_j. Also, the random parameters $V_j, j = 1,\dots,n$, are independent. Accordingly, the vulnerability random vector $\mathbf{V}$ is defined by $\mathbf{V} := (V_1,V_2,\dots,V_n)$ and its expectation is given by

$$\mathbf{E}(\mathbf{V}) := (\mathbf{E}(V_1),\mathbf{E}(V_2),\dots,\mathbf{E}(V_n)).$$

The defender has the limited budget $B_D > 0$, and $\mathbf{x} = (x_1,x_2,\dots,x_n) \in \mathcal{R}^n$ is the decision vector of the defender, where $x_j \ge 0$ is a part of the budget allocated to target j in order to defend this target. In our formulation, the effect of the defensive resource on each target is not the same and, for each $j \in \{1,\dots,n\}$, it is given by the following exponential function [19]

$$D_j(x) := \exp(-\zeta_j x),$$

where $\zeta_j > 0$ is the effectiveness parameter of the defense cost for target j. Obviously, the larger the parameter ζ_j, the more effective the defensive cost x. Accordingly, for any defender's decision vector $\mathbf{x} \in \mathbb{R}^n$, we define

$$\mathbf{D}(\mathbf{x}) := (D_1(x),D_2(x),\dots,D_n(x)) \in \mathcal{R}^n.$$

The attacker has the limited budget $B_A > 0$ with the decision vector $\mathbf{y} = (y_1,\dots,y_n) \in \mathcal{R}^n$, where $y_j \ge 0$ is the resource allocated to attack target j.

3.1. The Upper-level Problem

In the proposed formulation, the defender plays its role as the upper-level optimization problem. The main goal of the defender is to find an optimal allocation of the limited budget $B_D > 0$ in such a way that the conditional value at risk of the damage is minimized. Therefore, if we denote the defender's objective function by $F_u : \mathcal{R}^n \times \mathcal{R}^n \to \mathcal{R}$, our upper-level optimization problem is formulated as follows

$$\begin{aligned} \min_{(\mathbf{x},\mathbf{y})} \;& F_u(\mathbf{x},\mathbf{y}) = \mathrm{CVaR}_{1-\alpha}\left(\sum_{j=1}^{n} y_j D_j(x_j) V_j\right) \\ \text{s.t.} \quad & \mathbf{1}^T\mathbf{x} = B_D, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{3}$$

For convenience in formulation and calculations, for any upper-level decision vector $\mathbf{x} \in \mathcal{R}^n$, we define $\mathbf{z} \in \mathcal{R}^n$ by

$$\mathbf{z} := \mathbf{y} \odot \mathbf{D}(\mathbf{x}).$$

Then, one can rewrite $F_u(\mathbf{x},\mathbf{y})$ as

$$F_u(\mathbf{x},\mathbf{y}) = \mathrm{CVaR}_{1-\alpha}\left(\sum_{j=1}^{n} y_j D_j(x_j) V_j\right) = \mathrm{CVaR}_{1-\alpha}\left(\mathbf{z}^T\mathbf{V}\right). \tag{4}$$

To compute $\mathrm{CVaR}_{1-\alpha}\left(\mathbf{z}^T\mathbf{V}\right)$, assume that the discrete random variables $V_1,\dots,V_n$ take $r \in \mathcal{N}$ distinct values. Then, the joint distribution of $V_1,\dots,V_n$ takes r^n different scenarios, namely $\mathbf{s}_1,\dots,\mathbf{s}_{r^n}$ with probabilities $p_1,\dots,p_{r^n}$. Now, we have

$$\begin{aligned} \mathrm{CVaR}_{1-\alpha}\left(\mathbf{z}^T\mathbf{V}\right) &= \inf_{t\in\mathcal{R}} \left\{ t + \frac{1}{\alpha}\mathbf{E}(\mathbf{z}^T\mathbf{V} - t)^+ \right\} \\ &= \inf_{t\in\mathcal{R}} \left\{ t + \frac{1}{\alpha}\left(\sum_{l=1}^{r^n} p_l[\mathbf{z}^T\mathbf{s}_l - t]^+\right) \right\}. \end{aligned} \tag{5}$$

Thus, in view of (5), the upper-level problem (3) is represented as

$$\begin{aligned} \min_{(\mathbf{x},\mathbf{y},\mathbf{z},t)} \;& t + \frac{1}{\alpha}\left(\sum_{l=1}^{r^n} p_l[\mathbf{z}^T\mathbf{s}_l - t]^+\right) \\ \text{s.t.} \;& \mathbf{z} = \mathbf{y} \odot \mathbf{D}(\mathbf{x}), \\ & \mathbf{1}^T\mathbf{x} = B_D, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{6}$$

Since problem (6) is not a differentiable minimization problem, we consider its smoothen form [4]. To this end, let $\mathbf{h} := (h_1,\dots,h_{r^n}) \in \mathcal{R}^{r^n}$, then the smoothen form of problem (6) is given by

$$\begin{aligned} \min_{(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{h},t)} \;& t + \frac{1}{\alpha}\sum_{l=1}^{r^n} p_l h_l \\ \text{s.t.} \;& \mathbf{z} = \mathbf{y} \odot \mathbf{D}(\mathbf{x}), \\ & \mathbf{1}^T\mathbf{x} = B_D, \\ & \mathbf{z}^T\mathbf{s}_l - t \leq h_l, \quad l = 1,\dots,r^n, \\ & \mathbf{h} \geq \mathbf{0}, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{7}$$

3.2. The Lower-level Problem

After observing the defender's decision vector $\mathbf{x} \in \mathbb{R}^n$, the attacker tries to find an optimal allocation of limited budget $B_A > 0$ in such a way that the expected damage is maximized. In this respect, the lower-level problem is given as

$$\begin{aligned} \max_{\mathbf{y}} \;\; & F_l(\mathbf{x},\mathbf{y}) = \mathbf{E}\left(\sum_{j=1}^{n} y_j D_j(x_j) V_j\right) \\ \text{s.t.} \;\; & \mathbf{1}^T\mathbf{y} = B_A, \\ & \mathbf{y} \geq \mathbf{0}. \end{aligned} \tag{8}$$

Taking $\mathbf{z} = \mathbf{y} \odot \mathbf{D}(\mathbf{x})$ into account, the above problem is rewritten as

$$\begin{aligned} \max_{(\mathbf{y},\mathbf{z})} \;\; & \mathbf{z}^T\mathbf{E}(\mathbf{V}) \\ \text{s.t.} \;\; & \mathbf{z} = \mathbf{y} \odot \mathbf{D}(\mathbf{x}), \\ & \mathbf{1}^T\mathbf{y} = B_A, \\ & \mathbf{y} \geq \mathbf{0}. \end{aligned} \tag{9}$$

Clearly, the above problem is a linear optimization problem in $(\mathbf{y},\mathbf{z})$.

