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Original Article

On general (α, β)-metrics with Cartan torsion, mean Cartan torsion and Landsberg curvature

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ABSTRACT: In this paper, we derive a formula for the (mean) Cartan torsion of a class of general (α, β)-metrics. Also, we study weak Landsberg general (α, β)-metrics under a certain condition.

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1. Introduction

A great differential geometer of twentieth century, Chern, say that Finsler geometry is only Riemannian geometry without the quadratic restriction on its metrics [4]. In study of Finsler geometry, we mostly encounter long and tangled calculations. However, often we consider Finsler metrics with certain symmetries, that would make things much easier.

The concept of (α, β)-metrics was introduced by M. Matsumoto in 1972 [6] and studied by many authors ([11], [12] and [17]), as a generalization of Randers metrics, and the Randers metrics was introduced by Randers [7]. The (α, β)-metrics are of the form $F = \alpha\phi(s)$, where ϕ is a C^∞ positive function and $s = \frac{\beta}{\alpha}$.

The metrics in the form

$$F = \alpha\phi(b^2, s), s := \frac{\beta}{\alpha} \tag{1}$$

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are called general (α, β)-metrics, which were firstly introduced in 2012 by C. Yu and H. Zhu in [13], where $\phi = \phi(b^2, s)$ is a C^∞ positive function, $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form on an n-dimensional manifold M, and $b := b(x) = ||\beta(x)||_\alpha$ is a norm of β with respect to α, $b^2 := ||\beta||^2_\alpha$.
In particular, if $\phi_1 = 0$, then $F = \alpha\phi(s)$ are said to be (α, β)-metrics [5], where ϕ_1 means derivative of ϕ w. r. to first variable $t := b^2$. We will denote by ϕ_2 derivative of ϕ w. r. to second variable s.

This class of metrics not only generalize (α, β)-metrics, but also connect spherically symmetric metrics [8]. It is to note that the general (α, β) metric includes an interesting family of metric constructed by Bryant [2]. Bryant metrics are rectilinear Finsler metrics on the unit sphere S^n with flag curvature $K = 1$.

2. Preliminaries

Suppose M be n-dim C^∞-manifold. Tangent space denoted by T_xM at $x \in M$ and on M tangent bundle are denoted by $TM := \bigcup_{x\in M} T_xM$. The element of TM is of type (x, y), as $x \in M$ and $y \in T_xM$. Let $TM_0 = TM \setminus \{0\}$.

Definition 2.1. *If a metric function $F : TM \to [0, \infty)$ on M satisfy the following conditions:*
(i) F is C^∞ on TM_0,
(ii) F is positively 1-homogeneous on fibers of tangent bundle TM and
(iii) the Hessian of $\frac{F^2}{2}$ with components $g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ is positive definite on TM_0, then pair $F^n = (M, F)$ is called a Finsler space of dimension n. F is called fundamental function and the tensor g with components g_{ij} is called the fundamental tensor of the Finsler space.

Proposition 2.2. *[13] $F = \alpha\phi(b^2, s)$ is a Finsler metric on M, which is n-dimentional manifold, for some Riemannian metric α and 1-form β with $||\beta||_\alpha < b_0$ iff $\phi = \phi(b^2, s)$ is a positive C^∞ function satisfying*

$$\phi - s\phi_s > 0, \phi - s\phi_s + (b^2 - s^2)\phi_{ss} > 0, \quad \text{when} \quad n \geq 3 \tag{2}$$

or $\phi - s\phi_s + (b^2 - s^2)\phi_{ss} > 0$, when $n = 2$, where s and b are arbitrary numbers with $|s| \leq b < b_0$.

Firstly L.Zhou [15] introduced spherically symmetric metrics, which are a specific class of general (α, β)-metrics manifested by $F = \alpha\phi(b^2, s)$, where $\alpha = |y|$ is Euclidean metric and $\beta =< x, y >$ is a 1-form in $\mathbb{R}^n$. Nowdays, few development has been made on spherically symmetric metrics ([9], [15] and [16]). It is clear that Berwald metric, Funk metric and Bryant's metric belong to spherically symmetric metrics with constant flag curvature $K = 0$, $K = -1$ and $K = 1$ respectively and they are locally projectively flat ([14] and [3]).

Proposition 2.3. *[13] For a general (α, β)-metric $F = \alpha\phi(b^2, s)$, the fundamental tensor is written as*

$$g_{ij} = \rho a_{ij} + \rho_0 b_i b_j + \rho_1(b_i \alpha_{y^j} + b_j \alpha_{y^i}) + \rho_2 \alpha_{y^i}\alpha_{y^j}, \tag{3}$$

where

$\rho = \phi(\phi - s\phi_s)$, $\rho_0 = \phi\phi_{ss} + \phi_s^2$, $\rho_1 = (\phi - s\phi_s)\phi_s - s\phi\phi_{ss}$, $\rho_2 = -s\rho_1$.

