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Original Article

# A new characterization of some characteristically simple groups

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**ABSTRACT:** Let $G$ be a finite group and $\mathrm{cd}(G)$ be the set of irreducible complex character degrees of $G$. It was proved that some finite simple groups are uniquely determined by their orders and their degree graphs. Recently, in [Behravesh, et al., Recognition of Janko groups and some simple $K_4$-groups by the order and one irreducible character degree or character degree graph, Int. J. Group Theory, DOI: 10.22108/ijgt.2019.113029.1502.] new characterizations for some finite simple groups are given. Also, in [Qin, et al., Mathieu groups and its degree prime-power graphs, *Comm. Algebra*, 2019] the degree prime-power graph of a finite group is introduced and it is proved that the Mathieu groups are uniquely determined by order and degree prime-power graph. In this paper we continue this work and we characterize some simple groups and some characteristically simple groups by their orders and some vertices of their degree prime-power graphs.

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*(Dedicated to Professor Jamshid Moori)*

## 1. Introduction

Throughout this paper, all groups are finite and all characters referred to are complex characters. We denote the set of irreducible characters of $G$ by $\mathrm{Irr}(G)$, and assume $\mathrm{cd}(G)$ denotes the set of character degrees of $G$. Also, we denote the fitting subgroup of $G$ by $F(G)$ and the direct product of $n$ copies of $G$ is denoted by $G^n$. For a natural number $n$ and a prime number $p$, we denote by $n_p$, the $p$-part of $n$; i.e. $n_p = p^\alpha$, where $p^\alpha | n$ and $p^{\alpha+1} \nmid n$. Also, for a natural number $n$, $\pi(n)$ is the set of prime divisors of $n$. If $G$ is a finite group, $\pi(|G|)$ is denoted by $\pi(G)$. A finite group $G$ is called a $K_n$-group if $|\pi(G)| = n$ $|G|$ has exactly $n$ distinct prime divisors. Let $\rho(G)$ be the set of primes dividing an element in $\mathrm{cd}(G)$. A graph is named the *character degree graph* of $G$ and is denoted by $\Delta(G)$, if the vertex set is $\rho(G)$ and two different vertices $p, q \in \rho(G)$ are connected if $pq$ divides an element in $\mathrm{cd}(G)$.

In [6], Lewis and White proved that if $G$ is a group, then $\Delta(G)$ has three connected components if and only if $G = \mathrm{PSL}_2(2^n) \times A$, for an integer $n \geq 2$ and $A$ is an abelian group. In [1], it is proved that if $\Delta(G) = \Delta(\mathrm{PSL}_2(q))$ and $|G| = |\mathrm{PSL}_2(q)|$, where $q \geq 4$ is a prime power, then $G \cong \mathrm{PSL}_2(q)$. Some authors in [5] and [9] proved that some finite simple groups are characterizable by their character degree graphs and orders. In [8], some $K_3$-simple groups are characterized by their orders and the largest and the second largest irreducible character degrees of $G$. Recently, Behravesh et al. in [2] proved that Janko groups $J_1, J_3, J_4$ and some $K_4$-simple groups are characterized by order and one irreducible character degree or character degree graph. In [7], Qin et al. introduced the degree prime-power graph as follows:

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**Definition 1.1.** *Let $G$ be a group. Let $p^{e_p(G)} = max\{\chi(1)_p \ : \ \chi \in \mathrm{Irr}(G)\}$, for all primes $p \in \rho(G)$. A graph is called the degree prime-power graph and is denoted by $\Gamma(G)$, if the vertex set is $V(G) = \{p^{e_p(G)} \ : \ p \in \rho(G)\}$ and two different vertices $m, n \in V(G)$ are adjacent if there is an element in $\mathrm{cd}(G)$ divisible by $mn$.*

Then they proved that the Mathieu simple groups are characterizable by order and the degree prime-power graph. In this paper, we show that some simple groups and some characteristically simple groups are characterizable by their orders and only some vertices of the degree prime-power graphs. These results are generalizations of some results in [2], [5] and [8]. It is worth mentioning that in case $G = S^k$ where $S$ is a finite simple groups whose order is divided by at most four prime divisors, we tried to prove $G$ is characterizable by its order and the largest vertex of the degree prime-power graph. But it turns out that it is not correct (See Remark 2.16). We also tried to characterize all characteristically simple groups $S^k$ where $S$ is a simple group of Lie type, by the degree of the Steinberg character as an element of the vertices of the degree prime-power graph and order of the group, but we found an example which shows that it is not correct as well (See Remark 2.16). At last, we characterize some characteristically simple groups $G$ by a limited subset of $V(G)$ and the order of the groups and in each case we made an effort to select a collection of $V(G)$ as small as possible.