3.3. The Bi-level Problem

For any upper-level decision vector $\mathbf{x} \in \mathcal{R}^n$, the solution set of problem (9) is denoted by $\mathbf{S}(\mathbf{x})$, i.e.,

$$\mathbf{S}(\mathbf{x}) := \arg\max_{(\mathbf{z},\mathbf{y})} \left\{ \mathbf{z}^T\mathbf{E}(\mathbf{V}) : \mathbf{1}^T\mathbf{y} = B_A,\; \mathbf{z} = \mathbf{y} \odot \mathbf{D}(\mathbf{x}),\; \mathbf{y} \geq \mathbf{0} \right\}.$$

Eventually, our bi-level formulation for the stochastic attacker-defender game can be given by

$$\begin{aligned} \min_{(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{h},t)} \;\; & t + \frac{1}{\alpha}\sum_{l=1}^{r^n} p_l h_l \\ \text{s.t.} \;\; & \mathbf{1}^T\mathbf{x} = B_D, \\ & \mathbf{z}^T\mathbf{s}_l - t \leq h_l, \quad l = 1,\ldots,r^n, \\ & (\mathbf{z},\mathbf{y}) \in \mathbf{S}(\mathbf{x}), \\ & \mathbf{h} \geq \mathbf{0}, \\ & \mathbf{x} \geq \mathbf{0}. \end{aligned} \tag{10}$$

It is emphasized that the mean vector $\mathbf{E}(\mathbf{V})$ and different scenarios $\mathbf{s}_l$ with probabilities $p_l, l = 1,\ldots,r^n$, are the inputs of problem (10). In addition, $\mathbf{x},\mathbf{h},t$ are the decision vectors of the defender's problem, and $\mathbf{y},\mathbf{z}$ are the decision vectors of the attacker's problem.

4. Karush-Kuhn-Tucker Transformation

As the lower-level problem (9) is linear and regular, the **K**arush-**K**uhn-**T**ucker (**KKT**) conditions are necessary and sufficient optimality conditions for this problem [3]. Therefore, we make use of these optimality conditions to convert the bi-level optimization problem (10) to a single-level one. Before preceding any further, in the following remark we show that, for any upper-level decision vector $\mathbf{x} \in \mathcal{R}^n$, the lower-level problem is regular.

Remark 4.1. *For any upper-level decision vector $\mathbf{x} \in \mathcal{R}^n$, the* Abadie constraint qualification *[2] is satisfied for lower-level problem (9), as all constraints in the lower-level problem are linear.*

The **KKT** optimality conditions for lower-level problem (9) are as follows

$$\begin{aligned} & \nabla_{(\mathbf{z},\mathbf{y})}\mathcal{L}(\mathbf{z},\mathbf{y},\mathbf{x},\mu,\boldsymbol{\mu},\boldsymbol{\lambda}) = \mathbf{0}, \\ & \boldsymbol{\lambda}^T\mathbf{y} = 0, \quad \boldsymbol{\lambda} \geq \mathbf{0}, \\ & \mathbf{1}^T\mathbf{y} = B_A, \quad \mathbf{z} = \mathbf{y} \odot \mathbf{D}(\mathbf{x}), \quad \mathbf{y} \geq \mathbf{0}, \end{aligned} \tag{11}$$

where $\boldsymbol{\mu},\boldsymbol{\lambda} \in \mathcal{R}^n$, $\mu \in \mathcal{R}$, and

$$\mathcal{L}(\mathbf{z},\mathbf{y},\mathbf{x},\mu,\boldsymbol{\mu},\boldsymbol{\lambda}) = -\mathbf{z}^T\mathbf{E}(\mathbf{V}) + \boldsymbol{\mu}^T\left(\mathbf{z} - \mathbf{y} \odot \mathbf{D}(\mathbf{x})\right) + \mu\left(\mathbf{1}^T\mathbf{y} - B_A\right) - \boldsymbol{\lambda}^T\mathbf{y},$$

denotes the Lagrange function for the lower-level problem (9). Next, we replace the lower-level problem (9) by the optimality conditions (11), and consequently the bi-level problem (10) turns to the following single-level optimization problem:

$$\begin{aligned}
&\min_{(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{h},t,\mu,\boldsymbol{\mu},\boldsymbol{\lambda})} \; t+\frac{1}{\alpha}\sum_{l=1}^{r^n} p_l h_l \\
&\text{s.t.} \quad \mathbf{1}^T\mathbf{x}=B_D,\\
&\qquad \mathbf{z}^T\mathbf{s}_l - t \le h_l, \quad l=1,\dots,r^n,\\
&\qquad \nabla_{(\mathbf{z},\mathbf{y})}\mathcal{L}(\mathbf{z},\mathbf{y},\mathbf{x},\mu,\boldsymbol{\mu},\boldsymbol{\lambda})=\mathbf{0},\\
&\qquad \boldsymbol{\lambda}^T\mathbf{y}=0,\\
&\qquad \mathbf{1}^T\mathbf{y}=B_A,\\
&\qquad \mathbf{z}=\mathbf{y}\odot\mathbf{D}(\mathbf{x}),\\
&\qquad \mathbf{x}\ge\mathbf{0}, \quad \boldsymbol{\lambda}\ge\mathbf{0}, \quad \mathbf{y}\ge\mathbf{0}, \quad \mathbf{h}\ge\mathbf{0}.
\end{aligned} \tag{12}$$

In the rest of this section, we explore some relations between the solutions of bi-level problem (10) and single-level problem (12).

Theorem 4.2. *Suppose that* $(\mathbf{x}^*,\mathbf{y}^*,\mathbf{z}^*,\mathbf{h}^*,t^*,\mu^*,\boldsymbol{\mu}^*,\boldsymbol{\lambda}^*)$ *is a global optimal solution for problem* (12). *Then,* $(\mathbf{x}^*,\mathbf{y}^*,\mathbf{z}^*,\mathbf{h}^*,t^*)$ *is a global optimal optimal solution of bi-level optimization problem* (10).

