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Original Article

Inverse minimax circle location problem with variable coordinates

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ABSTRACT: Traditionally, the minimax circle location problems concern finding a circle C in the plane such that the maximum distance from the given points to the circumference of the circle is minimized. The radius of the circle can be fixed or variable. In this paper we consider the inverse case, that is: a circle C with radius r_0 is given and we want to modify the coordinate of existing points with the minimum cost such that the given circle becomes optimal. Mathematical models and some properties of the cases that circle C becomes optimal with comparing to all other circles, and circle C becomes the best circle with comparing to the circles with radius r_0 are presented.

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1. Introduction

Today, the circle location models have found a lot of interest. They have been applied to make many decisions in the real world such as locating circular facilities, e.g. a circular irrigation pipe, circular conveyor belts, or ring roads and out-of-roundness problem (see e.g. Drezner et al. [12]). In a circle location problem a set of points, that are supposed to be the location of clients, are given and the goal is finding a circle in the plane such that the distances from the given points to the circumference of the circle is minimized. The radius of circle can be fixed or variable. Two important objective functions in the circle location problems are: 1-minimizing the sum of distances from clients to the circumference of the circle (the minisum model), 2-minimizing the maximum of these distances (the minimax model).

Brimberg et al. [7] considered the minisum circle location problem. They showed that if the radius of the circle is variable, then there exists an optimal circle passing through at least two of the existing facilities. The discrete case of minisum circle location problem was investigated by Labbé et al. [19]. Recently, Gholami and Fathali [16] studied the minisum circle location problem with positive and negative weights on the existing points. Using some properties on this problem they presented a meta-heuristic algorithm to solve it.

The minimax circle location problem in the plane has been investigated by Brimberg et al. [6]. They considered the problems with fix and variable radius. Drezner et al. [12] proposed the apply of the minimax circle location model to the out-of-roundness problem. They suggested an approximate solution method to solve this problem.

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In the most classical location problems, finding the optimal locations of the facilities with respect to different criteria such as time, cost and distances between the clients and facilities are considered. Among them the Fermat-Weber, p-median and p-center problems are three basic classic facility location models. In the Fermat-Weber problem the goal is finding the location of a new facility in the plane such that the sum of weighted distances from the new facility to the clients is minimized. The p-median and p-center problems ask to select the location of p facilities from a set of candidate points such that respectively the sum and maximum weighted distances from the clients to the closest facility is minimized.

In many real applications of location models, the facilities may already exist and the goal is changing the parameters of the problem with minimum cost, such that the given locations are optimal. These kind of location models are concerned to inverse optimization and called inverse location problems. Among many researches in this area, Cai et al. [11] showed that the inverse center problem is NP-hard. The inverse median problem has been considered by Burkard et al. [9], [8] and Baroughi-Bonab et al. [5]. Burkard et al. [9] suggested an $O(n \log n)$ algorithm for the inverse 1-median problem on a tree. Then Galavii [14] improved the time complexity of this problem to linear time. The inverse 1-median problem on a cycle has been investigated by Burkard et al. [10]. They presented an $O(n^2)$ algorithm for this problem. Alizadeh et al. [3] presented an $O(n \log n)$ time algorithm for the inverse 1-center problem with edge length augmentation on trees. Later, Alizadeh and Burkard [2] showed that the inverse 1-center problem can be solved in $O(n^2)$ time. Guan and Zhang [18] and Nguyen and Sepasian [26] considered the inverse 1-median and 1-center problems on trees with Chebyshev and Hamming norms, respectively. Sepasian and Rahbarnia [29] solved the inverse 1-median problem with varying vertex and edge length on trees. When the underlying network is a block graph, Nguyan [25] proposed a solution algorithm for the inverse 1-median problem with variable vertex weight. Nazari et al. [24] and Nazari and Fathali [23] investigated the inverse of backup 2-median problem with variable edge length and vertex weight. Alizadeh et al. [1] and Alizadeh and Etemad [4] developed some combinatorial algorithms for the inverse obnoxious median and center problems, respectively. Omidi et al. [28] and Nazari and Fathali [22] investigated the inverse balanced facility location problem with varying edge lengths and varing vertex weights on trees, respectively. Omidi and Fathali [27] considered the inverse model of the single facility location problem on a tree with balancing on the distance of server to clients. Fathali [13] proposed an algorithm for general case of inverse continuous facility location models. Recently, Golami and Fathali in [15] and [17] presented mathematical models for inverse minimax and minisum circle location problems with variable weights of existing points, respectively.

In this paper, we consider the inverse minimax circle location problem with variable coordinates. Let a circle with radius r_0 be given. Two cases are considered: 1- The fixed radius problem. In this case, we want to modify the coordinate of the given points with minimum cost such that the given circle becomes the optimal with comparing to all other circles with radius r_0. 2- The problem with variable radius. This problem asks to modify the coordinates of the given points with minimum cost such that the given circle becomes optimal with comparing to all other circles with arbitrary positive radius. Using some properties on minimax circle location problems we present mathematical models for solving the inverse problems.

