AUT Journal of Mathematics and Computing

AUT J. Math. Comput., 4(1) (2023) 57-61
DOI: 10.22060/AJMC.2022.21224.1080

Original Article

Some properties of the finite Frobenius groups

Mohammadreza Darafsheh*[a]

[a] *College of Science, University of Tehran, Tehran, Iran*

ABSTRACT: The Frobenius group was defined more than 120 years ago and has been the center of interest for researchers in the field of group theory. This group has two parts, complement and kernel. Proving that the kernel is a normal subgroup has been a challenging problem and several attempts have been done to prove it. In this paper we prove some character theory properties of finite Frobenius groups and also give proofs of normality of the kernel in special cases.

Review History:

Received:16 March 2022
Revised:10 May 2022
Accepted:16 May 2022
Available Online:01 February 2023

Keywords:

Frobenius group
Frobenius complement
Frobenius kernel
Character
Permutation character

AMS Subject Classification (2010):

20C15; 20H20; 20F50

(*Dedicated to Professor Jamshid Moori*)

1. Introduction

Let G be a finite group acting on a finite set Ω, $|\Omega| > 1$. Then G is called a Frobenius group if

(a) G acts transitively on Ω,
(b) $G_\alpha \neq 1$ for any $\alpha \in \Omega$,
(c) $G_\alpha \cap G_\beta = 1$, for all $\alpha, \beta \in \Omega$, $\alpha \neq \beta$.

Let $H = G_\alpha$, for some $\alpha \in \Omega$, then for any $\beta \in \Omega$, G_β is conjugate to G_α, i. e. $G_\beta = G_\alpha^g = H^g$, for some $g \in G$. Therefore $F = G \setminus \bigcup_{g \in G} H^g$ is the set of elements of G that do not fix any element of Ω. We set $K = F \cup \{1\}$, hence $K = (G \setminus \bigcup_{g \in G} H^g) \cup \{1\}$. The subgroup H is called a Frobenius complement and it is clear that all subgroups of G conjugate to H are Frobenius complement. It is shown in ([10]pp.195) that all Frobenius complements of G are conjugate. The set K is called the Frobenius kernel of G. An equivalent definition of a Frobenius group is the following: G is called a Frobenius group with complement H if $1 \neq H \lneqq G$, and $H \cap H^g = 1$ for all $g \in G \setminus H$. In [4] it is proved that the Frobenius kernel K is a normal subgroup of G. The proof by Frobenius uses the theory

**Corresponding author.*
E-mail addresses: darafsheh@ut.ac.ir

of characters in [4]. But since then attempts have been made to prove the normality of K without using character theoretic methods. To see proofs of normality of K in G one is referred to [10], [5], [11], [8], [2], [6], [7] and [1], where they use ideas of Frobenius in [4]. As we mentioned earlier no group theoretic method is known to prove that K is a subgroup of G. But in a special cases that $2\Big||H|$ or H is a solvable group one can be referred to [10] and [12]. Using a different approach in [3] it is proved K is a subgroup of G without using character theory. If we consider the Frobenius group G as a permutation group of degree n, then in [9] using group theoretic arguments it is proved that if the rank of G, $rank(G) \leq 3 + \sqrt{n+1}$, then K is a subgroup of G. But another character free group that K is a subgroup of G uses the Fourier-analytic approach given in [13].

In this paper our aim is to prove some character properties of the Frobenius groups and also prove the set K is a subgroup of G using elementary group theory.

2. Character properties of the Frobenius group

Let G be a finite Frobenius group acting transitively on a set Ω of size n. Set

$$1 \neq H = G_\alpha \lneqq G, \, \alpha \in \Omega$$

and $F = G \setminus \bigcup_{g\in G} H^g$, $K = F \cup \{1\}$. Here we assume K is a subset of G. From the definition of K it is clear that K is a normal subset of G, i. e. $g^{-1}Kg \subseteq K$ for all $g \in G$. It is easy to prove that $N_G(H) = H$, hence $|\bigcup_{g\in G} H^g| = |H-1| \times [G:H] + 1$ from which it follows that $|K| = |G| + 1 - |H-1| \times [G:H] - 1 = [G:H] = |\Omega| = n$. Therefore $|G| = |K||H|$.