**Main Theorem.** *Let $G$ be a finite group. Then for $n = 1, 2$:*

(1) $G \cong A_6$ *if and only if* $|G| = |A_6|$ *and* $3^2 \in V(G)$.

(2) $G \cong A_6^k$ *if and only if* $|G| = |A_6^k|$ *and* $\{3^{2k}, 5^k\} \subseteq V(G)$, *for* $2 \leq k \leq 4$.

(3) $G \cong A_7$ *if and only if* $|G| = |A_7|$ *and* $\{5, 7\} \subseteq V(G)$.

(4) $G \cong A_7 \times A_7$ *if and only if* $|G| = |A_7 \times A_7|$ *and* $\{3^4, 5^2, 7^2\} \subseteq V(G)$.

(5) $G \cong A_8$ *or* $G \cong \mathrm{PSL}_3(4)$ *if and only if* $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ *and* $\{2^6, 5\} \subseteq V(G)$.

(6) $G \cong A_8 \times A_8, G \cong \mathrm{PSL}_3(4) \times \mathrm{PSL}_3(4)$ *or* $G \cong A_8 \times \mathrm{PSL}_3(4)$ *if and only if* $|G| = 2^{12} \cdot 3^4 \cdot 5^2 \cdot 7^2$ *and* $\{2^{12}, 5^2\} \subseteq V(G)$.

(7) $G \cong \mathrm{PSL}_2(16)^n$ *if and only if* $|G| = |\mathrm{PSL}_2(16)^n|$ *and* $17^n \in V(G)$.

(8) $G \cong \mathrm{PSL}_2(25)^n$ *if and only if* $|G| = |\mathrm{PSL}_2(25)^n|$ *and* $13^n \in V(G)$.

(9) $G \cong \mathrm{PSL}_2(32)^n$ *if and only if* $|G| = |\mathrm{PSL}_2(32)^n|$ *and* $11^n \in V(G)$ *or* $2^{5n} \in V(G)$.

(10) $G \cong \mathrm{PSL}_2(64)^n$ *if and only if* $|G| = |\mathrm{PSL}_2(64)^n|$ *and* $\{2^{6n}, 3^{2n}, 13^n\} \subseteq V(G)$.

(11) $G \cong \mathrm{PSL}_2(81)^n$ *if and only if* $|G| = |\mathrm{PSL}_2(81)^n|$ *and* $41^n \in V(G)$.

(12) $G \cong \mathrm{PSU}_3(4)^n$ *if and only if* $|G| = |\mathrm{PSU}_3(4)^n|$ *and* $\{2^{6n}, 13^n\} \subseteq V(G)$ *or* $\{2^{6n}, 5^{2n}\} \subseteq V(G)$.

(13) $G \cong \mathrm{PSU}_3(7)^n$ *if and only if* $|G| = |\mathrm{PSU}_3(7)^n|$ *and* $43^n \in V(G)$.

(14) $G \cong \mathrm{PSU}_3(8)^n$ *if and only if* $|G| = |\mathrm{PSU}_3(8)^n|$ *and* $19^n \in V(G)$.

## 2. MAIN RESULTS

We break the proof of the main results into several short Lemmas so that following them be easier for the readers.

**Lemma 2.1.** *Let $T \cong R_1^{k_1} \times R_2^{k_2} \times \cdots \times R_t^{k_t}$, where $R_i$'s are nonisomorphic finite simple groups. If $p$ is a prime and $r^p \nmid |T|$, for every $r \in \pi(T)$, then $p \nmid |\mathrm{Out}(T)|$.*

*Proof.* We know that $\mathrm{Aut}(T) \cong (\mathrm{Aut}(R_1) \wr S_{k_1}) \times (\mathrm{Aut}(R_2) \wr S_{k_2}) \times \cdots \times (\mathrm{Aut}(R_t) \wr S_{k_t})$. If $p \in \pi(\mathrm{Out}(T))$, then $p \in \pi(\mathrm{Out}(R_i))$ or $p \in \pi(S_{k_i})$, for some $1 \leq i \leq t$. Let $p \in \pi(S_{k_i})$, then $k_i \geq p$ and hence for every $r \in \pi(R_i)$ we have $r^p \Big| |T|$, a contradiction. Thus, $p \in \pi(\mathrm{Out}(R_i))$. If $R_i \cong A_n(q)$, where $q = q_0^f$ for some prime $q_0$ and integer $f$, then by [3, Tables 5,6, page xvi], we know that $|\mathrm{Out}(R_i)| = (2, q-1) \cdot f$, for $n = 1$; $|\mathrm{Out}(R_i)| = (n+1, q-1) \cdot f \cdot 2$, for $n > 1$, where $|R_i| = \dfrac{q^{1/2(n(n+1))}}{(n+1, q-1)} \displaystyle\prod_{i=1}^{n} \left(q^{i+1} - 1\right)$. Since $q_0^p \big| |R_i|$, we get a contradiction. Similarly, by checking the outer automorphisms of other finite simple groups, we get that there exists $r \in \pi(R_i)$ such that $r^p \Big| |R_i|$, a contradiction. $\square$