In what follows, the definition of minimax circle location problem with fixed and variable radius and some previously presented properties on these problems are given in sections 2 and 4, respectively. In sections 3 and 5 mathematical models for the fixed and variable radius cases of inverse model are presented, respectively. Section 6 contains the summary and some conjectures for the future works on the inverse minimax circle location problem.

2. The minimax circle location with fixed radius

Let the location of n clients $\mathbf{p}_1 = (p_{11}, p_{12}), \mathbf{p}_2 = (p_{21}, p_{22}), \ldots, \mathbf{p}_n = (p_{n1}, p_{n2})$, and a fixed radius $r_0 > 0$ be given in the plane. The minimax circle location problem asks to find the circle $C = C(\mathbf{x}, r_0)$ with center $\mathbf{x}$ and radius r_0 such that the largest distance between the circumference of the circle C and the clients is minimized, i.e.

$$\min_{\mathbf{x}} g(\mathbf{x}) = \max_{j=1,\ldots,n} |d(\mathbf{x}, \mathbf{p}_j) - r_0|, \tag{1}$$

where $d(\mathbf{x}, \mathbf{p}_j)$ represents the distance between the point $\mathbf{p}_j$ and the center of circle C. Let $d_j(C) = |d(\mathbf{x}, \mathbf{p}_j) - r_0|$, for $j = 1, \ldots, n$, be the distance between the circumference of the circle C and client j.

In the cases $r_0 = 0$ and $r_0 = \infty$, the problem reduces to the minimax single facility location and minimax line location problems, respectively. These two cases have been already studied in the facility location literature, e.g., see Love et al. [21] for the minimax single facility location and Lee and Wu [20] for the minimax line location problem.

The proof of the following basic properties on minimax circle location problem with fixed radius can be found in [6].

Theorem 2.1. *Let $\mathbf{x}_p$ denote the solution of minimax single facility location problem, also suppose $r_{\max} = \max_j\{d(\mathbf{x}_p, \mathbf{p}_j)\}$ and $r_{\min} = \min_j\{d(\mathbf{x}_p, \mathbf{p}_j)\}$, as well as let $d_{ij} = d(\mathbf{p}_i, \mathbf{p}_j)$ for all pairs (i, j), and $d_{\max} = \max_{i<j}\{d_{ij}\} = d_{rs}$, and $\mathbf{x}_m$ be the mid-point of the line segment $[\mathbf{p}_r, \mathbf{p}_s]$,*

1. *If $r_0 \in [0, r_m]$, where $r_m = \frac{r_{\min}+r_{\max}}{2}$, then $\mathbf{x}_p$ is the optimal solution of problem (1).*
2. *If $r_0 > r_m$ and $g(\mathbf{x}_m) = \frac{d_{\max}}{2} - r_0$, then $\mathbf{x}_m$ is the optimal solution of problem (1).*
3. *Let C^* be the optimal circle. If $g(\mathbf{x}_m) > \frac{d_{\max}}{2} - r_0$, then at least three extreme points, $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$ exist such that $d_i(C^*) = d_j(C^*) = d_k(C^*)$; that is, the subset of points at maximum distance from an optimal circle must have cardinal of at least three.*

Using Theorem 2.1, to find the optimal solution of the Minimax Circle Location Problem with Fixed radius (MCLPF), we need to consider all triples.

Brimberg et al. [6] presented a method for finding the optimal solution of MCLPF based on enumerating all triples $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k$ for all (i, j, k). They first check the optimal solution of the associated minimax single facility location problem. If this solution satisfies the first part of Theorem 2.1, then the algorithm is ended. Otherwise they determine the subset of extreme points, associated with the solution of the minimax single facility location problem, satisfied the third part of Theorem 2.1, to obtain the new solution that is the best one between solutions obtained from all triples. If the new solution is the same as the previouse then the algorithm ended. Otherwise all triples associated with the new solution are considered to obtain another new solution. This procedure continues until can not find a new solution. In the next section we use this method to present an approach for solving the Inverse Minimax Circle Location Problem with Fixed radius (IMCLPF).

3. The inverse minimax circle with fixed radius

In the inverse case of the minimax circle location problem with fixed radius, a circle $C^* = C(\mathbf{p}^*, r_0)$ is given and we want to modify the coordinates of the existing points $\mathbf{p}_i$, for $i = 1, \dots, n$, to $\mathbf{p}_i^* = \mathbf{p}_i + \mathbf{q}_i^+ - \mathbf{q}_i^-$ with minimum cost, such that the circle C^* is optimal compared to any other circle with radius r_0. Where $\mathbf{q}_i^+ \geq 0$ and $\mathbf{q}_i^- \geq 0$ are the vectors of augmenting and reduction of the coordinates of point $\mathbf{p}_i$ and bounded from above by $\mathbf{u}_i^+$ and $\mathbf{u}_i^-$, respectively. Furthermore, suppose that $\mathbf{c}_i^+$ and $\mathbf{c}_i^-$ denote the cost vectors of augmenting and reduction of per unit of coordinates of point $\mathbf{p}_i$, respectively. Then IMCLPF can be modeled as follow.

