



Original Article

## Characterization of some alternating groups by order and largest element order

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**ABSTRACT:** The prime graph (or Gruenberg-Kegel graph) of a finite group is a well-known graph. In this paper, first, we investigate the structure of the finite groups with a non-complete prime graph. Then as an application, we prove that every alternating group  $A_n$ , where  $n \leq 31$  is determined by its order and its largest element order. Also, we show that  $A_{32}$  is not characterizable by order and the largest element order.

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### 1. Introduction

Throughout this paper,  $n$  and  $G$  denote a natural number and a finite group, respectively. For a given prime number  $p$ , we let  $n_p$  denote the  $p$ -part of  $n$ ; i.e.,  $n_p = p^k$  if  $p^k \mid n$  but  $p^{k+1} \nmid n$ . The set of all prime divisors of  $|G|$  is denoted by  $\pi(G)$ . Also, the set of all element orders of  $G$  is denoted by  $\pi_e(G)$ . The *prime graph* (or *Gruenberg-Kegel graph*) of  $G$ , which is denoted by  $\Gamma(G)$  is a simple graph whose vertex set is  $\pi(G)$  and two distinct primes  $p$  and  $q$  are adjacent in  $\Gamma(G)$  if and only if  $pq \in \pi_e(G)$ . A subset  $\rho$  of vertices of  $\Gamma(G)$  is called an independent subset (or an isolated point set) of  $\Gamma(G)$ , whenever every two distinct primes in  $\rho$  are non-adjacent in  $\Gamma(G)$ .

Let  $m_1(G)$  be the largest element order of  $G$ , in the other word,  $m_1(G)$  is the maximum of  $\pi_e(G)$ . In general, if  $k = |\pi_e(G)|$ , then for  $2 \leq i \leq k$ , we define  $m_i(G)$  as follows:

$$m_i(G) = \max\{a \mid a \in \pi_e(G) \setminus \{m_1(G), \dots, m_{i-1}(G)\}\}$$

For a finite simple group  $S$  there are a lot of results about the numbers  $m_1(S)$ ,  $m_2(S)$  and  $m_3(S)$  (see [7, 12]). Also, the characterization of finite simple groups by their arithmetical properties has been researched widely. For instance, Mazurov et al. in [16], show that every finite simple group  $S$  can be determined by  $|S|$  and  $\pi_e(S)$ . Then some authors tried to investigate the characterization of finite simple groups by using fewer conditions. In [8, 17], it is proved that there are some finite simple groups  $S$ , which are determined by  $|S|$  and  $m_1(S)$ . For more results see [1, 3, 11, 13, 9]. However, the main result in [4], is not true in general (it is enough to consider the classical simple groups  $B_4(3^4)$  and  $C_4(3^4)$ ).

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In this paper, first, we consider the finite groups whose prime graphs are not complete. Then as an application we prove the following theorem:

**Theorem 1.1.** *Let  $G$  be a finite group and  $A_n$  be an alternating group such that  $n \leq 31$ . Then  $G$  is isomorphic to  $A_n$  if and only if  $|G| = |A_n|$  and  $m_1(G) = m_1(A_n)$ .*

By the above theorem, one may ask is any alternating group characterizable by the order and the largest element order? The following proposition gives a negative answer to this question.

**Proposition 1.2.** *Let  $Z_2$  be the cyclic group of order 2. If  $G = S_{31} \times Z_2 \times Z_2 \times Z_2 \times Z_2$ , then  $|G| = |A_{32}|$  and  $m_1(G) = m_1(A_{32})$ . In particular,  $A_{32}$  is not characterizable by the order and the largest element order.*

**Proof.** We know that  $|G| = |A_{32}|$  and  $\pi_e(G) \subseteq \{\text{lcm}(a, 2) \mid a \in \pi_e(S_{31})\}$ . In Appendix, it is shown that  $m_1(A_{32}) = m_1(S_{31}) = 4620$  and also if  $a \in \pi_e(S_{31})$  is an odd number, then  $a \leq 1365$ . Thus  $m_1(G) \geq 2a$  and this implies that  $m_1(G) = m_1(S_{31}) = m_1(A_{32})$ .  $\square$

We note that our main tool for considering Theorem 1.1 is the fact that when  $n \leq 20$  or  $n \in \{23, 24\}$ , since  $m_1(G) = m_1(A_n)$ ,  $\rho := \{p \mid n/2 \leq p \leq n\} \cap \pi(G)$  would be an independent subset of  $\Gamma(G)$  and so  $G$  has a non-complete prime graph. Also, if  $n \in \{21, 22\}$  or  $25 \leq n \leq 31$ , then we use a method, which is inspired by [6, Page 8]. We note that in the appendix, there are two procedures by Maple software for computing  $\pi_e(A_n)$  and  $\pi_e(S_n)$ .

Recall that  $\text{Soc}(G)$  denotes the socle of  $G$  (the subgroup generated by all the minimal nontrivial normal subgroups of  $G$ ). The other notation and terminologies in this paper are standard and the reader is referred to [2, 10] if necessary.