**Lemma 2.2.** *Let $G$ be a finite group and $|G| = p^{\alpha}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$, where $p$ and $p_i$ are distinct primes, for $1 \leq i \leq k$ and $\alpha_i$'s and $\alpha$ are natural numbers. Suppose $p^{\alpha}, p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_t^{\alpha_t} \in V(G)$, where $1 \leq t \leq k$. Then the following hold.*

*(a) If $H$ is any normal subgroup of $G$ such that $|H|_p = p^m$, then $|H| > p^{2m}$.*

*(b) If $p^{\alpha} \nmid \prod_{i=t+1}^{k} |\mathrm{GL}_{\alpha_i}(p_i)|$, then $G$ is nonsolvable.*

*(c) If $p \nmid \prod_{i=t+1}^{k} |\mathrm{GL}_{\alpha_i}(p_i)|$, then $p^{\alpha}\Big||G/H|$, where $H$ is the solvable radical of $G$.*

*Proof.* The proof of $(a)$ is clear.

(b) By the way of contradiction, assume that $G$ is solvable. Then $|G|\Big||F(G)||\mathrm{Aut}(F(G))|$. Since $O_p(G) = 1$ and $p^{\alpha}\Big||G|$, we get that $p^{\alpha}\Big||F(G)||\mathrm{Aut}(F(G))|\ \Big||F(G)|\prod_{i=t+1}^{k}|\mathrm{GL}_{\alpha_i}(p_i)|$, a contradiction by our hypothesis.

(c) It suffices to show that $p \nmid |H|$. If $p\Big||H|$, since $|H|\Big||F(H)||\mathrm{Aut}(F(H))|\Big||F(H)|\prod_{i=t+1}^{k}|\mathrm{GL}_{\alpha_i}(p_i)|$, we get a contradiction by our hypothesis. So, $p \nmid |H|$ and hence $p^{\alpha}\Big||G/H|$. □

In the following Lemma we give a proof for Part (1) and (2) of the Main Theorem.

**Lemma 2.3.** *Let $G$ be a group. Then*

*(a) $G \cong A_6$ if and only if $|G| = |A_6|$ and $3^2 \in V(G)$.*

*(b) $G \cong A_6^n$ if and only if $|G| = |A_6^n|$ and $\{3^{2n}, 5^n\} \subseteq V(G)$, for $2 \leq n \leq 4$.*

*Proof.* Let $1 \leq n \leq 4$. Obviously, we just need to prove sufficiency. By [3], $|G| = 2^{3n}\cdot 3^{2n}\cdot 5^n$ and $O_3(G) = 1$. Also, if $2 \leq n \leq 4$, then $O_5(G) = 1$. If $n = 1$, then $3^2 \nmid |\mathrm{GL}_3(2)||\mathrm{GL}_1(5)|$ and if $2 \leq n \leq 4$, then $5^n \nmid |\mathrm{GL}_{3n}(2)|$. So, by Lemma 2.2(b), $G$ is nonsolvable. Let $H$ be the solvable radical of $G$ and $K/H$ be the socle of $G/H$.

(a) By [4], $K/H \cong A_5$ or $A_6$. By Lemma 2.2(a), $K/H \ncong A_5$. So, $K/H \cong A_6$ and so $G \cong A_6$, as desired.