$$\min \sum_{i=1}^{n} ({\mathbf{c}_i^+}^T \mathbf{q}_i^+ + {\mathbf{c}_i^-}^T \mathbf{q}_i^-) \tag{2}$$
$$s.t. \tag{3}$$
$$\max_i\{|d(\mathbf{p}^*, \mathbf{p}_i^*) - r_0|\} \leq \max_i\{|d(\mathbf{x}, \mathbf{p}_i^*) - r_0|\} \qquad \forall \mathbf{x} \in \mathbb{R}^2 \tag{4}$$
$$\mathbf{p}_i^* = \mathbf{p}_i + \mathbf{q}_i^+ - \mathbf{q}_i^- \qquad i = 1, \dots, n, \tag{5}$$
$$0 \leq \mathbf{q}_i^+ \leq \mathbf{u}_i^+ \qquad i = 1, \dots, n, \tag{6}$$
$$0 \leq \mathbf{q}_i^- \leq \mathbf{u}_i^- \qquad i = 1, \dots, n. \tag{7}$$

This model is nonlinear with infinity constraints. Therefor, we are going to present a simpler model for finding the optimal solution by considering the model of Brimberg et al. [6].

Note that, by Theorem 2.1 the center of optimal circle is either the solution of the minimax single facility location problem or a point $\mathbf{p}^*$ associated to at least three extreme points such that the value of the objective function for $\mathbf{p}^*$ is better than any other point associated to three extreme points. We consider these cases in separate models.

First, let $\mathbf{p}^*$ be the solution of the minimax single facility location problem, therefore, we consider the following model.

$$(P_0) \qquad \min \sum_{i=1}^{n} ({\mathbf{c}_i^+}^T \mathbf{q}_i^+ + {\mathbf{c}_i^-}^T \mathbf{q}_i^-) \tag{8}$$
$$s.t. \tag{9}$$
$$\max_i\{d(\mathbf{p}^*, \mathbf{p}_i^*)\} \leq \max_i\{d(\mathbf{x}, \mathbf{p}_i^*)\} \qquad \forall \mathbf{x} \in \mathbb{R}^2 \tag{10}$$
$$\mathbf{p}_i^* = \mathbf{p}_i + \mathbf{q}_i^+ - \mathbf{q}_i^- \qquad i = 1, \dots, n, \tag{11}$$
$$0 \leq \mathbf{q}_i^+ \leq \mathbf{u}_i^+ \qquad i = 1, \dots, n, \tag{12}$$
$$0 \leq \mathbf{q}_i^- \leq \mathbf{u}_i^- \qquad i = 1, \dots, n. \tag{13}$$

Since the optimal solution of minimax single facility location problem has at least two extreme points with the maximum distance to the optimal point, it is sufficient to consider all pairs $(\mathbf{p}_j, \mathbf{p}_k)$ for all $j < k = 1, \ldots, n$. If the pair $(\mathbf{p}_j, \mathbf{p}_k)$ has the maximum distance to the optimal solution, then there is a point $\mathbf{x}_{jk}$ which is a candidate point for the optimal solution, such that satisfies the following constraints.

$$d(\mathbf{x}_{jk}, \mathbf{p}_j) = d(\mathbf{x}_{jk}, \mathbf{p}_k), \tag{14}$$

$$\max_{i=1,\ldots,n} \{d(\mathbf{x}_{jk}, \mathbf{p}_i)\} = d(\mathbf{x}_{jk}, \mathbf{p}_j). \tag{15}$$

Note that, $\mathbf{p}^*$ can only be equal to the one of all $\mathbf{x}_{jk}$ that satisfy in (14) and (15), so we define the following binary variable.

$$y_{jk} = \begin{cases} 1 & \text{If } \mathbf{x}_{jk} \text{ is the optimal solution,} \\ 0 & \text{Otherwise.} \end{cases}$$

Thus the model (P_0) is converted to the following model:

$$
\begin{aligned}
(P_0) \qquad & \min \sum_{i=1}^{n} (\mathbf{c}_i^{+T} \mathbf{q}_i^{+} + \mathbf{c}_i^{-T} \mathbf{q}_i^{-}) && (16)\\
& s.t. && (17)\\
& \quad d(\mathbf{x}_{jk}, \mathbf{p}_j^*) = d(\mathbf{x}_{jk}, \mathbf{p}_k^*) \qquad \forall\ j < k = 1, \ldots, n, && (18)\\
& \quad \max_{i=1,\ldots,n} \{d(\mathbf{x}_{jk}, \mathbf{p}_i^*)\} = d(\mathbf{x}_{jk}, \mathbf{p}_j^*) \qquad \forall\ j < k = 1, \ldots, n, && (19)\\
& \quad \max_{i=1,\ldots,n} \{d(\mathbf{p}^*, \mathbf{p}_i^*)\} \le \max_i \{d(\mathbf{x}_{jk}, \mathbf{p}_i^*)\} \qquad \forall\ j < k = 1, \ldots, n, && (20)\\
& \quad \mathbf{p}^* = \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} y_{jk} \mathbf{x}_{jk}, && (21)\\
& \quad \sum_{j=1}^{n-1} \sum_{k=j+1}^{n} y_{jk} = 1, && (22)\\
& \quad \mathbf{p}_i^* = \mathbf{p}_i + \mathbf{q}_i^{+} - \mathbf{q}_i^{-} \qquad i = 1, \ldots, n, && (23)\\
& \quad 0 \le \mathbf{q}_i^{+} \le \mathbf{u}_i^{+} \qquad i = 1, \ldots, n, && (24)\\
& \quad 0 \le \mathbf{q}_i^{-} \le \mathbf{u}_i^{-} \qquad i = 1, \ldots, n, && (25)\\
& \quad y_{jk} \in \{0, 1\} \qquad \forall\ j < k = 1, \ldots, n. && (26)
\end{aligned}
$$