## 2. Preliminary Results

**Lemma 2.1.** [18, Lemma 4] *In  $S_m$  (resp. in  $A_m$ ) there is an element of order  $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$ , where  $p_1, p_2, \dots, p_s$  are distinct primes and  $\alpha_1, \alpha_2, \dots, \alpha_s$  are naturals, if and only if  $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq m$  (resp.  $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq m$  for odd  $n$  and  $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq m - 2$  for even  $n$ ).*

**Lemma 2.2.** [15, Lemma 1] *Let a finite group  $G$  have a normal series of subgroups  $1 \leq K \leq M \leq G$ , and the primes  $p, q$  and  $r$  are such that  $p$  divides  $|K|$ ,  $q$  divides  $|M/K|$ , and  $r$  divides  $|G/M|$ . Then  $p, q$ , and  $r$  cannot be pairwise nonadjacent in  $\Gamma(G)$ .*

**Lemma 2.3.** (See, for example, [10]) *Let  $G = F \rtimes H$  be a Frobenius group with kernel  $F$  and complement  $H$ . Then  $|H|$  divides  $|F| - 1$ .*

**Corollary 2.4.** *Let  $G$  be a finite group and  $N$  be a normal subgroup of  $G$ . Then the following assertions hold:*

- 1) *Let  $p$  and  $q$  be two distinct primes in  $\pi(G)$ . If  $p \in \pi(N)$ ,  $q \in \pi(G/N)$  and  $\{p, q\}$  is an independent subset of  $\Gamma(G)$ , then  $q \mid (|N|_p - 1)$ .*
- 2) *Let  $p, q$  and  $r$  be three pairwise distinct primes in  $\pi(G)$ . If  $p \in \pi(N)$  and  $\{q, r\} \subseteq \pi(G/N)$  and  $G/N$  is solvable, then  $p, q$  and  $r$  cannot be pairwise nonadjacent in  $\Gamma(G)$ .*

**Proof.** 1) Let  $P$  be a Sylow  $p$ -subgroup of  $N$ . By Frattini's argument,  $G/N \cong N_G(P)/N_N(P)$ . In view of the hypothesis, we conclude that  $N_G(P)$  contains an element of order  $q$ . So  $N_G(P)$  contains a subgroup isomorphic to the semidirect product  $P \rtimes Q$  where  $Q$  is a cyclic subgroup of order  $q$ . On the other hand, by the assumption,  $G$  does not contain any element of order  $pq$ . Hence,  $Q$  acts fixed point freely on  $P$ . Thus,  $P \rtimes Q$  is a Frobenius group and so by Lemma 2.3,  $q \mid (|P| - 1)$ , which implies that  $q \mid (|N|_p - 1)$ .

2) Put  $\bar{G} = G/N$  and  $\rho = \{q, r\}$ . Recall that  $\bar{G}$  is a solvable group and  $\rho \subseteq \pi(\bar{G})$ . Take a Hall  $\rho$ -subgroup  $\bar{H}$  of  $\bar{G}$ . We know that  $O_q(\bar{H}) \neq 1$  or  $O_r(\bar{H}) \neq 1$ . So without loss of generality, we may assume that  $\bar{G}$  contains a subgroup isomorphic to the semidirect product  $\bar{H}_1 \rtimes \bar{H}_2$  in which  $\pi(\bar{H}_1) = \{q\}$  and  $\pi(\bar{H}_2) = \{r\}$ .

Now let  $P$  be a Sylow  $p$ -subgroup of  $N$ . Similar to the previous case, it follows that  $\bar{G} \cong N_G(P)/N_N(P)$ . Recall that  $\bar{H}_1 \rtimes \bar{H}_2$  is a subgroup of  $\bar{G}$ . Consequently,  $N_G(P)/N_N(P)$  contains a subgroup isomorphic to  $\bar{H}_1 \rtimes \bar{H}_2$ . Hence, there is a normal series  $1 < N_N(P) < T_1 < T_2$  in  $N_G(P)$  such that  $T_1/N_N(P) \cong \bar{H}_1$  and  $T_2/N_N(P) \cong \bar{H}_1 \rtimes \bar{H}_2$ . Also, by the above argument,  $p \in \pi(N_N(P))$ ,  $\pi(T_1/N_N(P)) = \{q\}$  and  $\pi(T_2/T_1) = \{r\}$ . Therefore, by Lemma 2.2, we get that the subset  $\{p, q, r\}$  can not be an independent subset of  $\Gamma(G)$ , which completes the proof.  $\square$

**Lemma 2.5.** *Let  $G$  be a finite group,  $M$  be a normal subgroup of  $G$  and  $G/M$  contain a subgroup  $S$ , which is isomorphic to a simple group. If  $R$  is a Sylow  $r$ -subgroup of  $M$ , then one of the following assertions holds:*

- 1)  $|S| \mid |\text{Aut}(R)|$ ,
- 2) *If  $a \in \pi_e(S)$  and  $r^\alpha \in \pi_e(R)$ , then  $\text{lcm}(r^\alpha, a) \in \pi_e(G)$ .*

**Proof.** Put  $N = N_G(R)$ ,  $L = N_M(R)$  and  $C = C_G(R)$ . By Frattini's argument,  $G/M \cong N/L$ . Hence by the assumption we get that  $N/L$  contains a subgroup isomorphic to the simple group  $S$ . Let  $K$  be a subgroup of  $N$  such that  $K/L \cong S$  is a simple group. Since  $K/L$  is a simple subgroup of  $N/L$  and  $CL/L$  is a normal subgroup of  $N/L$ , it follows that either  $K/L \cap CL/L = 1$  or  $K/L \leq CL/L$ . We consider each possibilities:

1) Let  $K/L \cap CL/L = 1$ . Then we obtain the following relation:

$$K/L \cong \frac{(K/L)(CL/L)}{CL/L} \leq \frac{N/L}{CL/L} \cong N/CL.$$

So  $|K/L| \mid |N/CL|$ . On the other hand:

$$|N/CL| = \left| \frac{N/C}{CL/C} \right| \mid |\text{Aut}(R)|.$$

Therefore,  $|K/L| \mid |\text{Aut}(R)|$  and consequently,  $|S| \mid |\text{Aut}(R)|$ .