(b) Let $n = 2$. By [4], $K/H \cong A_5, A_5 \times A_5, A_6, A_6 \times A_6, \mathrm{PSU}_4(2)$ or $A_5 \times A_6$. Note that by our hypothesis, $|G/K|\Big||\mathrm{Out}(K/H)|$. If $K/H \cong A_5$, then $|H| = 2^i \cdot 3^3 \cdot 5$, for $i = 3$ or $4$. If $K/H \cong A_5 \times A_5$, then $|H| = 2^i \cdot 3^2$, for $i = 0, 1$ or $2$. If $K/H \cong A_6$, then $|H| = 2^i \cdot 3^2 \cdot 5$, for $i = 1, 2$ or $3$. If $K/H \cong \mathrm{PSU}_4(2)$, then $|H| = 5$. If $K/H \cong A_5 \times A_6$, then $|H| = 2^i \cdot 3$, for $i = 0$ or $1$. In all the above cases, by Lemma 2.2(b), we get a contradiction. So, $K/H \cong A_6 \times A_6$ and hence $G \cong A_6 \times A_6$, as desired. Similarly to the above argument, for $n = 3, 4$, we obtain the result. □

**Remark 2.4.** *We note that $A_5 \times S_3$ has irreducible characters of degrees $2^3$ and $5$, which shows that $A_6$ is not characterizable by its order and these vertices of the degree prime-power graph. In [5], $A_6$ is characterized by its order and the character degree graph.*

The validation of Parts (3) and (4) of the Main Theorem follows from the next Lemma.

**Lemma 2.5.** *Let $G$ be a group. Then*

*(a) $G \cong A_7$ if and only if $|G| = |A_7|$ and $\{5, 7\} \subseteq V(G)$.*

*(b) $G \cong A_7 \times A_7$ if and only if $|G| = |A_7 \times A_7|$ and $\{3^4, 5^2, 7^2\} \subseteq V(G)$.*

*Proof.* (a) By [3], $|G| = 2^3\cdot 3^2\cdot 5\cdot 7$ and $O_5(G) = O_7(G) = 1$. As $5 \nmid |\mathrm{GL}_3(2)|\ |\mathrm{GL}_2(3)|$, by Lemma 2.2(b), we get that $G$ is nonsolvable. Let $H$ be the solvable radical of $G$ and $K/H$ be the socle of $G/H$. By [4], $K/H$ is isomorphic to one of the following groups: $A_5, A_6, \mathrm{PSL}_2(7), \mathrm{PSL}_2(8), A_7$. If $K/H \cong A_5$, then $21\Big||H|$, a contradiction, by Lemma 2.2(a). If $K/H \cong A_6$, then $|H| = 7$, a contradiction, as $O_7(G) = 1$. By Lemma 2.2(c), $K/H \ncong \mathrm{PSL}_2(7), \mathrm{PSL}_2(8)$. Hence $K/H \cong A_7$ and so $G \cong A_7$, as desired.

(b) Similarly to the above argument, we obtain the result. □

**Remark 2.6.** *Considering the groups $A_6 \times A$, $\mathrm{PSL}_2(8) \times A$, where $A$ is an abelian group, we get that $A_7$ is not characterizable by its order and any set of vertices $\{2^3, 5\}$, $\{3^2, 5\}$ $\{2^3, 3^2\}$, $\{2^3, 7\}$ or $\{3^2, 7\}$ of degree prime-power graph. In [5], $A_7$ is characterized by its order and the character degree graph.*

The next Lemma proves Parts (5) and (6) of the Main Theorem.

**Lemma 2.7.** *Let $G$ be a group. Then* *(a)* $G \cong A_8$ *or* $G \cong \mathrm{PSL}_3(4)$ *if and only if* $|G| = 2^6 \cdot 3^2 \cdot 5 \cdot 7$ *and* $\{2^6, 5\} \subseteq V(G)$.

*(b)* $G \cong A_8 \times A_8, G \cong \mathrm{PSL}_3(4) \times \mathrm{PSL}_3(4)$ *or* $G \cong A_8 \times \mathrm{PSL}_3(4)$ *if and only if* $|G| = 2^{12} \cdot 3^4 \cdot 5^2 \cdot 7^2$ *and* $\{2^{12}, 5^2\} \subseteq V(G)$.

*Proof.* By assumptions, $O_2(G) = O_5(G) = 1$. Since $5^n \nmid |\mathrm{GL}_{2n}(3)||\mathrm{GL}_n(7)|$ for $n = 1, 2$, by Lemma 2.2(b), $G$ is nonsolvable. Let $H$ be the solvable radical of $G$ and $K/H$ be the socle of $G/H$.