Moreover,

$$g^{ij} = \rho^{-1}\left\{a^{ij} + \eta b^i b^j + \eta_0\alpha^{-1}(b^i y^j + b^j y^i) + \eta_1\alpha^{-2}y^i y^j\right\}, \tag{4}$$

where

$$(g^{ij}) = (g_{ij})^{-1}, (a^{ij}) = (a_{ij})^{-1}, b^i = a^{ij}b_j,$$

$$\eta = -\frac{\phi_{ss}}{(\phi - s\phi_s + (b^2 - s^2)\phi_{ss})},$$

$$\eta_0 = -\frac{(\phi - s\phi_s)\phi_s - s\phi\phi_{ss}}{\phi(\phi - s\phi_s + (b^2 - s^2)\phi_{ss})},$$

$$\eta_1 = -\frac{(s\phi + (b^2 - s^2)\phi_s)(\phi - s\phi_s)\phi_s - s\phi\phi_{ss}}{\phi^2(\phi - s\phi_s + (b^2 - s^2)\phi_{ss})}.$$

The spray coefficients G^i of general (α,β)-metric $F=\alpha\phi(b^2,s)$ and spray coefficients G^i_α of Riemannian metric α are related by [13],

$$\begin{aligned} G^i = G^i_\alpha + \alpha Q s^i_0 + \left\{\Theta(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha\Omega(r_0+s_0)\right\}\frac{y^i}{\alpha} \\ + \left\{\Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha \mathit{\Pi}(r_0+s_0)\right\} b^i - \alpha^2 R(r^i + s^i), \end{aligned} \tag{5}$$

where

$$Q = \frac{\phi_s}{\phi - s\phi_s}, \qquad \Theta = \frac{(\phi - s\phi_s)\phi_s - s\phi\phi_{ss}}{2\phi(\phi - s\phi_s + (b^2-s^2)\phi_{ss})},$$

$$R = \frac{\phi_b}{\phi - s\phi_s}, \qquad \mathit{\Pi} = \frac{(\phi - s\phi_s)\phi_{bs} - s\phi_b\phi_{ss}}{(\phi - s\phi_s)(\phi - s\phi_s + (b^2-s^2)\phi_{ss})},$$

$$\Omega = \frac{2\phi_b}{\phi} - \frac{s\phi + (b^2-s^2)\phi_s}{\phi}\mathit{\Pi},$$

$$\Psi = \frac{\phi_{ss}}{2\phi(\phi - s\phi_s + (b^2-s^2)\phi_{ss})}.$$

As β is a closed and conformal 1-form, that is satisfy $b_{i|j} = c a_{ij}$, where $b_{i|j}$ is covarient derivative of β w. r. to α and $c = c(x) \neq 0$ is a scalar function, then

$$r_{00} = c\alpha^2, \quad r_0 = c\beta, \quad r = cb^2, r^i = cb^i, \quad s^i_0 = s_0 = s^i = 0. \tag{6}$$

Substituting equation (6) in equation (5), we have

$$G^i = G^i_\alpha + c\alpha^2 E l^i + c\alpha^2 H b^i, \tag{7}$$

where

$$H = \frac{\phi_{ss} - 2(\phi_1 - s\phi_{bs})}{2(\phi - s\phi_s + (b^2-s^2)\phi_{ss})},$$

$$E = \frac{\phi_s + 2s\phi_b}{2\phi} - H\frac{s\phi + (b^2-s^2)\phi_s}{\phi}.$$

3. The Cartan torsion of a general (α,β)-metric

Suppose

$$\rho_s = \rho_1(\rho_1)_s = -sT,$$

where

$$\rho = \phi(\phi - s\phi_s), \quad \rho_1 = (\phi - s\phi_s)\phi_s - s\phi\phi_{ss}, \quad T = 3\phi_s\phi_{ss} + \phi\phi_{sss}.$$

Let

$$(\rho_0)_s = T(\rho_2)_s = s^2 T - \rho_1,$$

where $\rho_0 = \phi\phi_{ss} + \phi_s^2, \rho_2 = -s\rho_1$.