Proof. Let us denote the objective function of bi-level problem (10) by

$$f_u(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{h},t).$$

By indirect proof, suppose $(\mathbf{x}^*,\mathbf{y}^*,\mathbf{z}^*,\mathbf{h}^*,t^*)$ is not a global optimal solution for bi-level problem (10). Thus, there is a point $(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{h},t)$ such that

$$\begin{aligned}
&\mathbf{1}^T\mathbf{x}=B_D,\\
&\mathbf{z}^T\mathbf{s}_l - t \le h_l, \quad l=1,\dots,r^n,\\
&(\mathbf{z},\mathbf{y})\in\mathbf{S}(\mathbf{x}),\\
&\mathbf{x}\ge\mathbf{0}, \quad \mathbf{h}\ge\mathbf{0},
\end{aligned}$$

with

$$f_u(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{h},t) < f_u(\mathbf{x}^*,\mathbf{y}^*,\mathbf{z}^*,\mathbf{h}^*,t^*). \tag{13}$$

In view of $(\mathbf{y},\mathbf{z})\in\mathbf{S}(\mathbf{x})$ along with the fact that lower-level problem (9) is regular for any upper-level decision vector $\mathbf{x}\in\mathcal{R}^n$ (see Remark 4.1), the **KKT** optimality conditions guarantee the existence of $\boldsymbol{\mu}$, $\boldsymbol{\lambda}\in\mathcal{R}^n$ and $\mu\in\mathcal{R}$ such that

$$\begin{aligned}
&\nabla_{(\mathbf{z},\mathbf{y})}\mathcal{L}(\mathbf{z},\mathbf{y},\mathbf{x},\mu,\boldsymbol{\mu},\boldsymbol{\lambda})=\mathbf{0},\\
&\boldsymbol{\lambda}^T\mathbf{y}=0,\\
&\mathbf{1}^T\mathbf{y}=B_A,\\
&\mathbf{z}=\mathbf{y}\odot\mathbf{D}(\mathbf{x}),\\
&\boldsymbol{\lambda}\ge\mathbf{0}, \quad \mathbf{y}\ge\mathbf{0},
\end{aligned}$$

meaning that $(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{h},t,\mu,\boldsymbol{\mu},\boldsymbol{\lambda})$ is feasible for single-level problem (12). This fact along with (13) implies that $(\mathbf{x}^*,\mathbf{y}^*,\mathbf{z}^*,\mathbf{h}^*,t^*\mu^*,\boldsymbol{\mu}^*,\boldsymbol{\lambda}^*)$ is not a global solution to problem (12), which violates the assumption. $\square$

Theorem (4.2) relates the global solution of the single-level problem to a global solution of the bi-level problem. However, locating a global minimizer of single-level problem (12) is cumbersome, as it is nonconvex. Fortunately, the next theorem provides an interesting relation between the local solutions of problems (12) and (10). Before it, we need a crucial auxiliary result, which is provided in the following lemma.

Lemma 4.3. *For any upper-level decision vector* $\mathbf{x}\in\mathcal{R}^n$, *let* $(\mathbf{z},\mathbf{y})\in\mathbf{S}(\mathbf{x})$. *If the set of Lagrange multipliers of lower-level problem* (9) *at* $(\mathbf{z},\mathbf{y})$ *is defined by*

$$\mathcal{M}_{\mathbf{x}}(\mathbf{z},\mathbf{y}) := \left\{(\mu,\boldsymbol{\mu},\boldsymbol{\lambda}) \,:\, \nabla_{(\mathbf{z},\mathbf{y})}\mathcal{L}(\mathbf{z},\mathbf{y},\mathbf{x},\mu,\boldsymbol{\mu},\boldsymbol{\lambda})=\mathbf{0},\ \boldsymbol{\lambda}^T\mathbf{y}=0,\ \boldsymbol{\lambda}\ge\mathbf{0}\right\},$$

then $\mathcal{M}_{\mathbf{x}}(\mathbf{z},\mathbf{y})$ *is a singleton set.*

Proof. By Remark 4.1, the lower-level problem is regular for any upper-level decision vector $\mathbf{x} \in \mathcal{R}^n$. Thus, since $(\mathbf{z}, \mathbf{y}) \in \mathbf{S}(\mathbf{x})$, one can conclude $\mathcal{M}_{\mathbf{x}}(\mathbf{z}, \mathbf{y})$ is nonempty, i.e., there is $(\mu, \boldsymbol{\mu}, \boldsymbol{\lambda})$ such that

$$\nabla \mathcal{L}_{(\mathbf{z},\mathbf{y})} = \begin{bmatrix} -\mathbf{E}(\mathbf{V}) + \boldsymbol{\mu} \\ \mu \mathbf{1} - \boldsymbol{\mu} \odot \mathbf{D}(\mathbf{x}) - \boldsymbol{\lambda} \end{bmatrix} = \mathbf{0}, \quad \boldsymbol{\lambda}^T \mathbf{y} = 0, \quad \boldsymbol{\lambda} \geq \mathbf{0}. \tag{14}$$

The first equation of system (14) yields $\mathbf{E}(\mathbf{V}) = \boldsymbol{\mu}$, and hence this system can be represented by

$$\begin{bmatrix} \mathbf{1} & -\mathbf{e}_1 \; -\mathbf{e}_2 \; \dots - \mathbf{e}_n \\ 0 & \mathbf{y}^T \end{bmatrix} \begin{bmatrix} \mu \\ \boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{E}(\mathbf{V}) \odot \mathbf{D}(\mathbf{x}) \\ 0 \end{bmatrix}, \quad \boldsymbol{\lambda} \geq \mathbf{0}. \tag{15}$$

Let us denote

$$\mathcal{A} := \begin{bmatrix} \mathbf{1} & -\mathbf{e}_1 \; -\mathbf{e}_2 \; \dots - \mathbf{e}_n \\ 0 & \mathbf{y}^T \end{bmatrix} \in \mathcal{R}^{(n+1)\times(n+1)}.$$

It is verified that matrix $\mathcal{A}$ is row equivalent to

$$\mathcal{A}' := \begin{bmatrix} \mathbf{e}_1 & -\mathbf{e}_1 + \mathbf{e}_2 & -\mathbf{e}_2 + \mathbf{e}_3 & \dots & -\mathbf{e}_{n-1} + \mathbf{e}_n & -\mathbf{e}_n \\ 0 & 0 & 0 & \dots & 0 & \mathbf{1}^T \mathbf{y} \end{bmatrix},$$

in which $\mathbf{1}^T \mathbf{y} > 0$. Clearly, $\mathcal{A}'$ is an upper triangular matrix with nonzero diagonal entries. Consequently, $\mathcal{A}$ is a nonsingular matrix, and system (14) has the unique solution $(\mu, \boldsymbol{\mu}, \boldsymbol{\lambda})$. □

Theorem 4.4. *Let $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \mathbf{h}^*, t^*, \mu^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ be a local optimal solution for single-level problem (12) with*

$$(\mu^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathcal{M}_{\mathbf{x}^*}(\mathbf{z}^*, \mathbf{y}^*).$$

Then, $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, h^*, t^*)$ is a local optimal solution for bi-level problem (10).