(a) By [4], $K/H \cong A_5, A_6, A_7, A_8, \mathrm{PSL}_2(7), \mathrm{PSL}_2(8)$, $\mathrm{PSL}_3(4)$ or $A_5 \times \mathrm{PSL}_2(7)$. Let $K/H \cong A_5$ and so $|H| = 2^m \cdot 3 \cdot 7$, where $m = 3$ or $m = 4$. Since $H$ is solvable and $2^3 \nmid |F(H)||\mathrm{Aut}(F(H))|$, we get a contradiction. If $K/H \cong A_6$, then $|H| \Big| 2^i \cdot 7$, where $1 \le i \le 3$. If $|H| = 2^3 \cdot 7$ or $|H| = 2^2 \cdot 7$, then similarly to the above we get a contradiction. If $|H| = 2 \cdot 7$, as $O_2(G) = 1$, $H \cong D_{14}$. Note that $G/H \cong \mathrm{Aut}(A_6)$. Let $P$ be the Sylow 7-subgroup of $G$. As $|G/C_G(P)|$ is a divisor of 6 and it is divided by 2. We know that $C_G(P)/P \cong C_G(P)H/H \trianglelefteq G/H$. Therefore, by Burnside theorem, $C_G(P) \cong P \times T$, where $T \cong \mathrm{Aut}(A_6)$. Therefore $G \cong T \times H$, as $T$ is a normal subgroup of $G$. Hence we get a contradiction, as $2^6 \in V(G)$. By Lemma 2.2(c), $K/H \ncong \mathrm{PSL}_2(7), \mathrm{PSL}_2(8)$. If $K/H \cong A_7$, then $H$ is a normal 2-subgroup of $G$, a contradiction as $O_2(H) = 1$. If $K/H \cong A_5 \times \mathrm{PSL}_2(7)$, then $|H| \Big| 2$ and so $|H| = 1$. Then $G \hookrightarrow \mathrm{Aut}(A_5 \times \mathrm{PSL}_2(7)) \cong S_5 \times \mathrm{PGL}(2, 7)$, a contradiction as $2^6 \notin V(\mathrm{Aut}(A_5 \times \mathrm{PSL}_2(7)))$. Hence $K/H \cong A_8$ or $K/H \cong \mathrm{PSL}_3(4)$ and so $G \cong A_8$ or $G \cong \mathrm{PSL}_3(4)$, as desired.

(b) Similarly to the above argument, we get the result. $\square$

**Remark 2.8.** *We see that* $G_1 \cong \mathrm{PSL}_2(7) \times S_3 \times \mathrm{SmallGroup}(20, 3)$*,* $G_2 \cong \mathrm{PSL}_2(8) \times \mathrm{SmallGroup}(20, 3) \times D_{14}$ *and* $G_3 \cong \mathrm{PSL}_2(7) \times A_5 \times C_2$*, have the same orders* $|A_8| = |\mathrm{PSL}_3(4)|$*. Also,* $\Gamma(G_1)$ *has vertices* $2^6$ *and* 7*. In* $\Gamma(G_2)$ *there are vertices* $2^6$ *and* $3^2$*. Moreover,* $\{3^2, 5\} \in V(G_3)$*. Therefore, the result of Lemma 2.7 just hold for two vertices* $2^6$ *and* 5*. In [5],* $A_8$ *is characterized by its order and the character degree graph.*

Part (7) of the Main Theorem is proven in the following Lemma.

**Lemma 2.9.** *Let* $G$ *be a group. Then,* $G \cong \mathrm{PSL}_2(16)^n$ *if and only if* $|G| = |\mathrm{PSL}_2(16)^n|$ *and* $17^n \in V(G)$*, for* $n = 1, 2$*.*

*Proof.* For $n = 1, 2$, by [3], $|G| = 2^{4n} \cdot 3^n \cdot 5^n \cdot 17^n$ and so $O_{17}(G) = 1$. Since $17^n \nmid |\mathrm{GL}_{4n}(2)|\ |\mathrm{GL}_n(3)||\mathrm{GL}_n(5)|$, for $n = 1, 2$, by Lemma 2.2(b), $G$ is nonsolvable. Let $H$ be the solvable radical of $G$ and $K/H$ be the socle of $G/H$. For $n = 1$, by Lemma 2.1, $17 \nmid |\mathrm{Out}(K/H)|$ and by Lemma 2.2(c), $17 \Big| |K/H|$. Then by [4], $K/H \cong \mathrm{PSL}_2(16)$ and so $G \cong \mathrm{PSL}_2(16)$, as desired. For $n = 2$, similarly to the above argument, we get the result. $\square$

**Remark 2.10.** *Considering* $A_5 \times A$ *or* $S_3 \times D_{17} \times \mathrm{SmallGroup}(20, 3)$*, where* $A$ *is an abelian group, we get that* $\mathrm{PSL}_2(16)$ *is not characterizable by its order and vertices* $3, 5$ *or* $2^4$ *of degree prime-power graph. In [5],* $\mathrm{PSL}_2(16)$ *is characterized by its order and the character degree graph.*

In the next Lemma we give a proof of the Main Theorem Part (8).