The Cartan torsion is defined in [5] as

$$C_{ijk} = \frac{1}{2}\frac{g_{ij}}{\partial y^k} \text{ and given by with the help of equation (3).}$$

$$C_{ijk} = \frac{1}{2\alpha}\left[\left\{\rho_1(b_k - s\frac{y_k}{\alpha})a_{ij} - sTb_ib_j\frac{y_k}{\alpha} + (s^2T - \rho_1)b_i\frac{y_k}{\alpha}\frac{y_j}{\alpha}\right\} + Tb_ib_jb_k + s(3\rho_1 - s^2T)\frac{y_i}{\alpha}\frac{y_j}{\alpha}\frac{y_k}{\alpha}\right]. \tag{8}$$

Proposition 3.1. *Suppose $F = \alpha\phi(b^2,s)$ be a general (α,β)-metric on M, then the Cartan torsion is given by equation (8).*

4. Mean Cartan torsion of a general (α, β)-metric

The Mean Cartan torsion is defined in [5] as

$I_i = g^{jk} C_{ijk}$ and given by with the help of Cartan torsion (8) and equation (4).

$$I_i = \frac{\rho^{-1}}{2\alpha}\left\{a^{jk} + \eta b^j b^k + \eta_0 \alpha^{-1}(b^j y^k + b^k y^j) + \eta_1 \alpha^{-2} y^j y^k\right\}$$
$$\left\{\left(\rho_1 (b_k - s\frac{y_k}{\alpha}) a_{ij} - sTb_i b_j \frac{y_k}{\alpha} + (s^2 T - \rho_1) b_i \frac{y_k}{\alpha}\frac{y_j}{\alpha}\right) + Tb_i b_j b_k + s(3\rho_1 - s^2 T)\frac{y_i}{\alpha}\frac{y_j}{\alpha}\frac{y_k}{\alpha}\right\}.$$

Thus

$$I_i = \frac{1}{2\alpha}\left\{\rho^{-1}\rho_1\left[(n+1) + 3\eta(b^2 - s^2)\right] + (b^2 - s^2)\rho^{-1} T\left[1 + \eta(b^2 - s^2)\right]\right\}\left\{b_i - s\frac{y_i}{\alpha}\right\}. \tag{9}$$

Proposition 4.1. *Let $F = \alpha\phi(b^2, s)$ be a general (α, β)-metric on M, then mean Cartan torsion is given by equation (9).*

5. Landsberg curvature for a general (α, β)-metric

The Landsberg curvature defined in [1] and expressed by

$$L_{jkl} := -\frac{1}{2} y^m g_{im} B^i_{jkl},$$

where

$$B^i_{jkl} := \frac{\partial^3 G^i}{\partial y^j \partial y^k \partial y^l}.$$

A metric is a Landsberg metric iff $L_{jkl} = 0$.

$$\begin{aligned} L_{ijk} = -\frac{1}{2} c\phi\Big[& L_1 b_i b_j b_k + s(3L_2 - s^2 L_1)\frac{y_i}{\alpha}\frac{y_j}{\alpha}\frac{y_k}{\alpha} \\ & + \left\{L_2 (b_i - s\frac{y_i}{\alpha}) a_{jk} - sL_1 b_i b_j b_k + (s^2 L_1 - L_2) y_i \frac{y_j}{\alpha}\frac{y_k}{\alpha}\right\}\Big], \end{aligned} \tag{10}$$

where

$$\begin{aligned} L_1 &= \phi P_{sss} + Q_{sss}\left[\phi s + \phi_s (b^2 - s^2)\right] + 3P_{ss}\phi_s, \\ L_2 &= -s\phi P_{ss} + \phi_s (P - sP_s) + \left[\phi s + \phi_s (b^2 - s^2)\right](Q_s - sQ_{ss}), \\ Q &= \frac{\phi_{ss} - 2(\phi_b - s\phi_{bs})}{2[\phi - s\phi_s + (b^2 - s^2)\phi_{ss}]} \end{aligned}$$

and

$$P = \frac{\phi_s + 2s\phi_b}{2\phi} - Q\frac{s\phi + (b^2 - s^2)\phi_s}{\phi}.$$

Proposition 5.1. *Suppose $F = \alpha\phi(b^2, s)$ be a general (α, β)-metric on M, then Landsberg curvature is given by equation (10).*

6. Weak Landsberg curvature for a general (α,β)-metric

A weaker non-Riemannian quantity $J = J_i dx^i$ than Landsberg curvature L, where

$$J_i = g^{jk} L_{ijk},$$

here J mean Landsberg curvature. A metric is said to be weak Landsberg metric if its mean Landsberg curvature vanishes [10].
That is a metric is called Weak Landsberg if $J = 0$.