Now consider the case that $\mathbf{p}^*$ is not the solution of minimax single facility location problem. In this case we should consider all triples $(\mathbf{p}_j, \mathbf{p}_k, \mathbf{p}_l)$ and find the set of candidate points for each triple. For this purpose we should consider the following four cases:

1. The three points all are inside or outside the circle. Therefore, there exist a point $\mathbf{x}_{jkl}^{(1)}$ such that

$$d(\mathbf{x}_{jkl}^{(1)}, \mathbf{p}_j) = d(\mathbf{x}_{jkl}^{(1)}, \mathbf{p}_k) = d(\mathbf{x}_{jkl}^{(1)}, \mathbf{p}_l). \tag{27}$$

   In this case, for each triple $(\mathbf{p}_j, \mathbf{p}_k, \mathbf{p}_l)$ we define the following binary variable.

$$y_{jkl}^{(1)} = \begin{cases} 0 & \text{The three points } \mathbf{p}_j,\ \mathbf{p}_k \text{ and } \mathbf{p}_l \text{ are located on the same side of the circle,} \\ 1 & \text{Otherwise.} \end{cases}$$

   Then the following constraints with minimizing $My_{jkl}^{(1)}$ should be considered in the model.

$$|d(\mathbf{x}_{jkl}^{(1)}, \mathbf{p}_j) - d(\mathbf{x}_{jkl}^{(1)}, \mathbf{p}_k)| \le My_{jkl}^{(1)}, \tag{28}$$

$$|d(\mathbf{x}_{jkl}^{(1)}, \mathbf{p}_k) - d(\mathbf{x}_{jkl}^{(1)}, \mathbf{p}_l)| \le My_{jkl}^{(1)}, \tag{29}$$

   where M is a big enough value.

2. The points $\mathbf{p}_j$ and $\mathbf{p}_k$ are inside and the point $\mathbf{p}_l$ is outside the circle (or the points $\mathbf{p}_j$ and $\mathbf{p}_k$ are outside and the point $\mathbf{p}_l$ is inside the circle). In this case we should find the point $\mathbf{x}_{jkl}^{(2)}$ such that

$$r_0 - d(\mathbf{x}_{jkl}^{(2)}, \mathbf{p}_j) = r_0 - d(\mathbf{x}_{jkl}^{(2)}, \mathbf{p}_k) = d(\mathbf{x}_{jkl}^{(2)}, \mathbf{p}_l) - r_0,$$

or equivalently,

$$d(\mathbf{x}_{jkl}^{(2)}, \mathbf{p}_j) = d(\mathbf{x}_{jkl}^{(2)}, \mathbf{p}_k),$$
$$d(\mathbf{x}_{jkl}^{(2)}, \mathbf{p}_j) + d(\mathbf{x}_{jkl}^{(2)}, \mathbf{p}_l) = 2r_0.$$

Using these equations we will obtain a second order equation (see [6]) which may have two, one, or none root, depending on whether $\min_{\mathbf{x}\in B_{jk}}\{d(\mathbf{x}, \mathbf{p}_l) + d(\mathbf{x}, \mathbf{p}_k)\}$ is less than, equal to, or greater than $2r_0$, respectively. To enforce these cases to the model, the same as the first case, we define the binary variable $y_{jkl}^{(2)}$ as follow:

$$y_{jkl}^{(2)} = \begin{cases} 0 & \text{The points } \mathbf{p}_j \text{ and } \mathbf{p}_k \text{ are inside and the point } \mathbf{p}_l \text{ is outside the circle,} \\ 1 & \text{Otherwise.} \end{cases}$$

Then the following constraints with minimizing $My_{jkl}^{(2)}$ should be added to the model.

$$|d(\mathbf{x}_{jkl}^{(2)}, \mathbf{p}_j) + d(\mathbf{x}_{jkl}^{(2)}, \mathbf{p}_k) - 2r_0| \leq My_{jkl}^{(2)}, \tag{30}$$
$$|d(\mathbf{x}_{jkl}^{(2)}, \mathbf{p}_k) - d(\mathbf{x}_{jkl}^{(2)}, \mathbf{p}_l)| \leq My_{jkl}^{(2)}. \tag{31}$$