By definition, the rank of a transitive permutation group on a set Ω is the number of orbits of a point stabilizer on Ω.

Proposition 1. *Let χ be the permutation character of H acting on K by conjugation. Then $\chi = s\rho_H + 1_H$, where ρ_H and 1_H are the regular character and the identity character of H respectively and $s = \dfrac{|K|-1}{|H|}$.*

Proof. By definition of the Frobenius group H fixes $1 \in K$ and is fixed point free on $K - \{1\} = F$. The size of each orbit of H on $K - \{1\}$ is $|H|$, hence $|H|\Big||K| - 1$.

For $h \in H$, $\chi(h)$ is equal to the number of elements of K fixed by h, hence

$$\chi(h) = \begin{cases} |K| & \text{, if} \quad h = 1; \\ 1 & \text{, if} \quad h \neq 1. \end{cases}$$

Let 1_H denote the identity character of H, then

$$(\chi, 1_H)_H = \frac{1}{|H|}\sum_{h\in H}\chi(h) = \frac{1}{|H|}(|K| + |H| - 1) = \frac{|K|-1}{|H|} + 1 = s + 1$$

and this is the number of orbits of H acting on K by conjugating which is the rank of G again acting by conjugation on K. Further computation shows:

$$\begin{aligned}(\chi, \chi)_H &= \frac{1}{H}\sum_{h\in H}\chi(h)\overline{\chi(h)} = \frac{1}{H}(|K|^2 + |H| - 1) \\ &= \frac{|K|^2 - 1}{|H|} + 1 = \text{rank of } H \text{ acting on } K \text{ by conjugation}\end{aligned}$$

Therefore $rank(H) = s(n+1) + 1 = sn + rank(G)$. The character χ is faithful but not irreducible. Let ψ be a non-identity irreducible character of H, then

$$\begin{aligned}(\psi, \chi)_H &= \frac{1}{|H|}\sum_{h\in H}\psi(h)\overline{\chi(h)} \\ &= \frac{1}{|H|}(\psi(1)|K|) + \frac{1}{|H|}\sum_{1\neq h\in H}\psi(h) \\ &= \frac{|K|}{|H|}\psi(1) - \frac{1}{|H|}\psi(1) \\ &= \frac{|K|-1}{|H|}\psi(1) = s\psi(1)\end{aligned}$$

since $(\psi, 1) = 0$, we obtain $\frac{1}{|H|}\sum_{h\in H}\psi(h) = 0$ which implies $\sum_{1\neq h\in H}\psi(h) = -\psi(1)$. Summing up the above calculations, we obtain

$$\chi = (s+1)1_H + s\sum_{1_H\neq\psi\in Irr(H)}\psi(1)\psi = s\rho_H + 1_H.$$

□

Let G be a Frobenius group with complement H, and $[G : H] = n$. Let

$$\Omega = \{Hx \mid x \in G\}$$

be the set of left cosets of H in G. Let π be the permutation character of G acting by right multiplication on Ω. Then clearly $(\pi, 1_G)_G = 1$, $(\pi, \pi) = rank(G)$ and for $g \in G$, $\pi(g)$ is the number of fixed points of g on Ω. Using the properties of the Frobenius groups, we obtain $\pi(1) = [G : H] = n$ and $\pi(g) = 0$ if $g \in K - \{1\}$. With the above notation and character χ of H in Proposition 1 we obtain:

Proposition 2. *Let ρ_H^G and χ denote the induction of the character ρ_H and χ of H to G respectively. Then:*

(a) $(\rho_H^G, \pi)_G = n$
(b) $\chi^G = s\rho_H^G + \pi$
(c) $(\chi^G, \pi)_G = sn + rank(G)$

Proof. Since ρ_H is the regular character of H, Then

$$\rho_H(h) = \begin{cases} |H| & \text{, if } \quad h = 1; \\ 0 & \text{, otherwise.} \end{cases}$$

Calculation show that:

$$(\rho_H^G, \pi)_G = (\rho_H, \pi\mid_H)_H = \frac{1}{|H|}\sum_{h\in H}\rho_H(h)\pi\mid_H(h) = \frac{1}{|H|}|H|n = n$$

and this proves (a).