**Lemma 2.11.** *Let* $G$ *be a group. Then* $G \cong \mathrm{PSL}_2(25)^n$ *if and only if* $|G| = |\mathrm{PSL}_2(25)^n|$ *and* $13^n \in V(G)$*, for* $n = 1, 2$*.*

*Proof.* For $n = 1, 2$, by [3], $|G| = 2^{3n} \cdot 3^n \cdot 5^{2n} \cdot 13^n$ and also $O_{13}(G) = 1$. By Lemma 2.2(b), $G$ is nonsolvable. Assume $H$ is the solvable radical of $G$ and $K/H$ is the socle of $G/H$. Let $n = 1$. By Lemma 2.1, $13 \nmid |\mathrm{Out}(K/H)|$ and by Lemma 2.2(c), $13 \Big| |K/H|$. Therefore, by [4], $K/H \cong \mathrm{PSL}_2(25)$ and so $G \cong \mathrm{PSL}_2(25)$. For $n = 2$, similarly to the above argument we get the result.

$\square$

**Remark 2.12.** *Similarly to the above we can prove that* $\mathrm{PSL}_2(25)$ *is characterized by its order and* $5^2 \in V(G)$*. But considering* $A_5 \times D_{10} \times A$*, where* $A$ *is an abelian group, we get that* $\mathrm{PSL}_2(25)$ *is not characterizable by its order and vertices* $2^3$ *or* 3 *of degree prime-power graph. In [2],* $\mathrm{PSL}_2(25)$ *is characterized by its order and the character degree graph.*

Nest Lemma, is the proof of Part (9) of the Main Theorem.

**Lemma 2.13.** *Let* $G$ *be a group. Then* $G \cong \mathrm{PSL}_2(32)^n$ *if and only if* $|G| = |\mathrm{PSL}_2(32)^n|$ *and* $11^n \in V(G)$ *or* $2^{5n} \in V(G)$*, for* $n = 1, 2$*.*

*Proof.* By [3], $|G| = 2^{5n} \cdot 3^n \cdot 11^n \cdot 31^n$ for $n = 1, 2$. First, assume $11^n \in V(G)$. By hypothesis, $O_{11}(G) = 1$ and by Lemma 2.2(b), $G$ is nonsolvable. Assume $H$ is the solvable radical of $G$ and $K/H$ is the socle of $G/H$. By [4], $K/H \cong \mathrm{PSL}_2^n(32)$, for $n = 1, 2$ and hence $G \cong \mathrm{PSL}_2^n(32)$, as wanted.

Let $2^{5n} \in V(G)$. By Lemma 2.2(b), $G$ is nonsolvable. Assume that $H$ is the solvable radical of $G$ and $K/H$ is the socle of $G/H$. By [4], $K/H \cong \mathrm{PSL}_2^n(32)$, for $n = 1, 2$ and so $G \cong \mathrm{PSL}_2^n(32)$, as desired. □

**Remark 2.14.** *Considering $A_4 \times A$ and* SmallGroup $(992, 194) \times A$*, where $A$ is an abelian group, we get that* $\mathrm{PSL}_2(32)$ *is not characterizable by its order and vertices* 3 *or* 31 *of degree prime-power graph. In [5],* $\mathrm{PSL}_2(32)$ *is characterized by its order and the character degree graph.*

Next, we see the proof of Main Theorem Part (10).

**Lemma 2.15.** *Let $G$ be a group. Then $G \cong \mathrm{PSL}_2(64)^n$ if and only if $|G| = |\mathrm{PSL}_2(64)^n|$ and $\{2^{6n}, 3^{2n}, 13^n\} \subseteq V(G)$, for $n = 1, 2$.*