3. The points $\mathbf{p}_j$ and $\mathbf{p}_l$ are inside and the point $\mathbf{p}_k$ is outside the circle. Similar to the second case we add the following constraints with minimizing $My_{jkl}^{(3)}$ in the model.

$$|d(\mathbf{x}_{jkl}^{(3)}, \mathbf{p}_j) + d(\mathbf{x}_{jkl}^{(3)}, \mathbf{p}_l) - 2r_0| \leq My_{jkl}^{(3)}, \tag{32}$$
$$|d(\mathbf{x}_{jkl}^{(3)}, \mathbf{p}_k) - d(\mathbf{x}_{jkl}^{(3)}, \mathbf{p}_l)| \leq My_{jkl}^{(3)}, \tag{33}$$

where

$$y_{jkl}^{(3)} = \begin{cases} 0 & \text{The points } \mathbf{p}_j \text{ and } \mathbf{p}_l \text{ are inside and the point } \mathbf{p}_k \text{ is outside the circle,} \\ 1 & \text{Otherwise.} \end{cases}$$

4. The points $\mathbf{p}_k$ and $\mathbf{p}_l$ are inside and the point $\mathbf{p}_j$ is outside the circle. Then the following constraints with minimizing $My_{jkl}^{(4)}$ should be added to the model.

$$|d(\mathbf{x}_{jkl}^{(4)}, \mathbf{p}_k) + d(\mathbf{x}_{jkl}^{(4)}, \mathbf{p}_l) - 2r_0| \leq My_{jkl}^{(4)}, \tag{34}$$
$$|d(\mathbf{x}_{jkl}^{(4)}, \mathbf{p}_k) - d(\mathbf{x}_{jkl}^{(4)}, \mathbf{p}_j)| \leq My_{jkl}^{(4)}, \tag{35}$$

where

$$y_{jkl}^{(4)} = \begin{cases} 0 & \text{The points } \mathbf{p}_k \text{ and } \mathbf{p}_l \text{ are inside and the point } \mathbf{p}_j \text{ is outside the circle,} \\ 1 & \text{Otherwise.} \end{cases}$$

Note that, $\mathbf{p}^*$ can be only associated to one of the four represented cases, so we define the following binary variables $t_{jkl}^{(m)}$, $m = 1, \ldots, 4$, for each triple.

$$t_{jkl}^{(m)} = \begin{cases} 0 & \mathbf{p}^* \text{ is the solution of the problem in } m\text{th case,} \\ 1 & \text{Otherwise.} \end{cases}$$

Then if the mth case happen, i.e. $y_{jkl}^{(m)} = 0$, then $t_{jkl}^{(m)}$ can be equal to 0 or 1. Also, if the mth case doesn't happen, i.e. $y_{jkl}^{(m)} = 1$, then we will have $t_{jkl}^{(m)} = 1$. So we consider the following constraints in the model.

$$y_{jkl}^{(m)} \leq t_{jkl}^{(m)}, \quad \text{for all } j < k < l, \quad m = 1, \ldots, 4. \tag{36}$$

In addition, $\mathbf{p}^*$ can only be equal to one of $t_{jkl}^{(m)}$'s. Then we should put the following constraints to the model.

$$\mathbf{p}^* = \sum_{j=1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{l=k+1}^{n} \sum_{m=1}^{4} (1 - t_{jkl}^{(m)}) \mathbf{x}_{jkl}^{(m)}, \tag{37}$$
$$\sum_{j=1}^{n-2} \sum_{k=j+1}^{n-1} \sum_{l=k+1}^{n} \sum_{m=1}^{4} (1 - t_{jkl}^{(m)}) = 1. \tag{38}$$

Therefore using the equations (27 to 38) we obtain the following model.

$$(P_1)\ \min\Big\{\sum_{i=1}^{n}(\mathbf{c}_i^{+T}\mathbf{q}_i^{+}+\mathbf{c}_i^{-T}\mathbf{q}_i^{-})+M\sum_{j=1}^{n-2}\sum_{k=j+1}^{n-1}\sum_{l=k+1}^{n}\sum_{m=1}^{4}y_{jkl}^{(m)}\Big\} \tag{39}$$

$$s.t. \tag{40}$$

$$|d(\mathbf{x}_{jkl}^{(1)},\mathbf{p}_j^*)-d(\mathbf{x}_{jkl}^{(1)},\mathbf{p}_k^*)|\le My_{jkl}^{(1)} \qquad \forall\ j<k<l, \tag{41}$$

$$|d(\mathbf{x}_{jkl}^{(1)},\mathbf{p}_l^*)-d(\mathbf{x}_{jkl}^{(1)},\mathbf{p}_k^*)|\le My_{jkl}^{(1)} \qquad \forall\ j<k<l, \tag{42}$$

$$|d(\mathbf{x}_{jkl}^{(2)},\mathbf{p}_j^*)+d(\mathbf{x}_{jkl}^{(2)},\mathbf{p}_k^*)-2r_0|\le My_{jkl}^{(2)} \qquad \forall\ j<k<l, \tag{43}$$