2) Let  $K/L \leq CL/L$ . Since  $CL/L \cong C/C_L(R)$ , it follows that  $C/C_L(R)$  contains a subgroup isomorphic to  $K/L$ . Recall that  $C = C_G(R)$ . Hence if  $a \in \pi_e(S) = \pi_e(K/L)$  and  $r^\alpha \in \pi_e(R)$ , then  $a \in \pi_e(C_G(R))$  and so  $\text{lcm}(r^\alpha, a) \in \pi_e(G)$ , which completes the proof.  $\square$

**Lemma 2.6.** Let  $G$  be a finite group,  $N_2 \leq N_1$  be some characteristic subgroups of  $G$  such that  $S_1 \leq G/N_1 \leq \text{Aut}(S_1)$  and  $S_2 \leq N_1/N_2 \leq \text{Aut}(S_2)$  where  $S_1$  and  $S_2$  are some non-abelian simple groups. Then  $G/N_2$  has a subgroup isomorphic to  $S_1 \times S_2$ .

**Proof.** Let  $K_1$  and  $K_2$  be the subgroups of  $G$  such that  $K_1/N_1 \cong S_1$  and  $K_2/N_2 \cong S_2$ . Put  $C/N_2 := C_{G/N}(K_2/N_2)$ . Note that since  $N_1$  and  $N_2$  are characteristic subgroups in  $G$ ,  $K_1$  and  $K_2$  are some normal subgroups of  $G$  and so  $C/N_2$  is a normal subgroup of  $G/N_2$ .

By the hypothesis,  $K_2/N_2$  is not abelian and  $K_2/N_2 \leq N_1/N_2 \leq \text{Aut}(K_2/N_2)$ . This shows that  $C_{N_1/N_2}(K_2/N_2) = 1$ . Hence because of  $C/N_2 \cap N_1/N_2 \leq C_{N_1/N_2}(K_2/N_2)$ , we have:

$$\frac{C}{N_2} \cap \frac{N_1}{N_2} = 1. \tag{1}$$

This yields that  $(C/N_2) \times (K_2/N_2) \leq G/N_2$ . So in the sequel, we show that  $C/N_2$  has a subgroup isomorphic to  $K_1/N_1$ , which implies that  $G/N_2$  has a subgroup isomorphic to  $S_1 \times S_2$ .

First, let  $K_1/N_1 \cap CN_1/N_1 = 1$ . Similar to the above argument, since  $K_1/N_1$  is a non-abelian simple and  $C_{G/N_1}(K_1/N_1) = 1$ ,  $CN_1/N_1 = 1$ . Hence  $C \leq N_1$  and so by Relation (1),  $C/N_2 = 1$ . This implies that:

$$S_2 \cong \frac{K_2}{N_2} \cong \frac{(K_2/N_2)(C/N_2)}{C/N_2} \leq \frac{G/N_2}{C/N_2} \cong \frac{G}{C} \leq \text{Aut}(S_2). \tag{2}$$

On the other hand, because of  $C/N_2 = 1$ , by Relation (2), we get that:

$$\frac{G}{K_2} \cong \frac{G/N_2}{K_2/N_2} \hookrightarrow \frac{\text{Aut}(S_2)}{S_2}. \tag{3}$$

This implies that  $G/K_2$  is a solvable group. Also, since  $G/N_1$  is a quotient of  $G/K_2$ , we deduce that  $G/N_1$  is solvable, too. On the other hand,  $K_1/N_1$  is a subgroup of the solvable group  $G/N_1$ , which is impossible since  $K_1/N_1 \cong S_1$  is a non-abelian simple group.

Therefore,  $K_1/N_1 \cap CN_1/N_1 \neq 1$  and this implies that  $K_1/N_1 \leq CN_1/N_1$ . We remark that by Relation (1),  $C/N_2 \cap N_1/N_2 = 1$ . So we have:

$$\frac{CN_1}{N_1} \cong \frac{CN_1/N_2}{N_1/N_2} \cong \frac{(C/N_2)(N_1/N_2)}{N_1/N_2} \cong \frac{C}{N_2} \tag{4}$$

Thus, since  $K_1/N_1 \leq CN_1/N_1$ , by the above relation we conclude that  $C/N_2$  has a subgroup isomorphic to  $K_1/N_1$ , which completes the proof.  $\square$

### 3. Finite groups with the non-complete prime graph

**Lemma 3.1.** Let  $G$  be a finite group,  $K_1$  and  $K_2$  two normal subgroups of  $G$  and  $\rho$  an independent subset of  $\Gamma(G)$ . Then either  $\pi(K_1) \cap \rho \subseteq \pi(K_2) \cap \rho$  or  $\pi(K_2) \cap \rho \subseteq \pi(K_1) \cap \rho$ . Moreover, if  $N$  is the product of all normal subgroups  $K$  of  $G$  such that  $|\pi(K) \cap \rho| \leq 1$ , then  $|\pi(N) \cap \rho| \leq 1$ .