Proof. As in the proof of Theorem 4.2, we denote the objective function of the bi-level problem (10) by f_u. By contradiction, suppose $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \mathbf{h}^*, t^*)$ is not a local solution for bi-level problem (10). Thus, one can find the convergent sequence

$$\{(\mathbf{x}_\nu, \mathbf{y}_\nu, \mathbf{z}_\nu, \mathbf{h}_\nu, t_\nu)\}_\nu \to (\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \mathbf{h}^*, t^*),$$

such that, for all ν, it satisfies

$$\begin{aligned} &\mathbf{1}^T \mathbf{x}_\nu = B_D, \\ &\mathbf{z}_\nu^T \mathbf{s}_l - t_\nu \leq h_l^\nu, \quad l = 1, \dots, r^n, \\ &(\mathbf{z}_\nu, \mathbf{y}_\nu) \in \mathbf{S}(\mathbf{x}_\nu), \\ &\mathbf{h}_\nu \geq \mathbf{0}, \quad \mathbf{x}_\nu \geq \mathbf{0}, \end{aligned}$$

with

$$f_u(\mathbf{x}_\nu, \mathbf{y}_\nu, \mathbf{z}_\nu, \mathbf{h}_\nu, t_\nu) < f_u(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \mathbf{h}^*, t^*), \quad \text{for all}\, \nu. \tag{16}$$

As $(\mathbf{z}_\nu, \mathbf{y}_\nu) \in \mathbf{S}(\mathbf{x}_\nu)$ along with the fact that the lower-level problem is regular for any upper-level decision vector $\mathbf{x}_\nu$, the **KKT** optimality conditions ensure the existence of the sequence $\{(\mu_\nu, \boldsymbol{\mu}_\nu, \boldsymbol{\lambda}_\nu)\}_\nu$ with

$$(\mu_\nu, \boldsymbol{\mu}_\nu, \boldsymbol{\lambda}_\nu) \in \mathcal{M}_{\mathbf{x}_\nu}(\mathbf{z}_\nu, \mathbf{y}_\nu).$$

Consequently, for any ν, the point $(\mathbf{x}_\nu, \mathbf{y}_\nu, \mathbf{z}_\nu, \mathbf{h}_\nu, t_\nu, \mu_\nu, \boldsymbol{\mu}_\nu, \boldsymbol{\lambda}_\nu)$ is feasible for single-level problem (12). By the upper semicontinuity of the map $\mathcal{M}_{\mathbf{x}}(\mathbf{z}, \mathbf{y})$ [12], the sequence $\{(\mu_\nu, \boldsymbol{\mu}_\nu, \boldsymbol{\lambda}_\nu)\}_\nu$ has the accumulation point

$$(\bar{\mu}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}) \in \mathcal{M}_{\mathbf{x}^*}(\mathbf{z}^*, \mathbf{y}^*).$$

Now, by Lemma 4.3, $\mathcal{M}_{\mathbf{x}^*}(\mathbf{z}^*, \mathbf{y}^*)$ is singleton, and hence

$$(\bar{\mu}, \bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\lambda}}) = (\mu^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*).$$

Therefore, we obtain the feasible point $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \mathbf{h}^*, t^*, \mu^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ for single-level problem (12) such that

$$f_u(\mathbf{x}_\nu, \mathbf{y}_\nu, \mathbf{z}_\nu, \mathbf{h}_\nu, t_\nu) < f_u(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \mathbf{h}^*, t^*), \quad \text{for all}\, \nu.$$

This means that the point $(\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \mathbf{h}^*, t^*, \mu^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$ with $(\mu^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*) \in \mathcal{M}_{\mathbf{x}^*}(\mathbf{z}^*, \mathbf{y}^*)$ is not a local solution for single-level problem (12), which is a contradiction. □

4.1. Solving the Single-level Optimization Problem

In light of Theorems 4.2 and 4.4, one can find a solution for bi-level problem (10) by solving single-level problem (12). However, due to the complementarity constraint

$$\boldsymbol{\lambda}^T \mathbf{y} = 0, \quad \boldsymbol{\lambda} \geq \mathbf{0}, \ \ \mathbf{y} \geq \mathbf{0}, \tag{17}$$

obtaining a global or local solution for problem (10) is not an easy task. In fact, because of the complementarity condition (17), our single-level problem is indeed a mixed-integer optimization problem, in which non of the regularity conditions that are commonly used in smooth optimization holds. To resolve this issue, we employ the Fischer-Burmeister function $\mathcal{FB}_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$, which is given by [1]

$$\mathcal{FB}_\varepsilon(a, b) := \sqrt{a^2 + b^2 + 2\varepsilon} - a - b.$$

For any $\varepsilon > 0$, $\mathcal{FB}_\varepsilon(a, b)$ is continuously differentiable in (a, b), and $\mathcal{FB}_\varepsilon(a, b) = 0$ iff $a > 0, b > 0$ and $ab = \varepsilon$. In this regard, we approximate the constraint (17) by using the following constraints

$$\mathcal{FB}_\varepsilon(\lambda_j, y_j) = \sqrt{\lambda_j^2 + y_j^2 + 2\varepsilon} - \lambda_j - y_j = 0, \quad j = 1, \ldots, n.$$

Eventually, one can approximate single-level problem (12) by

$$\begin{aligned}
\min_{(\mathbf{x},\mathbf{y},\mathbf{z},\mathbf{h},t,\mu,\boldsymbol{\mu},\boldsymbol{\lambda})} \quad & t + \frac{1}{\alpha}\sum_{l=1}^{r^n} p_l h_l \\
\text{s.t.} \quad & \mathbf{1}^T \mathbf{x} = B_D, \\
& \mathbf{z}^T \mathbf{s}_l - t \leq h_l, \quad l = 1, \ldots, r^n, \\
& \nabla_{(\mathbf{z},\mathbf{y})} \mathcal{L}(\mathbf{z}, \mathbf{y}, \mathbf{x}, \mu, \boldsymbol{\mu}, \boldsymbol{\lambda}) = \mathbf{0}, \\
& \sqrt{\lambda_j^2 + y_j^2 + 2\varepsilon} - \lambda_j - y_j = 0, \quad j = 1, \ldots, n, \\
& \mathbf{1}^T \mathbf{y} = B_A, \\
& \mathbf{z} = \mathbf{y} \odot \mathbf{D}(\mathbf{x}), \\
& \mathbf{x} \geq \mathbf{0}, \quad \mathbf{h} \geq \mathbf{0}.
\end{aligned} \tag{18}$$

The following theorem from [1] provides a relationship between the solutions of problems (18) and (12). Before it, we need to state the following definition.