Now, $J_i = g^{jk} L_{ijk}$ is given by with the help of equation (4) and (10).

$$J_i = g^{jk} L_{ijk} = -\frac{c\phi\rho^{-1}}{2}\left[a^{jk} + \eta b^j b^k + \eta_0 \alpha^{-1}(b^j y^k + b^k y^j) + \eta_1 \alpha^{-2} y^j y^k\right] \times \left[L_1 b_i b_j b_k + s(3L_2 - s^2 L_1)\frac{y_i}{\alpha}\frac{y_j}{\alpha}\frac{y_k}{\alpha} + L_2(b_k - s\frac{y_k}{\alpha})a_{ij} - sL_1 b_i b_j b_k + (s^2 L_1 - L_2) y_i \frac{y_j}{\alpha}\frac{y_k}{\alpha}\right].$$

Thus

$$J_i = -\frac{c\phi\rho^{-1}}{2}\left[L_2\left\{(n+1) + 3\eta(b^2 - s^2)\right\} + L_1(b^2 - s^2)(1 + \eta(b^2 - s^2))\right](b_i - \frac{sy^i}{\alpha}), \tag{11}$$

where

$$L_1 = \phi P_{sss} + Q_{sss}\left[\phi s + \phi_s(b^2 - s^2)\right] + 3P_{ss}\phi_s$$

and

$$L_2 = -s\phi P_{ss} + \phi_s(P - sP_s) + \left[\phi s + \phi_s(b^2 - s^2)\right](Q_s - sQ_{ss}).$$

Proposition 6.1. *Suppose $F = \alpha\phi(b^2, s)$ be a general (α,β)-metric on M, then mean Landsberg curvature is given by equation (11).*

Now, for $J_i = 0$, this implies from equation (11) that $L_1 = 0$ and $L_2 = 0$. Therefore

$$\phi P_{sss} + Q_{sss}\left[\phi s + \phi_s(b^2 - s^2)\right] + 3P_{ss}\phi_s = 0 \tag{12}$$

and

$$-s\phi P_{ss} + \phi_s(P - sP_s) + \left[\phi s + \phi_s(b^2 - s^2)\right](Q_s - sQ_{ss}) = 0. \tag{13}$$

Let,

$$\pi = \phi s + \phi_s(b^2 - s^2) \tag{14}$$

$$\pi_s = \phi - s\phi_s + \phi_{ss}(b^2 - s^2) \tag{15}$$

$$\tau = P - sP_s \tag{16}$$

$$\tau_s = -sP_{ss} \tag{17}$$

$$\tau_{ss} = -[sP_{sss} + P_{ss}] \tag{18}$$

$$(\phi\tau)_s = [\phi(P - sP_s)]_s = \phi_s(P - sP_s) - s\phi P_{ss} \tag{19}$$

$$(\phi\tau)_{ss} = \phi_{ss}(P - sP_s) - 2s\phi_s P_{ss} - \phi P_{ss} - s\phi P_{sss}. \tag{20}$$

Differentiating equation (13), we have

$$-\phi P_{ss} - 2s\phi_s P_{ss} - s\phi P_{sss} + \phi_{ss}(P - sP_s) + \left[\phi - s\phi_s + \phi_{ss}(b^2 - s^2)\right] \times (Q_s - sQ_{ss}) - s\left[\phi s + \phi_s(b^2 - s^2)\right] Q_{sss} = 0. \tag{21}$$

Using equations (14), (15), (20) and (21), we get

$$(\phi\tau)_{ss} + \pi_s(Q_s - sQ_{ss}) - s\pi Q_{sss} = 0.$$

That is

$$\pi_s(Q_s - sQ_{ss}) = s\pi Q_{sss} - (\phi\tau)_{ss}. \tag{22}$$

Using equations (20) and (22)

$$\pi_s(Q_s - sQ_{ss}) = s\pi Q_{sss} - \phi_{ss}(P - sP_s) + 2s\phi_s P_{ss} + \phi P_{ss} + s\phi P_{sss}$$
$$= s(\pi Q_{sss} + \phi P_{sss} + 3\phi_s P_{ss}) - \phi_{ss}(P - sP_s) + \phi P_{ss} - s\phi_s P_{ss}.$$

Using equation (12)

$$\pi_s(Q_s - sQ_{ss}) = -\phi_{ss}(P - sP_s) + P_{ss}(\phi - s\phi_s).$$
$$\pi_s(Q_s - sQ_{ss}) = -\left[\tau\phi_{ss} + \frac{(\phi - s\phi_s)}{s}\tau_s\right]. \tag{23}$$