**Proof.** For  $1 \leq i \leq 2$ , put  $\pi_i = \pi(K_i) \cap \rho$ . If  $\pi_1 \not\subseteq \pi_2$  and  $\pi_2 \not\subseteq \pi_1$ , then there exist two primes  $p_1$  and  $p_2$  such that  $p_1 \in \pi_1 \setminus \pi_2$  and  $p_2 \in \pi_2 \setminus \pi_1$ . This implies that  $p_1 \in \pi(K_1/(K_1 \cap K_2))$  and  $p_2 \in \pi(K_2/(K_1 \cap K_2))$ . By the following relation:

$$\frac{K_1 K_2}{K_1 \cap K_2} \cong \frac{K_1}{K_1 \cap K_2} \times \frac{K_2}{K_1 \cap K_2}$$

it follows that  $K_1 K_2$  contains an element of order  $p_1 p_2$ , which contradicts to the assumption. Therefore,  $\pi_1 \subseteq \pi_2$  or  $\pi_2 \subseteq \pi_1$  and consequently, there is  $i \in \{1, 2\}$ , such that  $\pi(K_1 K_2) \cap \rho \subseteq \pi_i$ . Also this implies that if  $|\pi_1| \leq 1$  and  $|\pi_2| \leq 1$ , then  $|\pi(K_1 K_2) \cap \rho| \leq 1$ .

Finally, let  $N$  be the product of all normal subgroups  $K$  of  $G$  such that  $|\pi(K) \cap \rho| \leq 1$ . Then by the above discussion,  $|\pi(N) \cap \rho| \leq 1$ , which completes the proof.  $\square$

We note that by the previous lemma, if  $\rho$  is an independent subset of  $\Gamma(G)$  such that  $|\rho| \geq 2$ , then  $G$  contains a normal subgroup  $N$ , which is the largest normal subgroup of  $G$  among the normal subgroups of  $G$  with the property  $|\pi(N) \cap \rho| \leq 1$ .

**Theorem 3.2.** *Let  $G$  be a finite group and  $\rho$  be an independent subset of  $\Gamma(G)$  such that  $|\rho| \geq 2$ . Then one of the following assertions holds:*

1)  $G$  has a normal series  $1 \leq N \leq L \leq G$ , where  $L/N = \text{Soc}(G/N)$  is the socle of  $G/N$ . Moreover, in this case  $\pi(N) \cap \rho = \{p\}$ ,  $\pi(L/N) \cap \rho = \{q\}$  and  $\rho = \{p, q\}$ .

2) There exists a normal subgroup  $N$  of  $G$  and a non-abelian simple group  $S$  such that

$$S \leq \frac{G}{N} \leq \text{Aut}(S),$$

where  $|\pi(N) \cap \rho| \leq 1$  and  $|\pi(S) \cap \rho| \geq 2$ . Moreover, if  $|\rho| \geq 3$ , then  $|\pi(S) \cap \rho| \geq |\rho| - 1$ .

**Proof.** Let  $G$  be a finite group,  $\rho$  be an independent subset of  $\Gamma(G)$  such that  $|\rho| \geq 2$  and  $N$  be the product of all normal subgroups  $K$  of  $G$  such that  $|\pi(K) \cap \rho| \leq 1$ . Also let  $L/N$  be the socle of  $G/N$ . By Lemma 3.1,  $|\pi(N) \cap \rho| \leq 1$ . Let  $M_1/N, \dots, M_t/N$  be the minimal normal subgroups of  $G/N$  such that  $L/N \cong M_1/N \times \dots \times M_t/N$ . We know that for each  $1 \leq i \leq t$ ,  $M_i/N$  is a direct product of some isomorphic simple groups. Also since  $N$  is a pure subgroup of  $M_i$ ,  $|\pi(M_i) \cap \rho| > 1$  and so  $|\pi(M_i/N) \cap \rho| \geq 1$ . In the sequel, we consider the following cases, seperately:

1) Let for every  $1 \leq i \leq t$ ,  $|\pi(M_i/N) \cap \rho| = 1$ . In view of the definition of  $N$ , we conclude that there exist two distinct primes  $p$  and  $q$  such that  $\pi(N) \cap \rho = \{p\}$  and for every  $1 \leq i \leq t$ ,  $\pi(M_i/N) \cap \rho = \{q\}$ . This implies that  $\pi(L/N) \cap \rho = \{q\}$ .

By the above discussion,  $\{p, q\} \subseteq \rho$ . Let there exist  $r \in \rho \setminus \{p, q\}$ . Recall that,  $\pi(N) \cap \rho = \{p\}$  and  $\pi(L/N) \cap \rho = \{q\}$ . This shows that  $r \in \pi(G/L)$ . On the other hand,  $\{p, q, r\}$  is an independent subset of  $\Gamma(G)$ , which contradicts to Lemma 2.2. Therefore,  $\rho = \{p, q\}$ , which get the assertion (1) in the theroem.

2) Let there exist  $1 \leq i \leq t$ , such that  $|\pi(M_i/N) \cap \rho| \geq 2$ . Without lose of generality, suppose that  $|\pi(M_1/N) \cap \rho| \geq 2$ . In this case, if  $t \geq 2$ , then  $M_1/N \times M_2/N$  contains an element of order  $pq$  where  $p \in \pi(M_1/N) \cap \rho$  and  $q \in \pi(M_2/N) \cap \rho$ , which is a contradiction. Thus,  $t = 1$ . Also since  $M_1/N$  is a direct product of some isomorphic simple groups, by a similar argument, we conclude that  $L/N = M_1/N$  is isomorphic to a non-abelian simple group. Then in this case,  $C_{G/N}(L/N) = 1$  since  $L/N$  is the socle of  $G/N$ . Let  $L/N$  be isomorphic to a non-abelian simple group  $S$ . So the following relation holds:

$$S \leq \bar{G} := \frac{G}{N} \leq \text{Aut}(S).$$

We recall that in this case,  $L/N = M_1/N \cong S$  and by the assumption  $|\pi(M_1/N) \cap \rho| \geq 2$ . So  $|\pi(S) \cap \rho| \geq 2$ .