By Proposition 1: $\chi = s\rho_H + 1_H$ which implies $\chi^G = s\rho_H^G + 1_H^G = s\rho_H^G + \pi$ proving (b).

Therefore $(\chi^G, \pi)_G = s(\rho_H^G, \pi) + (\pi, \pi) = sn + rankG$. Finally (c) is proved. □

The next result is devoted to introducing a character of G whose kernel is K in a special case proving K is a subgroup of G.

Theorem 3. *Let G be a Frobenius group with kernel K. Suppose all elements of K commute. Then $K \trianglelefteq G$.*

Proof. We consider K as a subset of G which is of course a normal subset, so that G acts on it by conjugation. Let η be the permutation character associated to this action. For $g \in G$ the value of $\eta(g)$ is the number of $k \in K$ such that $k^g = k$. We have $\eta(1) = |K|$ and in general if $g \neq 1$ and $k^g = k$, then $k \in C_G(g) \cap K = C_K(g)$. We know that $G = K \bigcup_{g\in G} H^g$, hence we distinguish the following cases:

(i) $1 \neq g \in K$, $k^g = k \Longrightarrow k \in C_K(g) \Longrightarrow \eta(g) = |C_K(g)|$
(ii) $1 \neq g \in \bigcup_{g\in G} H^g \Longrightarrow$ g belongs to some conjugate of H we may take $g \in H$, $k^g = k \Longrightarrow g = g^k \in H \cap H^k \Longrightarrow k \in H \cap K = 1 \Longrightarrow \eta(g) = 1$

Therefore

$$\eta(g) = \begin{cases} |K| & \text{, if } \quad g = 1; \\ 1 & \text{, if } \quad 1 \neq g \in \{H^x \mid x \in G\}, \\ |C_K(g)| & \text{, if } \quad 1 \neq g \in K. \end{cases}$$

Now by assumption all elements of K commute, from which it follows that:

$$\eta(g) = \begin{cases} |K| & \text{, if } \quad g \in K; \\ 1 & \text{, if } \quad 1 \neq g \in \{H^x \mid x \in G\}. \end{cases}$$

Now we see that $ker\eta = K \trianglelefteq G$ proving that K is a normal subgroup of G. □

3. A group theoretic proof that Frobenius kernel is a subgroup

As we mentioned earlier several attempts have been done to prove the Frobenius kernel is a normal subgroup. In this section we are going to prove the Frobenius kernel is a normal subgroup in a special case using only group theoretical arguments. Therefore we assume G is a finite Frobenius group with complement H and

$$K = (G \setminus \bigcup_{g \in G} H^g) \cup \{1\}.$$

We use the fact that $N_G(H) = H$ and $|K| = [G : H]$.

Lemma 4. *If $N \trianglelefteq G$, $G = NH$, $N \cap H = 1$, then $N \leq K$.*

Proof. Let $1 \neq n \in N$. Assume $n \notin K$. Then by definition of K we have $n \in H^g$ for some $g \in G$. Therefore $n^{g^{-1}} \in H \cap N = 1$ which imply $n^{g^{-1}} = 1$, hence $n = 1$ a contradiction. □

Next we will use a theorem of Burnside whose proof can be found in [**?**].