$$|d(\mathbf{x}_{jkl}^{(2)},\mathbf{p}_k^*)-d(\mathbf{x}_{jkl}^{(2)},\mathbf{p}_l^*)|\le My_{jkl}^{(2)} \qquad \forall\ j<k<l, \tag{44}$$

$$|d(\mathbf{x}_{jkl}^{(3)},\mathbf{p}_j^*)+d(\mathbf{x}_{jkl}^{(3)},\mathbf{p}_l^*)-2r_0|\le My_{jkl}^{(3)} \qquad \forall\ j<k<l, \tag{45}$$

$$|d(\mathbf{x}_{jkl}^{(3)},\mathbf{p}_k^*)-d(\mathbf{x}_{jkl}^{(3)},\mathbf{p}_l^*)|\le My_{jkl}^{(3)} \qquad \forall\ j<k<l, \tag{46}$$

$$|d(\mathbf{x}_{jkl}^{(4)},\mathbf{p}_k^*)+d(\mathbf{x}_{jkl}^{(4)},\mathbf{p}_l^*)-2r_0|\le My_{jkl}^{(4)} \qquad \forall\ j<k<l, \tag{47}$$

$$|d(\mathbf{x}_{jkl}^{(4)},\mathbf{p}_k^*)-d(\mathbf{x}_{jkl}^{(4)},\mathbf{p}_j^*)|\le My_{jkl}^{(4)} \qquad \forall\ j<k<l, \tag{48}$$

$$\max_{i=1,\dots,n}\{|d(\mathbf{p}^*,\mathbf{p}_i^*)-r_0|\}\le\max_{i=1,\dots,n}\{|d(\mathbf{x}_{jkl}^{(m)},\mathbf{p}_i^*)-r_0|\}+M(y_{jkl}^{(m)}+t_{jkl}^{(m)}) \qquad \forall\ j<k<l,\quad m=1,\dots,4, \tag{49}$$

$$\Big|\max_{i=1,\dots,n}\{|d(\mathbf{p}^*,\mathbf{p}_i^*)-r_0|\}-|d(\mathbf{x}_{jkl}^{(m)},\mathbf{p}_j^*)-r_0|\Big|\le M(y_{jkl}^{(m)}+t_{jkl}^{(m)}) \qquad \forall\ j<k<l,\quad m=1,\dots,4, \tag{50}$$

$$\mathbf{p}^*=\sum_{j=1}^{n-2}\sum_{k=j+1}^{n-1}\sum_{l=k+1}^{n}\sum_{m=1}^{4}(1-t_{jkl}^{(m)})\mathbf{x}_{jkl}^{(m)}, \tag{51}$$

$$\sum_{j=1}^{n-2}\sum_{k=j+1}^{n-1}\sum_{l=k+1}^{n}\sum_{m=1}^{4}(1-t_{jkl}^{(m)})=1, \tag{52}$$

$$y_{jkl}^{(m)}\le t_{jkl}^{(m)} \qquad \forall\ j<k<l,\quad m=1,\dots,4, \tag{53}$$

$$0\le\mathbf{q}_i^{+}\le\mathbf{u}_i^{+} \qquad i=1,\dots,n, \tag{54}$$

$$0\le\mathbf{q}_i^{-}\le\mathbf{u}_i^{-} \qquad i=1,\dots,n, \tag{55}$$

$$t_{jkl}^{(m)},y_{jkl}^{(m)}\in\{0,1\} \qquad \forall\ j<k<l,\quad m=1,\dots,4. \tag{56}$$

Finally, the solution of inverse problem is the solution of problem (P_0) or (P_1) with the least objective value.

4. The minimax circle with variable radius

In this section we review basic properties of Minimax Circle Location Problem with Variable radius (MCLPV), which will be used in the discussion of the inverse model. In the MCLPV we should find a circle $C=C(\mathbf{x},r)$ with center $\mathbf{x}$ and radius r such that the largest distance between the circumference of the circle C and the given points is minimized, i.e.

$$\min_{\mathbf{x},r} g(\mathbf{x},r)=\max_{j=1,\dots,n}|d(\mathbf{x},\mathbf{p}_j)-r|. \tag{57}$$

The proof of the following important property on MCLPV can be found in [6].

Theorem 4.1. *Let $C^*=(\mathbf{x}^*,r^*)$ be a solution of problem (57), and S be the associated set of extreme points, that are at the maximum distance from C^*. Then at least two extreme points are located on each side of C^*, and hence, $|S|\ge 4$.*

Using Theorem 4.1, to find the optimal solution of MCLPV, we need to consider all quadruples $(\mathbf{p}_j,\mathbf{p}_k,\mathbf{p}_l,\mathbf{p}_h)$, where $j<k$ and $l<h$. Let points $\mathbf{p}_j,\mathbf{p}_k$ be inside and points $\mathbf{p}_l,\mathbf{p}_h$ be outside of the circle, then the following constraints are established

$$\max_{i=1,\dots,n}|d(\mathbf{x},\mathbf{p}_i)-r|=r-d(\mathbf{x},\mathbf{p}_j)=r-d(\mathbf{x},\mathbf{p}_k)=d(\mathbf{x},\mathbf{p}_l)-r=d(\mathbf{x},\mathbf{p}_h)-r.$$