Finally, we prove that if  $|\rho| \geq 3$ , then  $|\pi(S) \cap \rho| \geq |\rho| - 1$ . On the contrary, let  $|\rho| \geq 3$  and  $|\pi(S) \cap \rho| \leq |\rho| - 2$ . This implies that there are two distinct primes  $p$  and  $q$  in  $\rho$  such that  $\{p, q\} \subseteq \pi(N) \cup \pi(\bar{G}/S)$  and  $\{p, q\} \cap \pi(S) = \emptyset$ . Since  $|\rho| \geq 3$ , if  $\{p, q\} \subseteq \pi(\bar{G}/S)$ , then by Corollary 2.4 (Assertion 2), we get a contradiction since  $\bar{G}/S$  is solvable. Similarly, if  $p \in \pi(N)$  and  $q \in \pi(\bar{G}/S)$ , then by Lemma 2.2, we arrive a contradiction. Therefore, when  $|\rho| \geq 3$ , we deduce that  $|\pi(S) \cap \rho| \geq |\rho| - 1$ , which completes the proof.  $\square$

**Example 3.1.** *Let  $G = 11^2 : \text{SL}_2(5)$ , which is a Frobenius group with kernel  $11^2$  and complement  $\text{SL}_2(5)$ . In the prime graph of  $G$ , the subsets  $\rho_1 = \{2, 11\}$  and  $\rho_2 = \{11, 3, 5\}$  are two independent subsets. If we choose  $\rho_1$  as the independent subset said in Theorem 3.2, then we have  $N = 11^2$  and  $L = 11^2 : 2$ , which shows that Case (1) of Theorem 3.2 holds. Also if we choose  $\rho_2$  as the independent subset  $\rho$  in Theorem 3.2, then  $N = 11^2 : 2$  and we have*

$$\text{PSL}_2(5) \leq G/N \leq \text{Aut}(\text{PSL}_2(5)),$$

which satisfies Case (2) of Theorem 3.2.

Now by Theorem 3.2, we can easily get the following two corollaries which modify [5, Lemma 10] and [13, Lemma 2.3]:

**Corollary 3.3.** *If  $G$  is a finite group and  $\rho$  an independent subset of  $\Gamma(G)$  such that  $|\rho| \geq 3$ , then there exists a nonabelian simple group  $S$  and a normal subgroup  $N$  of  $G$  such that*

$$S \leq \frac{G}{N} \leq \text{Aut}(S),$$

and also we have  $|\pi(S) \cap \rho| \geq |\rho| - 1$  and  $|\pi(N) \cap \rho| \leq 1$ .

**Corollary 3.4.** *Let  $G$  be a finite group,  $\rho$  be an independent subset of  $\Gamma(G)$  such that  $|\rho| \geq 2$ . Also let for every pair of distinct prime numbers  $p$  and  $q$  belong to  $\rho$  we have  $p \nmid (q^j - 1)$  and  $q \nmid (p^i - 1)$  where  $1 < p^i \leq |G|_p$  and  $1 < q^j \leq |G|_q$ . Then there exists a non-abelian simple group  $S$  such that*

$$S \leq \frac{G}{O_{\rho'}(G)} \leq \text{Aut}(S),$$

and also we have  $\rho \subseteq \pi(S)$  and  $\rho \cap \pi(\text{Out}(S)) = \emptyset$ .

**Proof.** It immediately comes from Theorem 3.2 and Corollary 2.4. □

#### 4. Proof of Theorem 1.1

Recall that in number theory  $\text{Landau}(n)$  is a familiar notation for  $m_1(S_n)$ .

**Lemma 4.1.** *Let  $A_n$  be an alternating group. If  $n \geq 25$  or  $n \in \{21, 22\}$ , then  $m_1(A_n) \geq pq$  for all distinct primes  $p$  and  $q$  in  $\pi(A_n)$ .*

**Proof.** Let  $p$  and  $q$  be two distinct primes in  $\pi(A_n)$ . By the definition of  $m_1(A_n)$  and Lemma 2.1,  $m_1(A_n) \geq m_1(S_{n-2}) = \text{Landau}(n-2)$ . In view of [14], if  $n \geq 906$ , then

$$\text{Landau}(n) \geq e^{\sqrt{n \ln(n)}}.$$

Hence,

$$m_1(A_n) \geq e^{\sqrt{(n-2) \ln(n-2)}}.$$

On the other hand, by the hypothesis,  $(n-2)^3 > n(n-2) \geq pq$ . Using an easy computation, we can show that if  $n \geq 906$ , then

$$e^{\sqrt{(n-2) \ln(n-2)}} \geq (n-2)^3,$$

Thus, by the above argument if  $n \geq 906$ , then  $m_1(A_n) \geq (n-2)^3$  and consequently,  $m_1(A_n) > pq$ . Finally, by the program in the appendix, and an easy computation we deduce that if  $25 \leq n \leq 905$  or  $n \in \{21, 22\}$ , then  $m_1(A_n) \geq pq$ , which completes the proof. □