Using equations (13) and (19), we have

$$(\phi\tau)_s + \pi(Q_s - sQ_{ss}) = 0. \tag{24}$$

Therefore

$$Q_s - sQ_{ss} = -\frac{(\phi\tau)_s}{\pi} = -\frac{\phi\tau_s + \phi_s\tau}{\pi}. \tag{25}$$

From (23) and (25), we have

$$\pi_s\frac{\phi\tau_s + \phi_s\tau}{\pi} = \left[\tau\phi_{ss} + \frac{(\phi - s\phi_s)}{s}\tau_s\right].$$
$$\pi\left(\tau\phi_{ss} + \frac{\phi - s\phi_s}{s}\tau_s\right) - \pi_s\left(\phi\tau_s + \phi_s\tau\right) = 0. \tag{26}$$

Case I: When $\tau = 0$, from equation (16), we have $P - sP_s = 0$.
After solving we obtain $P = st_0(b)$.
From equation (13) and (14), we get $\pi(Q_s - sQ_{ss}) = 0$.
We obtain $Q = s^2 t_1(b) + t_2(b)$.

Case II: When $\tau \neq 0$, (nowhere).
Multiplying equation (26) by $\frac{1}{\tau}$, we have

$$\left[(\phi - s\phi_s)\pi - s\pi_s\phi\right]\frac{\tau_s}{\tau} = s\left(\pi_s\phi_s - \pi\phi_{ss}\right).$$

Put the value of π and π_s, we have

$$\left(\phi\phi_s - s\phi_s^2 - s\phi\phi_{ss}\right)\left[(b^2 - s^2)\frac{\tau_s}{\tau} - s\right] = 0. \tag{27}$$

Case II(a): Let

$$\left(\phi\phi_s - s\phi_s^2 - s\phi\phi_{ss}\right) = 0.$$

Put

$$\mu = \phi^2 \quad \Rightarrow \quad \mu_s = 2\phi\phi_s \quad \Rightarrow \quad \mu_{ss} = 2\left[\phi\phi_{ss} + \phi_s^2\right].$$

Therefore

$$2\phi\phi_s - 2s(\phi\phi_{ss} + \phi_s^2) = 0.$$
$$\Rightarrow \mu_s - s\mu_{ss} = 0, \quad \mu = A(b) + B(b)s^2.$$

That is

$$\phi = \sqrt{A(b) + B(s)s^2}.$$

The corresponding general (α, β)-Finsler metric is Riemannian metric.
Put the value of ϕ in P and Q, where

$$P = \frac{\phi_s + 2s\phi_b}{2\phi} - Q\frac{s\phi + (b^2 - s^2)\phi_s}{\phi} \tag{28}$$

and

$$Q = \frac{\phi_{ss} - 2(\phi_b - s\phi_{bs})}{2[\phi - s\phi_s + (b^2 - s^2)\phi_{ss}]}. \tag{29}$$

Case II(b): Let

$$\left(\phi\phi_s - s\phi_s^2 - s\phi\phi_{ss}\right) \neq 0. \tag{30}$$

We have

$$P = \frac{c_2(b)\sqrt{b^2 - s^2}}{b^2} + sc_3(b). \tag{31}$$

$$Q = \frac{-\sqrt{b^2 - s^2}}{b^4} sc_2(b) + \frac{s^2}{2} c_0(b) + c_1(b). \tag{32}$$

From equations (28), (29) and (31), we have

$$\frac{\phi_s + 2s\phi_b}{2\phi} - \frac{\phi_{ss} - 2(\phi_b - s\phi_{bs})}{2[\phi - s\phi_s + (b^2 - s^2)\phi_{ss}]} \frac{s\phi + (b^2 - s^2)\phi_s}{\phi} = \frac{c_2(b)\sqrt{b^2 - s^2}}{b^2} + sc_3(b). \tag{33}$$

From equations (29) and (32), we have

$$\frac{\phi_{ss} - 2(\phi_b - s\phi_{bs})}{2[\phi - s\phi_s + (b^2 - s^2)\phi_{ss}]} = \frac{-\sqrt{b^2 - s^2}}{b^4} sc_2(b) + \frac{s^2}{2} c_0(b) + c_1(b). \tag{34}$$

From equations (33) and (34), we get

$$\begin{aligned} &\frac{\phi_s + 2s\phi_b}{2\phi} + \left[\frac{\sqrt{b^2 - s^2}}{b^4} sc_2(b) - \frac{s^2}{2} c_0(b) - c_1(b) \right] \frac{s\phi + (b^2 - s^2)\phi_s}{\phi} \\ &= \frac{c_2(b)\sqrt{b^2 - s^2}}{b^2} + sc_3(b). \end{aligned} \tag{35}$$

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