Definition 4.5. *Let $\boldsymbol{\xi} := (\mathbf{x}, \mathbf{y}, \mathbf{z}, \mathbf{h}, t, \mu, \boldsymbol{\mu}, \boldsymbol{\lambda})$ be a feasible point for problem (18). The point $\boldsymbol{\xi}$ is a regular point for problem (18) provided that the gradients of equality constraints and vanishing inequality constraints are linearly independent at this point.*

Theorem 4.6. *For a given sequence $\{\varepsilon_\nu\}_\nu$ with $\varepsilon_\nu \downarrow 0$, assume*

$$\boldsymbol{\xi}_\nu := (\mathbf{x}_\nu, \mathbf{y}_\nu, \mathbf{z}_\nu, \mathbf{h}_\nu, t_\nu, \mu_\nu, \boldsymbol{\mu}_\nu, \boldsymbol{\lambda}_\nu)$$

fulfills the second-order optimality conditions of problem (18), for $\varepsilon := \varepsilon_\nu$. If the sequence $\{\boldsymbol{\xi}_\nu\}_\nu$ converges to the regular point $\boldsymbol{\xi}^*$ as $\varepsilon_\nu \downarrow 0$, then $\boldsymbol{\xi}^*$ is a Bouligand stationary solution for problem (12).

Proof. Please see [1] for a comprehensive proof. □

5. Numerical Experiments

In this section, we consider some numerical instances of the stochastic attacker-defender game (10), and present the obtained results. Moreover, we analyze the sensitivity of the proposed formulation to parameters B_D, B_A, and $\zeta_j, j = 1, \ldots, n$. The following experiments have been implemented in MATLAB software.

In virtue of Theorem 4.6, to compute a (local) solution of bi-level problem (10), we need to solve a sequence of single-level optimization problems for a decreasing sequence $\varepsilon_\nu \downarrow 0$. For any $\varepsilon_\nu > 0$, let

$$\boldsymbol{\xi}_\nu := (\mathbf{x}_\nu, \mathbf{y}_\nu, \mathbf{z}_\nu, \mathbf{h}_\nu, t_\nu, \mu_\nu, \boldsymbol{\mu}_\nu, \boldsymbol{\lambda}_\nu)$$

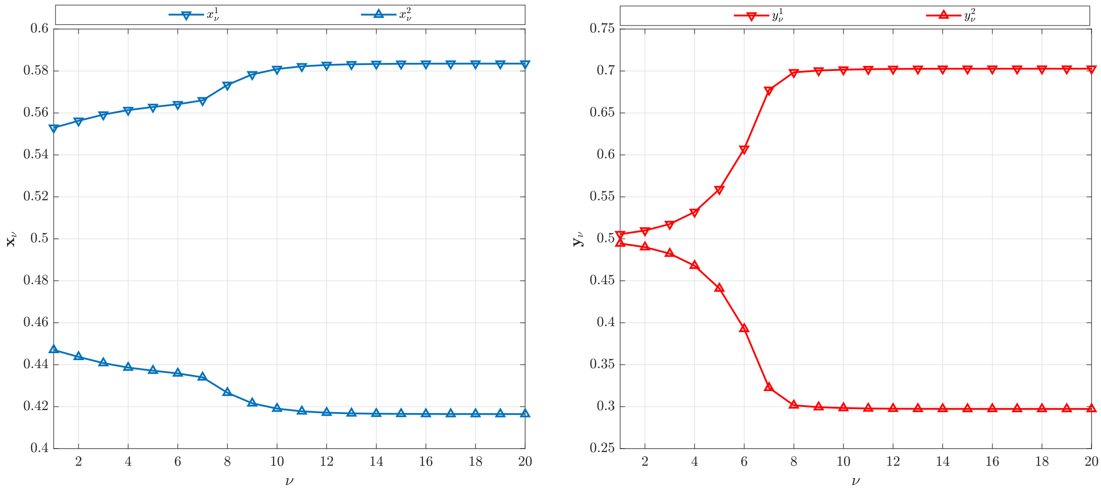

Figure 1: Left: Convergence of the defender's decision vector $\mathbf{x}_\nu$ as $\varepsilon_\nu \downarrow 0$. Right: Convergence of the attacker's decision vector $\mathbf{y}_\nu$ as $\varepsilon_\nu \downarrow 0$.

be a (local) solution of single-level problem (18) with $\varepsilon := \varepsilon_\nu$. Once

$$\|\boldsymbol{\xi}_\nu - \boldsymbol{\xi}_{\nu-1}\| \leq 10^{-6},$$

for some $\nu > 0$, we consider

$$\boldsymbol{\xi}_\nu =: \boldsymbol{\xi}^* = (\mathbf{x}^*, \mathbf{y}^*, \mathbf{z}^*, \mathbf{h}^*, t^*, \mu^*, \boldsymbol{\mu}^*, \boldsymbol{\lambda}^*)$$

as a (local) solution of single-level problem (12), which provides a (local) solution of bi-level problem (10). The sequence ε_ν is defined by $\varepsilon_{\nu+1} := 0.5\varepsilon_\nu$ with $\varepsilon_1 := 1$. For a given $\varepsilon_\nu > 0$, the optimization problem (18) is solved by using the `fmincon` solver. This solver employs an interior-point algorithm with the following options. The constraint and optimality tolerance are set to 10^{-6}, and the step tolerance is 10^{-10}. Moreover, the maximum number of function evaluations and iterations are limited to 3×10^3 and 10^3, respectively.

To provide a starting point for this solver, we employed some heuristic methods, and the best result was used as an initial guess.

5.1. Experiment 1.

Our first example is a two-dimensional instance of bi-level problem (10) with $\zeta_1 = \zeta_2 = 1, B_D = B_A = 1$, and $\alpha = 0.3$. The independent stochastic parameters V_1 and V_2 take two distinct values with mathematical expectation $\mathbf{E}(\mathbf{V}) = (\mathbf{E}(V_1), \mathbf{E}(V_2)) = (1.375, 1.100)$. Moreover, the joint distribution of V_1 and V_2 is given in Table 1.

$\mathbf{s}_l$	$(1, 1)$	$(1, 2)$	$(1.5, 1)$	$(1.5, 2)$
p_l	$9/40$	$1/40$	$27/40$	$3/40$

Table 1: Joint distribution of V_1 and V_2

One can observe from Figure 1 that $\mathbf{x}_\nu \to \mathbf{x}^* = (0.5835, 0.4165)$ and $\mathbf{y}_\nu \to \mathbf{y}^* = (0.7027, 0.2973)$ as $\varepsilon_\nu \downarrow 0$. In other words, the attacker invest almost 70% of its budget on target $j = 1$, where $\mathbf{E}(V_1) > \mathbf{E}(V_2)$.