**Lemma 4.2.** *If  $G$  is a finite group such that  $|G| = |A_n|$  and  $m_1(G) = m_1(A_n)$ , where  $n \leq 20$  or  $n \in \{23, 24\}$ , then  $G \cong A_n$ .*

**Proof.** If  $n \leq 4$ , then by using GAP we can see that  $G \cong A_n$ . Also, if  $n \in \{5, 6\}$ , then by [17, Theorem 1],  $G \cong A_n$ . So let  $7 \leq n \leq 20$  or  $n \in \{23, 24\}$ . By Table 1, there exists an independent subset  $\rho$  of  $\Gamma(G)$  such that  $\rho$  satisfies the conditions of Corollary 3.4, which implies that there is a non-abelian simple group  $S$  such that

$$S \leq \bar{G} := \frac{G}{M} \leq \text{Aut}(S)$$

where  $M = O_{\rho'}(G)$ ,  $\rho \subseteq \pi(S)$  and  $\pi(\bar{G}/S) \cap \rho = \emptyset$ . Moreover, by the assumption  $|S| \mid |A_n|$ . In view of [19, Table 1], the possible cases for  $S$  are indicated in Table 1. Hence, if  $n \in \{7, 13, 14, 17, 19, 23\}$ , then by Table 1,  $S \cong A_n$  and so  $G \cong A_n$  since  $|G| = |A_n|$ . In the sequel, for the other cases, we suppose that  $S$  is not isomorphic to  $A_n$ .

Let  $n = 8$ . By Table 1,  $S \cong A_7$  or  $L_3(4)$ . If  $S \cong A_7$ , then  $G/M$  is isomorphic to either  $A_7$  or  $S_7$  and  $|M| \mid 8$ . On the other hand,  $A_7$  and  $S_7$  do not contain any element of order 15, in while  $m_1(G) = m_1(A_8) = 15$ , which is a contradiction. If  $S \cong L_3(4)$ , then  $|S| = |A_8|$  and so  $G \cong L_3(4)$ , which is impossible since by [2],  $m_1(L_3(4)) = 7$ .

Let  $n = 9$ . By Table 1,  $S \cong A_8, A_7$  or  $L_3(4)$ . If  $S \cong A_8, A_7$  or  $L_3(4)$ , then  $7 \mid |G/M|$  and  $|M|_3 = 3$  or  $9$ . By Corollary 2.4, we get that  $G$  contains an element of order 21, which is a contradiction since  $m_1(G) = m_1(A_9) = 15$ .



Let  $n = 10$ . By Table 1,  $S \cong J_2$ . Then by [2], we deduce that  $|M| = 9$  and  $S$  contains an element of order 10. Hence by Lemma 2.5, we get that  $G$  contains an element of order 30, which is a contradiction since  $m_1(G) = m_1(A_{10}) = 21$ .

Let  $n = 11$ . By Table 1,  $S \cong M_{22}$ . Then  $11 \mid |S|$  and  $|M|_3 = 3^2$ . So by Corollary 2.4, we get that  $G$  contains an element of order 33, which is a contradiction since  $m_1(G) = m_1(A_{11}) = 21$ .

Let  $n = 12$ . By Table 1,  $S \cong A_{11}$  or  $M_{22}$ . Let  $S \cong M_{22}$ . Then  $11 \mid |S|$  and  $|M|_5 = 5$ . So by Lemma 2.5, we get that  $55 \in \pi_e(G)$ , which is impossible since  $m_1(G) = m_1(A_{12}) = 35$ . Let  $S \cong A_{11}$ . Then  $|M|_3 = 3$  and  $S$  contains an element of order 20. So by Lemma 2.5, we get that  $60 \in \pi_e(G)$ , which is a contradiction.

Let  $n = 14$ . By Table 1,  $S \cong A_{13}$ . Then  $|M|_7 = 7$ . So  $|M|_7 = 7$  and  $S$  contains an element of order 30. Hence by Lemma 2.5, we get that  $210 \in \pi_e(G)$ , which is a contradiction since  $m_1(G) = m_1(A_{14}) = 60$ .

Let  $n = 15$ . By Table 1,  $S \cong A_{14}$  or  $A_{13}$ . Let  $S \cong A_{13}$  or  $A_{14}$ . Then  $|M|_5 = 5$  and  $S$  contains an element of order 28. Hence by Lemma 2.5, we get that  $140 \in \pi_e(G)$ , which is a contradiction since  $m_1(G) = m_1(A_{15}) = 105$ .

Let  $n = 16$ . By Table 1,  $S \cong A_{15}, A_{14}$  or  $A_{13}$ . We note that  $m_1(G) = m_1(A_{16}) = 105$ . So if  $S \cong A_{13}$  or  $A_{14}$ , then similar to the case  $n = 15$ , we get that  $140 \in \pi_e(G)$ , which is a contradiction. Let  $S \cong A_{15}$ . In this case, we have  $S$  contains an element of order 105 and also  $|M| = 8$  or  $16$ . Thus, by Lemma 2.5, we get that  $210 \in \pi_e(G)$ , which is impossible.

Let  $n = 18$ . By Table 1,  $S \cong A_{17}$  and  $m_1(G) = m_1(A_{18}) = 140$ . If  $S \cong A_{17}$ , then  $|M|_3 = 9$  and  $70 \in \pi_e(S)$ . So by Lemma 2.5,  $210 \in \pi_e(G)$ , which is impossible.