Lemma 5 (Burnside)**.** *If G is a finite group and P is a Sylow p-subgroup of G such that $N_G(P) = C_G(P)$, then P has a normal complement in G, i.e. there is $N \trianglelefteq G$ such that $G = NP$, and $N \cap P = 1$.*

Theorem 6. *Let G be a finite Frobenius group with complement H and kernel K. Assume that H is an abelian p-group. Then K is a normal subgroup of G.*

Proof. From $N_G(H) = H$ and the fact that H is abelian we obtain

$$H = N_G(H) \geq C_G(H) \geq H.$$

Therefore $N_G(H) = C_G(H)$. But it is easy to see that $(|H| : [G : H]) = 1$ from which it follows that H is a Sylow p-subgroup of G. Hence by Burnside's theorem H has a normal complement N in G, i.e. $G = NH$, $N \cap H = 1$, and $N \trianglelefteq G$. Now by Lemma 4, $N \subseteq K$. But from $G = NH$ we obtain $|N| = [G : H]$, and since $|K| = [G : H]$, it follows $|N| = |K|$. This implies that $K = N \trianglelefteq G$. □

Corollary 7. *Let G be a finite Frobenius group with complement H and kernel K. Suppose H is centralized by a Sylow p-subgroup of G. Then K is a normal subgroup of G.*

Proof. By assumption $H \leq C_G(P)$ where P is a Sylow p-subgroup of G. But it can be proved that if $1 \neq x \in H$, then $C_G(\langle x \rangle) \leq H$, therefore $C_G(P) \leq H$, it follows that $C_G(P) = H = N_G(P)$. Now by Burnside's theorem there is a normal subgroup N of G such that $G = NP$. Therefore $|N| = [G : P]$, By Lemma 4, $N \subseteq K$, hence $|N| = [G : P] \leq |K| = [G : H]$. Therefore $|P| \geq |H|$ implying $P = H$ and by Theorem 6, K is a normal subgroup of G. □

Acknowledgement

The author would like to thank the referee for his/her comments which improved this paper.

References

[1] L. Dornhoff, *Group representation theory. Part A: Ordinary representation theory*, vol. 7 of Pure and Applied Mathematics, Marcel Dekker, Inc., New York, 1971.

[2] W. Feit, *On a conjecture of Frobenius*, Proc. Amer. Math. Soc., 7 (1956), pp. 177–187.

[3] P. Flavell, *A note on Frobenius groups*, J. Algebra, 228 (2000), pp. 367–376.

[4] G. Frobenius, *Über auflösbare Gruppen. IV.*, Berl. Ber., 1901 (1901), pp. 1216–1230.

[5] L. C. Grove, *Groups and characters*, Pure and Applied Mathematics, John Wiley & Sons, Inc., New York, 1997. A Wiley-Interscience Publication.

[6] M. Hall, Jr., *The theory of groups*, The Macmillan Company, New York, N.Y., 1959.

[7] B. Huppert, *Endliche Gruppen. I*, Die Grundlehren der mathematischen Wissenschaften, Band 134, Springer-Verlag, Berlin-New York, 1967.

[8] W. Knapp and P. Schmid, *A note on Frobenius groups*, J. Group Theory, 12 (2009), pp. 393–400.

[9] ——, *Frobenius groups of low rank*, Arch. Math. (Basel), 117 (2021), pp. 121–127.

[10] H. Kurzweil and B. Stellmacher, *The theory of finite groups*, Universitext, Springer-Verlag, New York, 2004. An introduction, Translated from the 1998 German original.

[11] D. Passman, *Permutation groups*, W. A. Benjamin, Inc., New York-Amsterdam, 1968.

[12] R. H. Shaw, *Remark on a theorem of Frobenius*, Proc. Amer. Math. Soc., 3 (1952), pp. 970–972.

[13] T. C. Tao, *A Fourier-analytic proof of Frobenius' theorem.* https://terrytao.wordpress.com/2013/05/24/a-fourier-analytic-proof-of-frobeniuss-theorem/, 2013.

Please cite this article using:

Mohammadreza Darafsheh, Some properties of the finite Frobenius groups, AUT J. Math. Comput., 4(1) (2023) 57-61
DOI: 10.22060/AJMC.2022.21224.1080