Let

$$f_u(\nu) := t_\nu + \frac{1}{\alpha}\sum_{l=1}^{4} p_l h_l^\nu \quad \text{and} \quad f_l(\nu) := \mathbf{z}_\nu^T \mathbf{E}(\mathbf{V}),$$

denote the value of the objective function of upper-level and lower-level problems at solution $\boldsymbol{\xi}_\nu$, respectively. As seen from Figure 2, $f_u(\nu) \to f_u^* = 0.8331$ and $f_l(\nu) \to f_l^* = 0.7253$ as $\varepsilon_\nu \downarrow 0$. Indeed, the maximum expected damage that can be achieved by the attacker is $f_l^* = 0.7253$. This value has been obtained as a consequence of reducing the conditional value at risk of the damage to $f_u^* = 0.8331$. As one would expect, if the defender works with a larger confidence level, then the conditional value at risk of the damage increases. In general, $\mathrm{CVaR}_{1-\alpha}(\cdot)$ is decreasing with respect to α, and for a given random variable $\mathbf{X}$

$$\mathbf{E}(\mathbf{X}) = \mathrm{CVaR}_0(\mathbf{X}).$$

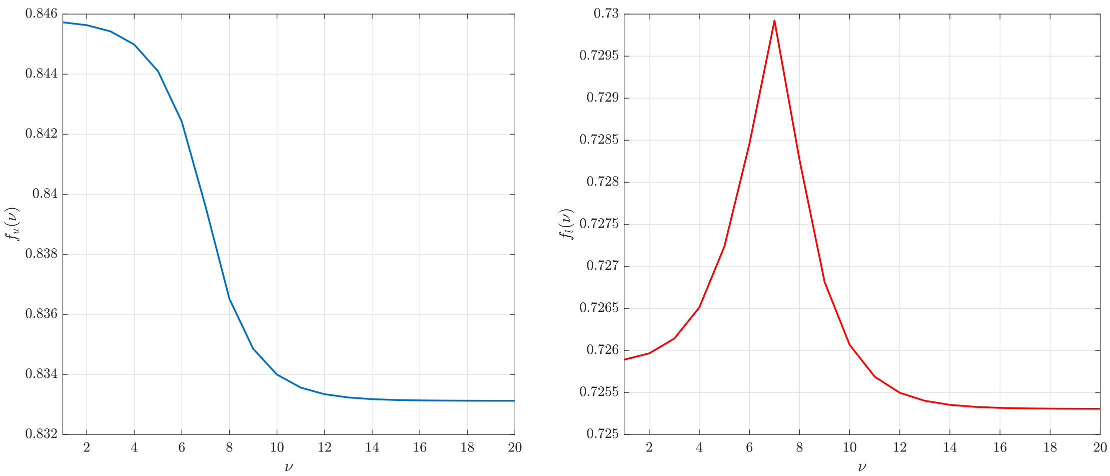

Figure 2: Left: Convergence of $f_u(\nu)$ as $\varepsilon_\nu \downarrow 0$. Right: Convergence of $f_l(\nu)$ as $\varepsilon_\nu \downarrow 0$.

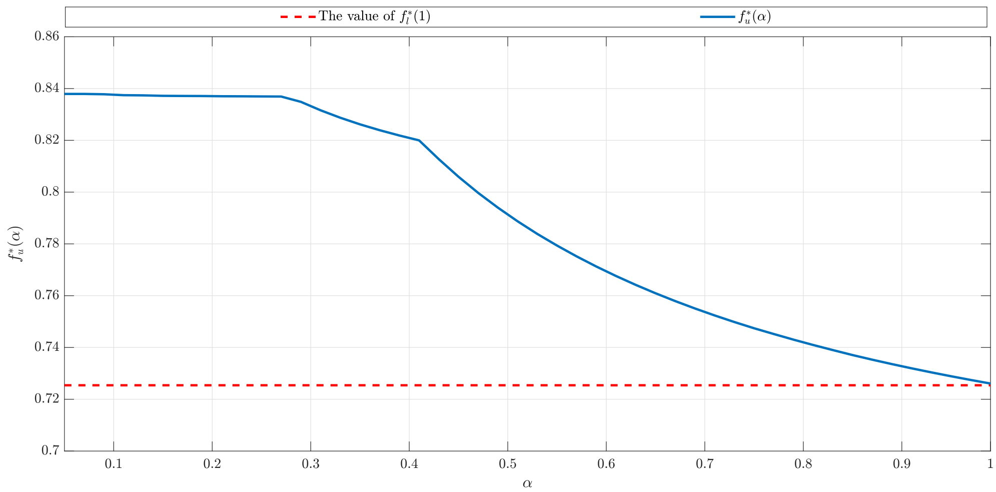

Figure 3: $f_u^*(\alpha)$ on the interval $(0, 1]$ illustrating $f_u^*(1) = f_l^*(1)$.

To observe this fact in practice, let $f_u^*(\alpha)$ and $f_l^*(\alpha)$ denote the optimal value of upper-level and lower-level problems using confidence level $1 - \alpha$. As seen from Figure 3, $f_u^*(\alpha)$ is decreasing with respect to α, which means conditional value at risk suggests smaller damages in return for lower confidence levels. In particular, $f_u^*(1) = f_l^*(1)$. In fact, for $\alpha = 1$, both the attacker and defender use the same tool to measure the risk of damage.

5.2. Experiment 2.

In this experiment, we consider a three-dimensional instance of bi-level problem (10) with $\zeta_1 = \zeta_2 = \zeta_3 = 1, B_D = B_A = 1$, and $\alpha = 0.4$. The stochastic parameters V_1, V_2 and V_3 take two distinct values with

$$\mathbf{E}(\mathbf{V}) = (\mathbf{E}(V_1), \mathbf{E}(V_2), \mathbf{E}(V_3)) = (1.375, 1.100, 1.350).$$

The joint distribution of stochastic parameters V_1, V_2 and V_3 is given in Table 2.

As observed from Figure 4, $\mathbf{x}_\nu \to (0.4821, 0.2589, 0.2590)$ and $\mathbf{y}_\nu \to (0.7142, 0.2856, 0.0000)$ as $\varepsilon_\nu \downarrow 0$. In fact, the defender allocates almost 50% of its budget to defend the most vulnerable target $j = 1$. Also, the attacker

$\mathbf{s}_l$	(1, 1, 1.2)	(1, 1, 1.5)	(1, 2, 1.2)	(1, 2, 1.5)	(1.5, 1, 1.2)	(1.5, 1, 1.5)	(1.5, 2, 1.2)	(1.5, 2, 1.5)
p_l	9/80	9/80	1/80	1/80	27/80	27/80	3/80	3/80

Table 2: Joint distribution of V_1, V_2 and V_3.