Let  $n = 20$ . By Table 1,  $S \cong A_{19}$  and  $m_1(G) = m_1(A_{20}) = 210$ . If  $S \cong A_{19}$ , then  $|M|_5 = 5$  and  $77 \in \pi_e(S)$ , and so by Lemma 2.5,  $5 \cdot 77 \in \pi_e(G)$ , which is impossible.

Let  $n = 24$ . By Table 1,  $S \cong A_{23}$  and  $m_1(G) = m_1(A_{24}) = 420$ . If  $S \cong A_{23}$ , then  $|M|_3 = 3$  and  $385 \in \pi_e(A_{23})$  and so by Lemma 2.5,  $3 \cdot 385 \in \pi_e(G)$ , which is impossible.

Finally, by the above discussions we conclude that if  $|G| = |A_n|$  and  $m_1(G) = m_1(A_n)$ , then  $S \cong A_n$  and consequently,  $G \cong A_n$ , which completes the proof.  $\square$

**Lemma 4.3.** *Let  $G$  be a finite group such that  $|G| = |A_n|$  and  $m_1(G) = m_1(A_n)$ , where  $n \in \{21, 22\}$ , then  $G \cong A_n$ .*

**Proof.** Let  $P \in \text{Syl}_{19}(G)$ ,  $C := C_G(P)/P$  and  $\rho := \{11, 13, 17\}$ . We consider the following cases:

Case 1. Let  $\rho \subseteq \pi(C)$ . Since  $m_1(G) = m_1(A_{21}) = m_1(A_{22}) = 420$  and  $19 \cdot 13 \cdot 11 > 420$ , we get that  $\rho$  is an independent subset of  $\Gamma(C)$ . So by Corollary 3.4, there exists a non-abelian simple group  $S$  such that  $S \leq C/N \leq \text{Aut}(S)$ , where  $\rho \subseteq \pi(S)$  and  $\pi(N) \cap \rho = \emptyset$ . Hence by [19],  $S$  is isomorphic to either  $A_{17}$  or  $A_{18}$ . On the other hand,  $m_1(A_{17}) = 105$ . This implies that  $m_1(G) \geq m_1(C) \geq 19 \cdot 105$ , which is impossible since  $m_1(G) = 405$ .

Case 2. Let  $\rho \not\subseteq \pi(C)$ . So there exists a prime  $p \in \rho$  such that  $p$  is not adjacent to 19 in  $\Gamma(G)$ . Thus by Corollary 3.4, there is a non-abelian simple group  $S_1$  such that  $S_1 \leq G/N_1 \leq \text{Aut}(S_1)$ , where  $\{p, 19\} \subseteq \pi(S_1)$  and  $\{p, 19\} \cap \pi(N_1) = \emptyset$ . By [19], we get that  $S_1 \cong J_3, J_1, HN, U_4(8)$  or  $A_m$  where  $19 \leq m \leq 22$ .

Let  $S_1 \cong J_3$ . In this case,  $\{17, 19\} \subseteq \pi(S_1)$  and  $\{11, 13\} \subseteq \pi(N_1)$ . By Lemma 2.4, it follows that  $G$  contains some elements of orders  $19 \cdot 11$  and  $19 \cdot 13$ . However,  $\{11, 13\}$  is an independent subset of  $\Gamma(N_1)$ , since  $m_1(G) < 19 \cdot 13 \cdot 11$ . So again by Corollary 3.4, there is a non-abelian simple group  $S_2$  such that  $S_2 \leq N_1/N_2 \leq \text{Aut}(S_2)$ , where  $\{11, 13\} \subseteq \pi(S_2)$ . Then by [19],  $S_2$  is isomorphic to  $A_{13}, A_{14}, A_{15}, A_{16}, Suz$  or  $Fi_{22}$ . By using GAP, we can see that in every possible cases for  $S_2$ ,  $m_1(S_2) \geq 24$ . Thus, by Lemma 2.6,  $m_1(G) \geq 19 \times m_1(S_2) \geq 19 \cdot 24 = 456$ , which is a contradiction.

Let  $S_1 \cong J_1$  or  $HN$ . In this case,  $\{13, 17\} \subseteq \pi(N_1)$ . Similar to the above discussion, we conclude that  $\{13, 17\}$  is an independent subset of  $\Gamma(N_1)$  and so there is a non-abelian simple group  $S_2$  such that  $S_2 \leq N_1/N_2 \leq \text{Aut}(S_2)$ , where  $\{13, 17\} \subseteq \pi(S_2)$ . Then by [19],  $S_2$  is isomorphic to  $U_4(4), L_3(16), A_{17}$  or  $A_{18}$ . By [12, Tables 1 and 2] and using GAP,  $m_1(S_2) \geq 65$ . So by Lemma 2.6,  $m_1(G) \geq 19 \times m_1(S_2) \geq 19 \cdot 65$ , which is a contradiction.

Let  $S_1 \cong U_4(8)$ . In this case,  $\{11, 17\} \subseteq \pi(N_1)$ . Thus  $\{11, 17\}$  is an independent subset of  $\Gamma(N_1)$  and so there is a non-abelian simple group  $S_2$  such that  $S_2 \leq N_1/N_2 \leq \text{Aut}(S_2)$ , where  $\{11, 17\} \subseteq \pi(S_2)$ . Similar to the previous case, we get a contradiction.