These constraints can be written as follows

$$\begin{cases} \max\limits_{i=1,\dots,n} d(\mathbf{x},\mathbf{p}_i) = d(\mathbf{x},\mathbf{p}_l), \\ \min\limits_{i=1,\dots,n} d(\mathbf{x},\mathbf{p}_i) = d(\mathbf{x},\mathbf{p}_j), \end{cases} \tag{58}$$

$$\begin{cases} d(\mathbf{x},\mathbf{p}_j) = d(\mathbf{x},\mathbf{p}_k), \\ d(\mathbf{x},\mathbf{p}_l) = d(\mathbf{x},\mathbf{p}_h), \\ r = \dfrac{d(\mathbf{x},\mathbf{p}_j) + d(\mathbf{x},\mathbf{p}_l)}{2}, \end{cases} \tag{59}$$

$$j,k \in J_-(C) = \{i | d(\mathbf{x},\mathbf{p}_i) < r\},\ l,h \in J_+(C) = \{i | d(\mathbf{x},\mathbf{p}_i) > r\}. \tag{60}$$

Brimberg et al. [6] presented a method for finding the optimal solution of MCLPV based on enumerating quadruples $\mathbf{p}_i, \mathbf{p}_j, \mathbf{p}_k, \mathbf{p}_l$ for all (i,j,k,l). They first find the optimal solution of the associated minimax single facility location problem $\mathbf{x}_p$ (i.e. the case $r = 0$) and compute

$$r_p = \frac{\max\limits_{i=1,\dots,n} d(\mathbf{x},\mathbf{p}_i) + \min\limits_{i=1,\dots,n} d(\mathbf{x},\mathbf{p}_i)}{2}.$$

Then determine the subset of extreme points, associated with $\mathbf{x}_p$, and their corresponding radius r_p. The circle associated to an extreme point with the best value of objective function is a candidate solution. Then $\mathbf{x}_p$ is updated by this solution and new subset of extreme points is determined. This procedure continues until a new candidate solution is reached. Finally the best candidate solution is chosen as the final solution.

In the next section we use this method to present an approach for solving the Inverse Minimax Circle Location Problem with Variable radius (IMCLPV).

5. The inverse minimax circle with variable radius

Let the circle $C^* = C(\mathbf{p}^*, r^*)$ be given and we want to modify the coordinates of the existing points $\mathbf{p}_i$, for $i = 1,\dots,n$, to $\mathbf{p}_i^* = \mathbf{p}_i + \mathbf{q}_i^+ - \mathbf{q}_i^-$ with minimum cost, such that the circle C^* is optimal compared to any other circle with positive radius. Where $\mathbf{q}_i^+ \geq 0$ and $\mathbf{q}_i^- \geq 0$ are the vectors of augmenting and reduction of the coordinates of point $\mathbf{p}_i$. These vectors are bounded from above by $\mathbf{u}_i^+$ and $\mathbf{u}_i^-$, respectively. Let $\mathbf{c}_i^+$ and $\mathbf{c}_i^-$ denote the cost vectors of augmenting and reduction of per unit of coordinates of point $\mathbf{p}_i$, respectively. Then the model of IMCLPF can be written as follow.

$$\min \sum_{i=1}^{n} (\mathbf{c}_i^{+T} \mathbf{q}_i^+ + \mathbf{c}_i^{-T} \mathbf{q}_i^-) \tag{61}$$

$$s.t. \tag{62}$$

$$\max_i \{|d(\mathbf{p}^*,\mathbf{p}_i^*) - r^*|\} \leq \max_i \{|d(\mathbf{x},\mathbf{p}_i^*) - r|\} \qquad \forall \mathbf{x} \in \mathbb{R}^2,\ r \in \mathbb{R}^+ \tag{63}$$

$$\mathbf{p}_i^* = \mathbf{p}_i + \mathbf{q}_i^+ - \mathbf{q}_i^- \qquad i = 1,\dots,n, \tag{64}$$

$$0 \leq \mathbf{q}_i^+ \leq \mathbf{u}_i^+ \qquad i = 1,\dots,n, \tag{65}$$

$$0 \leq \mathbf{q}_i^- \leq \mathbf{u}_i^- \qquad i = 1,\dots,n. \tag{66}$$

Again this model is nonlinear with infinity constraints. Therefore, with considering the method of Brimberg et al. [6], we are going to find a simpler model.