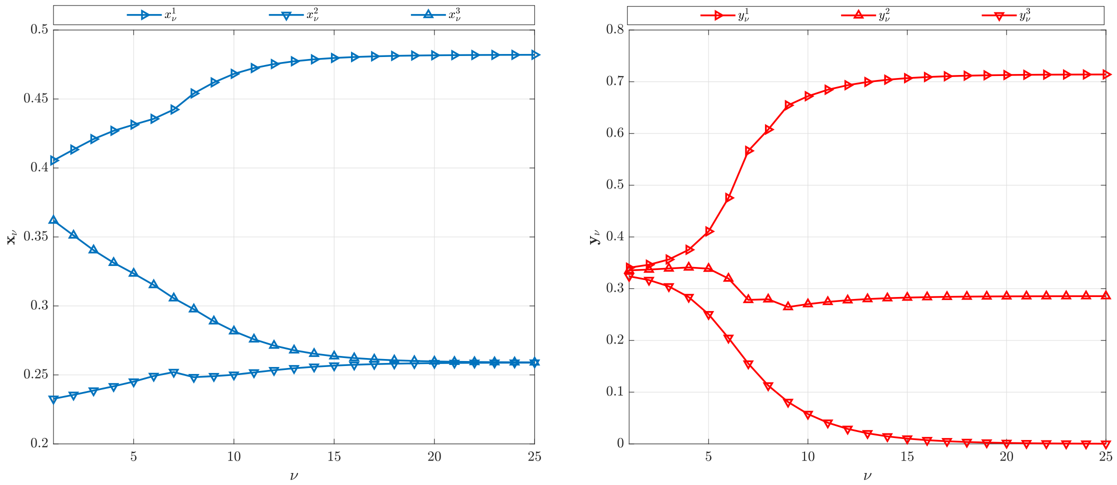

Figure 4: Left: Convergence of the defender's decision vector $\mathbf{x}_\nu$ as $\varepsilon_\nu \downarrow 0$. Right: Convergence of the attacker's decision vector $\mathbf{y}_\nu$ as $\varepsilon_\nu \downarrow 0$.

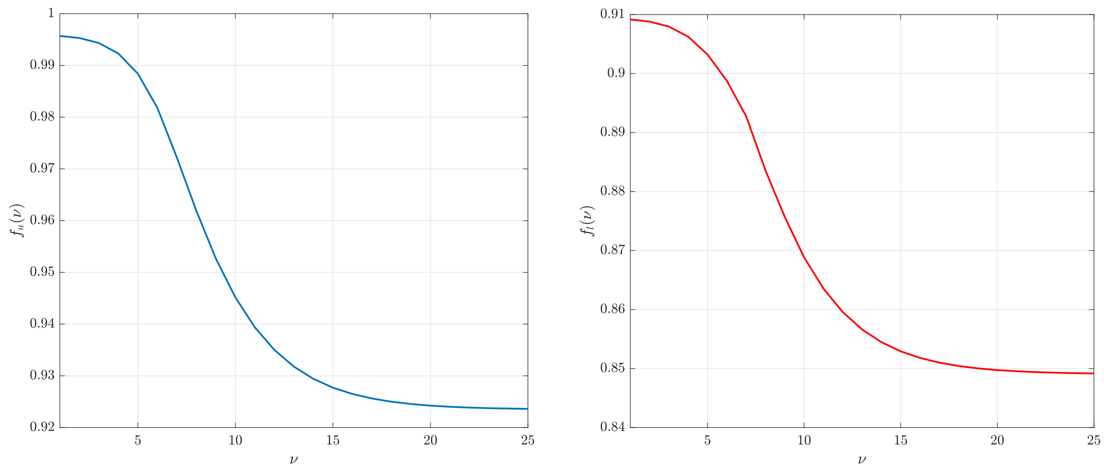

Figure 5: Left: Convergence of $f_u(\nu)$ as $\varepsilon_\nu \downarrow 0$. Right: Convergence of $f_l(\nu)$ as $\varepsilon_\nu \downarrow 0$.

invest more than 70% of its budget on target $j = 1$ to maximize the expected damage. In this scenario, as seen from Figure 5,

$$f_u(\nu) := t_\nu + \frac{1}{\alpha} \sum_{l=1}^{8} p_l h_l^\nu \to 0.9235 \quad \text{and} \quad f_l(\nu) := \mathbf{z}_\nu^T \mathbf{E}(\mathbf{V}) \to 0.8491,$$

as $\varepsilon_\nu \downarrow 0$. In other words, the maximum expected damage that can be achieved by the attacker is 0.8491. This value has been obtained as a result of reducing the conditional value at risk to 0.9235. Regarding the defender, there are two strategies to reduce the expected damage. The first strategy is to increase the defense cost effectiveness of each target, which are denoted by $\zeta_j, j = 1, \dots, n$. The second one is to increase the available budget $B_D > 0$. In the next experiment, this issue is discussed in more details.

5.3. Experiment 3.

In this experiment, we study the sensitivity of our formulation to the parameters $\zeta_j, j = 1, \dots, n$, B_D, and B_A. To this end, we consider the two-dimensional scenario which presented in Experiment 1. First, for the fixed parameters $B_D = B_A = 1$, let $f^*_{u,\zeta_1}(t)$ and $f^*_{l,\zeta_1}(t)$ denote the optimal values of the upper and lower-level problems when $\zeta_1 = t$ and ζ_2 is fixed to 1. Moreover, $f^*_{u,\zeta_2}(t)$ and $f^*_{l,\zeta_2}(t)$ are defined in a similar manner.

The left plot of Figure 6 illustrates the behavior of functions $f^*_{u,\zeta_i}(t)$ and $f^*_{l,\zeta_i}(t)$, $i = 1, 2$ on the interval $[0.1, 4]$. One can see that, as the defense cost effectiveness of each target increases, the expected damage reduces, and the

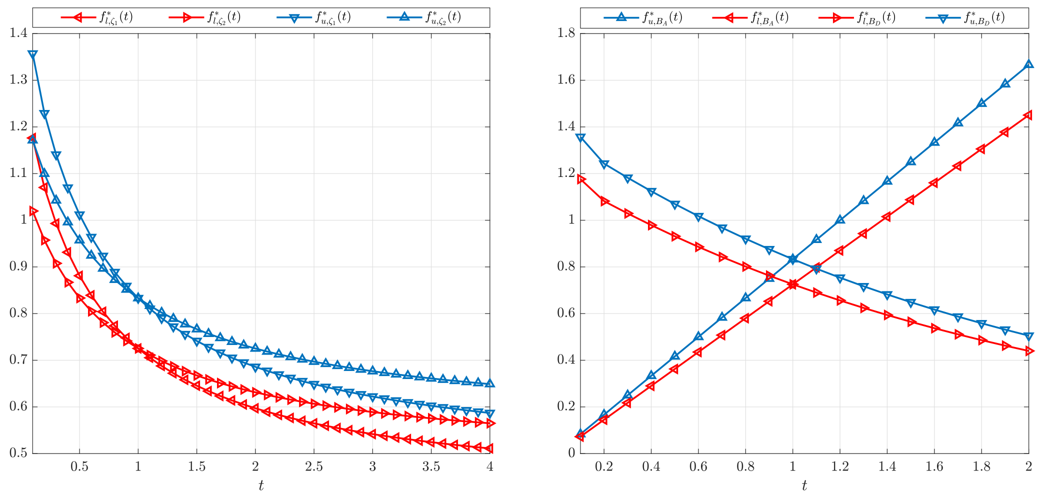

Figure 6: Left: functions $f^*_{u,\zeta_i}(t)$ and $f^*_{l,\zeta_i}(t)$, $i = 1, 2$ on interval $[0.1, 4]$. Right: functions $f^*_{u,B_A}(t)$, $f^*_{l,B_A}(t)$, $f^*_{u,B_D}(t)$, and $f^*_{l,B_D}(t)$ on interval $[0.1, 2]$.