*Proof.* For $n = 1, 2$, by [3], $|G| = 2^{6n} \cdot 3^{2n} \cdot 5^n \cdot 7^n \cdot 13^n$ and so by our hypothesis, $O_2(G) = O_3(G) = O_{13}(G) = 1$. By Lemma 2.2(b), $G$ is nonsolvable. Assume that $H$ is the solvable radical of $G$ and $K/H$ is the socle of $G/H$. Let $n = 1$. By Lemma 2.1, $13 \nmid |\mathrm{Out}(K/H)|$ and by Lemma 2.2(c), $13 \Big| |K/H|$. Therefore, by [4], $K/H \cong \mathrm{PSL}_2(13), \mathrm{PSL}_2(13) \times A_5, Sz(8)$ or $\mathrm{PSL}_2(64)$. Let $K/H \cong \mathrm{PSL}_2(13)$ and so $2^2 \cdot 3 \Big| |H|$, a contradiction as $H$ is solvable and $2^2 \nmid |F(H)||\mathrm{Aut}(F(H))|$. Suppose that $K/H \cong \mathrm{PSL}_2(13) \times A_5$. Then, $|H| \Big| 2^2$. If $|H| \neq 1$, then we get that $O_2(G) \neq 1$, a contradiction. If $|H| = 1$, again we get a contradiction as $2^6 \notin V(\mathrm{PGL}_2(13) \times S_5)$. If $K/H \cong Sz(8)$, then $|H| | 3^2$ which is impossible as $O_3(G) = 1$. Finally, $K/H \cong \mathrm{PSL}_2(64)$ and hence $G \cong \mathrm{PSL}_2(64)$, as wanted. For $n = 2$, similarly to the above argument, we get the result. □

**Remark 2.16.** *Considering $A_5 \times \mathrm{PSL}_2(7) \times D_{26}$ and $\mathrm{PSL}_2(13) \times A_5 \times A$, where $A$ is an abelian group, we get that* $\mathrm{PSL}_2(64)$ *is not characterizable by its order and any set of vertices* $\{2^6, 3^2, 5\}, \{2^6, 3^2, 7\}, \{3^2, 5, 13\}, \{3^2, 7, 13\}$ *or* $\{5, 7, 13\}$ *of degree prime-power graph.* In [5], $\mathrm{PSL}_2(64)$ is characterized by its order and the character degree graph.

The proof of the Main Theorem Part (11) comes as follows.

**Lemma 2.17.** *Let $G$ be a group. Then $G \cong \mathrm{PSL}_2(81)^n$ if and only if $|G| = |\mathrm{PSL}_2(81)^n|$ and $41^n \in V(G)$, for $n = 1, 2$.*

*Proof.* From [3], $|G| = 2^{4n} \cdot 3^{4n} \cdot 5^n \cdot 41^n$, for $n = 1, 2$. First, assume that $41^n \in V(G)$. So, $O_{41}(G) = 1$. By Lemma 2.2(b), $G$ is nonsolvable. Assume that $H$ is the solvable radical of $G$ and $K/H$ is the socle of $G/H$. Let $n = 1$. By Lemma 2.1, $41 \nmid |\mathrm{Out}(K/H)|$ and by Lemma 2.2(c), $41 \Big| |K/H|$. Therefore, by [4], $K/H \cong \mathrm{PSL}_2(81)$ and so $G \cong \mathrm{PSL}_2(81)$. For $n = 2$, similarly to the above argument, we get the result. □

**Remark 2.18.** *Similarly to the above we can prove that* $\mathrm{PSL}_2(81)$ *is characterized by its order and* $3^4 \in V(G)$*. But considering $S_3^4 \times A$ and $A_5 \times A$, where $A$ is an abelian group, we get that* $\mathrm{PSL}_2(81)$ *is not characterizable by its order and vertices* $2^4$ *or* 5 *of degree prime-power graph. In [2],* $\mathrm{PSL}_2(81)$ *is characterized by its order and the character degree graph.*

In the following Lemma we show that Main Theorem Part (12) is valid.

**Lemma 2.19.** *Let $G$ be a finite group. Then $G \cong \mathrm{PSU}_3(4)^n$ if and only if $|G| = |\mathrm{PSU}_3(4)^n|$ and $\{2^{6n}, 13^n\} \subseteq V(G)$ or $\{2^{6n}, 5^{2n}\} \subseteq V(G)$.*

*Proof.* By [3], $|G| = 2^{6n} \cdot 3^n \cdot 5^{2n} \cdot 13^n$, for $n = 1, 2$. First, assume $\{2^{6n}, 13^n\} \subseteq V(G)$. By hypothesis, $O_2(G) = O_{13}(G) = 1$. By Lemma 2.2(b), $G$ is nonsolvable. Let $H$ be the solvable radical of $G$ and $K/H$ be the socle of $G/H$. Let $n = 1$. By [4], $K/H \cong \mathrm{PSL}_2(25), \mathrm{PSU}_3(4)$. Let $K/H \cong \mathrm{PSL}_2(25)$ and so $|H| = 2^i$, for $i = 1, 2, 3$, a contradiction as $O_2(G) = 1$. So, $K/H \cong \mathrm{PSU}_3(4)$ and hence $G \cong \mathrm{PSU}_3(4)$, as desired. For $n = 2$, Similarly to the above argument, we get the result.