Note that, by Theorem 4.1 the center of optimal circle $\bar{C}$ of problem (57) is a point $\bar{\mathbf{p}}$ with at least four extreme points such that the value of the objective function for $\bar{C}$ is better than any other circle whose center has four extreme points. So we should check all quadruples $(\mathbf{p}_j, \mathbf{p}_k, \mathbf{p}_l, \mathbf{p}_h)$ where $j < k, l < h,\ j \neq l,\ k \neq h$ and $l,h \in J_+,\ \ j,k \in J_-$. Then the candidate circle associated to each extreme point $C_{jklh} = (\mathbf{x}_{jklh}, r_{jklh})$ is obtained by constraints (58) and (59). Not that, C^* can only be equal to one of these candidate circles, so we define the following binary variables.

$$y_{jklh} = \begin{cases} 0 \text{ If constraints (58) and (59) are solvable,} \\ 1 \text{ Otherwise,} \end{cases}$$

$$t_{jklh} = \begin{cases} 0 \text{ If } C_{jklh} \text{ is the optimal circle,} \\ 1 \text{ Otherwise.} \end{cases}$$

Therefore the inverse problem is modeled as follows

$$\min\Big\{\sum_{i=1}^{n}({\mathbf{c}_i^+}^T\mathbf{q}_i^+ + {\mathbf{c}_i^-}^T\mathbf{q}_i^-) + M\sum_{j=1,\ l=1}^{n-1}\ \sum_{\substack{k=j+1,\ h=l+1\\ j\neq l,\ k\neq h}}^{n} y_{jklh}\Big\} \tag{67}$$

$$s.t. \tag{68}$$

$$|d(\mathbf{x}_{jklh},\mathbf{p}_j^*) - d(\mathbf{x}_{jklh},\mathbf{p}_k^*)| \leq My_{jklh} \qquad \forall\ j<k,\ l<h,\ j\neq l,\ k\neq h, \tag{69}$$

$$|d(\mathbf{x}_{jklh},\mathbf{p}_l^*) - d(\mathbf{x}_{jklh},\mathbf{p}_h^*)| \leq My_{jklh} \qquad \forall\ j<k,\ l<h,\ j\neq l,\ k\neq h \tag{70}$$

$$\left|r_{jklh} - \frac{d(\mathbf{x}_{jklh},\mathbf{p}_j^*) + d(\mathbf{x},\mathbf{p}_l^*)}{2}\right| \leq My_{jklh} \qquad \forall\ j<k,\ l<h,\ j\neq l,\ k\neq h, \tag{71}$$

$$|\max_{i=1,\ldots,n}\{d(\mathbf{x}_{jklh},\mathbf{p}_i^*)\} - d(\mathbf{x}_{jklh},\mathbf{p}_l^*)| \leq My_{jklh} \qquad \forall\ j<k,\ l<h,\ j\neq l,\ k\neq h, \tag{72}$$

$$|\min_{i=1,\ldots,n}\{d(\mathbf{x}_{jklh},\mathbf{p}_i^*)\} - d(\mathbf{x}_{jklh},\mathbf{p}_j^*)| \leq My_{jklh} \qquad \forall\ j<k,\ l<h,\ j\neq l,\ k\neq h, \tag{73}$$

$$\left|\max_{i=1,\ldots,n}\{|d(\mathbf{p}^*,\mathbf{p}_i^*) - r^*|\} - \max_{i=1,\ldots,n}\{|d(\mathbf{x}_{jklh},\mathbf{p}_i^*) - r_{jklh}|\}\right| \leq M(y_{jklh} + t_{jklh})$$
$$\forall\ j<k,\ l<h,\ j\neq l,\ k\neq h, \tag{74}$$

$$\mathbf{p}^* = \sum_{j=1,\ l=1}^{n-1}\ \sum_{\substack{k=j+1,\ h=l+1\\ j\neq l,\ k\neq h}}^{n}(1-t_{jklh})\mathbf{x}_{jklh}, \tag{75}$$

$$r^* = \sum_{j=1,\ l=1}^{n-1}\ \sum_{\substack{k=j+1,\ h=l+1\\ j\neq l,\ k\neq h}}^{n}(1-t_{jklh})r_{jklh}, \tag{76}$$

$$\sum_{j=1,\ l=1}^{n-1}\ \sum_{\substack{k=j+1,\ h=l+1\\ j\neq l,\ k\neq h}}^{n}(1-t_{jklh}) = 1, \tag{77}$$

$$y_{jklh} \leq t_{jklh} \qquad \forall\ j<k,\ l<h,\ j\neq l,\ k\neq h, \tag{78}$$

$$t_{jklh}, y_{jklh} \in \{0,1\} \qquad \forall\ j<k,\ l<h,\ j\neq l,\ k\neq h, \tag{79}$$

$$0 \leq \mathbf{q}_i^+ \leq \mathbf{u}_i^+ \qquad i=1,\ldots,n, \tag{80}$$

$$0 \leq \mathbf{q}_i^- \leq \mathbf{u}_i^- \qquad i=1,\ldots,n. \tag{81}$$

6. Summary and conclusion

This paper was concern to the inverse minimax circle location problem with variable coordinates. In this problem, the coordinate of existing points should be modified with minimum cost such that a given circle becomes better than any other circle in the plane. The model of this problem is nonlinear with infinite constraints. Using some properties of classic minimax center location problem, we showed that the model can be converted to a model with finite constraints.

As the future works, the other kind of inverse circle location problems such as inverse circle location problem with hamming norm and inverse circle location on networks with variable edge lengths can be considered. Moreover, other kinds of inverse continuous location problems such as inverse line location model and inverse location models with limited budgeting can be investigated in the future works.

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