same happens for the conditional value at risk of the damage. Since $\mathbf{E}(V_1) > \mathbf{E}(V_2)$, we observe that the reduction rate of the $f^*_{u,\zeta_1}(t)$ and $f^*_{l,\zeta_1}(t)$ is more than $f^*_{u,\zeta_2}(t)$ and $f^*_{l,\zeta_2}(t)$, respectively.

Next, for the fixed parameters $\zeta_1 = \zeta_2 = 1$, let $f^*_{u,B_D}(t)$ and $f^*_{l,B_D}(t)$ denote the optimal values of the upper and lower-level problems when $B_D = t$ and B_A is fixed to 1. Furthermore, $f^*_{u,B_A}(t)$ and $f^*_{l,B_A}(t)$ are defined similarly. The right plot of Figure 6 shows the behavior of these functions on the interval $[0.1, 2]$. It is observed that, as t increases, $f^*_{u,B_A}(t)$ and $f^*_{l,B_A}(t)$ grow linearly. On the other hand, we see a slow reduction in $f^*_{u,B_D}(t)$ and $f^*_{l,B_D}(t)$, as t increases.

6. Conclusion

We have presented a bi-level formulation for a sequential stochastic attacker-defender game with multiple targets. Concerning the defender, for each allocation of the budget, the conditional value at risk was employed to measure the risk of damage. The attacker used the mathematical expectation to provide an optimal reaction to the defender's decisions. Convexity and regularity of the attacker's optimization problem prepared the ground to employ **KKT** transformations to reduce the proposed bi-level optimization problem to a single-level one. We have established some important relationships between the solutions of the bi-level and single-level optimization problems. The Fischer-Burmeister function was employed to handle the complementarity constraints of the single-level problem. To assess the efficiency of the proposed formulation in practice, we considered several stochastic attacker-defender games and reported the most important results.

References

[1] J. F. Bard, *Practical Bilevel Optimization: Algorithms and Applications*, Nonconvex Optimization and Its Applications (NOIA, volume 30), Springer, 1998.

[2] M. S. Bazaraa, H. D. Sherali, and C. M. Shetty, *Nonlinear Programming: Theory and Algorithms*, John Wiley & Sons, 3rd ed., 2006.

[3] J. F. Bonnans, J. C. Gilbert, C. Lemaréchal, and C. A. Sagastizábal, *Numerical Optimization: Theoretical and Practical Aspects*, Universitext (UTX), Springer Berlin, Heidelberg, 1998.

[4] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, 2004.

[5] S. Dempe and J. Dutta, *Is bilevel programming a special case of a mathematical program with complementarity constraints?*, Math. Program., 131 (2012), pp. 37–48.

[6] Q. Feng, H. Cai, and Z. Chen, *Using game theory to optimize the allocation of defensive resources on a city scale to protect chemical facilities against multiple types of attackers*, Reliab. Eng. Syst. Saf., 191 (2019), p. 105900.

[7] H.-M. Kaltenbach, *A concise guide to statistics*, SpringerBriefs in Statistics, Springer, Heidelberg, 2012.

[8] J. Kisiala, *Conditional value-at-risk: Theory and applications*, Master's thesis, University of Edinburgh, 2015.

[9] M. Ouyang, M. Xu, C. Zhang, and S. Huang, *Mitigating electric power system vulnerability to worst-case spatially localized attacks*, Reliab. Eng. Syst. Saf., 165 (2017), pp. 144–154.

[10] P. J. Phillips, *Applying modern portfolio theory to the analysis of terrorism. computing the set of attack method combinations from which the rational terrorist group will choose in order to maximise injuries and fatalities*, Defence and Peace Economics, 20 (2009), pp. 193–213.

[11] R. Powell, *Defending against terrorist attacks with limited resources*, American Political Science Review, 101 (2007), pp. 527–541.

[12] S. M. Robinson, *Generalized equations and their solutions, part II: Applications to nonlinear programming*, Springer Berlin Heidelberg, 1982, pp. 200–221.

[13] X. Shan and J. Zhuang, *Hybrid defensive resource allocations in the face of partially strategic attackers in a sequential defender–attacker game*, Eur. J. Oper. Res., 228 (2013), pp. 262–272.

[14] ——, *Modeling cumulative defensive resource allocation against a strategic attacker in a multi-period multi-target sequential game*, Reliab. Eng. Syst. Saf., 179 (2018), pp. 12–26.

[15] Z. Yazdaniyan, M. Shamsi, Z. Foroozandeh, and M. R. D. Pinho, *A numerical method based on the complementarity and optimal control formulations for solving a family of zero-sum pursuit-evasion differential games*, J. Comput. Appl. Math., 368 (2020), p. 112535.

[16] Z. Yazdaniyan, M. Shamsi, M. R. D. Pinho, and Z. Foroozandeh, *Heuristic artificial bee colony algorithm for solving the homicidal chauffeur differential game*, AUT J. Math. Com., 1 (2020), pp. 153–163.

[17] C. Zhang and J. E. Ramirez-Marquez, *Protecting critical infrastructures against intentional attacks: A two-stage game with incomplete information*, Iie Transactions, 45 (2013), pp. 244–258.

[18] C. Zhang, J. E. Ramirez-Marquez, and J. Wang, *Critical infrastructure protection using secrecy–a discrete simultaneous game*, Eur. J. Oper. Res., 242 (2015), pp. 212–221.

[19] J. Zhang and J. Zhuang, *Modeling a multi-target attacker-defender game with multiple attack types*, Reliab. Eng. Syst. Saf., 185 (2019), pp. 465–475.

[20] J. Zhang, J. Zhuang, and V. R. R. Jose, *The role of risk preferences in a multi-target defender-attacker resource allocation game*, Reliab. Eng. Syst. Saf., 169 (2018), pp. 95–104.

[21] J. Zhuang and V. M. Bier, *Balancing terrorism and natural disasters—defensive strategy with endogenous attacker effort*, Oper. Res., 55 (2007), pp. 976–991.

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