Let $\{2^{6n}, 5^{2n}\} \in V(G)$. By Lemma 2.2(b), $G$ is nonsolvable. Let $H$ be the solvable radical of $G$ and $K/H$ be the socle of $G/H$. Let $n = 1$. By [4], $K/H \cong A_5, \mathrm{PSL}_2(25), \mathrm{PSU}_3(4)$. If $K/H \cong A_5$, then $|H| = 2^i \cdot 5 \cdot 13$, for $i = 2, 3$, a contradiction, by Lemma 2.2(b). Let $K/H \cong \mathrm{PSL}_2(25)$ and so $|H| = 2^i$, for $i = 1, 2, 3$, a contradiction as $O_2(G) = 1$. So, $K/H \cong \mathrm{PSU}_3(4)$ and hence $G \cong \mathrm{PSU}_3(4)$, as desired. For $n = 2$, similarly to the above argument, we get the result. □

**Remark 2.20.** *Considering* $\mathrm{PSL}_2(25)\times A$ *and* $A_5\times\mathrm{SmallGroup}(20,3)\times\mathrm{SmallGroup}\ (52,3)$*, where* $A$ *is an abelian group, we get that* $\mathrm{PSU}_3(4)$ *is not characterizable by its order and any set of vertices* $\{3,5^2\}$*,* $\{3,13\}$*,* $\{5^2,13\}$ *or* $\{2^6,3\}$ *of degree prime-power graph.*

The next lemma gaurantees the Main Theorem Part (13).

**Lemma 2.21.** *Let* $G$ *be a group. Then* $G\cong\mathrm{PSU}_3(7)^n$ *if and only if* $|G|=|\mathrm{PSU}_3(7)^n|$ *and* $43^n\in V(G)$*, for* $n=1,2$.

*Proof.* By [3], $|G|=2^{7n}\cdot 3^n\cdot 7^{3n}\cdot 43^n$ and so $O_{43}(G)=1$. By Lemma 2.2(b), $G$ is nonsolvable. Suppose that $H$ is the solvable radical of $G$ and $K/H$ is the socle of $G/H$. Let $n=1$. By Lemma 2.1, $43\nmid|\mathrm{Out}(K/H)|$ and by Lemma 2.2(c), $43\Big||K/H|$ and hence $K/H\cong\mathrm{PSU}_3(7)$ and so $G\cong\mathrm{PSU}_3(7)$. □

**Remark 2.22.** *Considering* $\mathrm{PSL}_2(7)\times A$*,* $\mathrm{SmallGroup}(56,11)\times\mathrm{SmallGroup}\ (56,11)\times\mathrm{SmallGroup}(301,1)\times A$ *and* $\mathrm{PSL}_2(7)\times A$*, where* $A$ *is an abelian group. we get that* $\mathrm{PSU}_3(7)$ *is not characterizable by its order and vertices* $2^3,7^3$ *or* $3$ *of degree prime-power graph. In [2],* $\mathrm{PSU}_3(7)$ *is characterized by its order and the character degree graph.*

The last Lemma Shows that Part (14) of the Main Theorem is correct.

**Lemma 2.23.** *Let* $G$ *be a group. Then* $G\cong\mathrm{PSU}_3(8)^n$ *if and only if* $|G|=|\mathrm{PSU}_3(8)^n|$ *and* $19^n\in V(G)$*, for* $n=1,2$.

*Proof.* We know that by [3], $|G|=2^{9n}\cdot 3^{4n}\cdot 7^n\cdot 19^n$, for $n=1,2$ and $O_{19}(G)=1$. By Lemma 2.2(b), $G$ is nonsolvable. Assume $H$ is the solvable radical of $G$ and $K/H$ is the socle of $G/H$. Let $n=1$. By Lemma 2.1, $19\nmid|\mathrm{Out}(K/H)|$ and by Lemma 2.2(c), $19\Big||K/H|$ and hence by [4], $G\cong\mathrm{PSU}_3(8)$. For $n=2$, similarly to the above argument, we get the result. □

**Remark 2.24.** *Considering* $\mathrm{PSL}_2(7)\times A_4\times A_4\times A$ *and* $\mathrm{PSL}_2(8)\times A_4\times A_4\times A$*, where* $A$ *is an abelian group, we get that* $\mathrm{PSU}_3(8)$ *is not characterizable by its order and vertices* $3^4$ *or* $7$*. In [2],* $\mathrm{PSU}_3(8)$ *is characterized by its order and the character degree graph.*

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