Hence  $S \cong A_m$  where  $m \in \{19, 20, 21, 22\}$ . Let  $m = 19$  or  $m = 20$ . Then  $|N|_7 = 7$  and  $165 \in \pi_e(S)$ . So by Lemma 2.5,  $7 \cdot 165 \in \pi_e(G)$ , which is impossible. If  $n = 22$  and  $m = 21$ , then  $|N|_{11} = 11$  and by Lemma 2.5,  $11 \cdot 165 \in \pi_e(G)$ , which is impossible. Therefore, if  $n \in \{21, 22\}$ , then  $G$  is isomorphic  $A_n$ .  $\square$

**Lemma 4.4.** *Let  $n$  be an integer such that  $25 \leq n \leq 28$ . If  $G$  is a group such that  $|G| = |A_n|$  and  $m_1(G) = m_1(A_n)$ , then  $G \cong A_n$ .*

**Proof.** Let  $P \in \text{Syl}_{23}(G)$ ,  $C := C_G(P)/P$  and  $\rho := \{13, 17, 19\}$ . We consider the following cases:

Case 1. Let  $\rho \subseteq \pi(C)$ . Since  $m_1(G) \leq m_1(A_{28}) = 1365$ ,  $m_1(G) < 23 \cdot 17 \cdot 13$ . So  $\rho$  is an independent subset of  $\Gamma(C)$ . So by Corollary 3.4, there exists a non-abelian simple group  $S$  such that  $S \leq C/N \leq \text{Aut}(S)$ , where  $\rho \subseteq \pi(S)$  and  $\pi(N) \cap \rho = \emptyset$ . Since  $|S|_2 \leq |G|_2 \leq |A_{28}|_2 = 2^{24}$  by [19],  $S$  is isomorphic to either  $A_m$  where

$19 \leq m \leq 22$ . On the other hand  $m_1(A_{19}) = 210$ . This implies that  $m_1(G) \geq m_1(C) \geq 23 \cdot 210$ , which is impossible since  $m_1(G) \leq 1365$ .

Case 2. Let  $\rho \not\subseteq \pi(C)$ . So there exists a prime  $p \in \rho$  such that  $p$  is not adjacent to 23 in  $\Gamma(G)$ . Thus by Corollary 3.4, there is a non-abelian simple group  $S_1$  such that  $S_1 \leq G/N_1 \leq \text{Aut}(S_1)$ , where  $\{p, 23\} \subseteq \pi(S_1)$  and  $\{p, 23\} \cap \pi(N_1) = \emptyset$ . By [19], we get that  $S_1 \cong Co_1, Fi_{23}$  or  $A_m$  where  $23 \leq m \leq 28$ .

Let  $S_1 \cong Co_1$ . Thus by the order of  $S_1$ ,  $\{19, 17\} \subseteq \pi(N_1)$ . By Lemma 2.4,  $\{19, 17\}$  is an independent subset of  $\Gamma(N_1)$ . So by Corollary 3.4, there is a non-abelian simple group  $S_2$  such that  $S_2 \leq N_1/N_2 \leq \text{Aut}(S_2)$ , where  $\{19, 17\} \subseteq \pi(S_2)$ . Then by [19],  $S_2$  is isomorphic to  $J_3, A_{19}, A_{20}, A_{21}$  or  $A_{23}$ . In each case,  $|S_2|_2 \geq 2^7$ . So  $|G|_2 \geq |S_1|_2 |S_2|_2 \geq 2^{21} \cdot 2^7$ , which is a contradiction.

Let  $S_1 \cong Fi_{23}$ . In this case,  $\{11, 19\} \subseteq \pi(N_1)$ . Similar to the above discussion, we conclude that  $\{11, 19\}$  is an independent subset of  $\Gamma(N_1)$  and so there is a non-abelian simple group  $S_2$  such that  $S_2 \leq N_1/N_2 \leq \text{Aut}(S_2)$ , where  $\{11, 19\} \subseteq \pi(S_2)$ . Then by [19],  $S_2$  is isomorphic to  $J_1$ . So  $|G|_3 \geq |S_1|_3 |S_2|_3 \geq 3^{13} \cdot 3$ , which is a contradiction since  $|G|_3 \leq |A_{28}|_3$ .

Hence  $S \cong A_m$  where  $23 \leq m \leq 28$ . Easily we can show that  $m = n$ . Therefore, since  $|G| = |A_n|$ , where  $25 \leq n \leq 28$ ,  $G$  is isomorphic  $A_n$ .  $\square$

**Lemma 4.5.** *Let  $n$  be an integer such that  $n \in \{29, 30\}$ . If  $G$  is a group such that  $|G| = |A_n|$  and  $m_1(G) = m_1(A_n)$ , then  $G \cong A_n$ .*

**Proof.** Let  $P \in \text{Syl}_{29}(G)$ ,  $C := C_G(P)/P$  and  $\rho := \{17, 19, 23\}$ . We consider the following cases:

Case 1. Let  $\rho \subseteq \pi(C)$ . Since  $m_1(G) \leq m_1(A_{30}) = 2310$ ,  $m_1(G) < 29 \cdot 19 \cdot 17$ . So  $\rho$  is an independent subset of  $\Gamma(C)$ . By Corollary 3.4, there exists a non-abelian simple group  $S$  such that  $S \leq C/N \leq \text{Aut}(S)$ , where  $\rho \subseteq \pi(S)$  and  $\pi(N) \cap \rho = \emptyset$ . By [19],  $S$  is isomorphic to either  $A_m$  where  $23 \leq m \leq 28$ . On the other hand  $m_1(A_{23}) = 420$ . This implies that  $m_1(G) \geq m_1(C) \geq 29 \cdot 420$ , which is impossible since  $m_1(G) \leq 2310$ .

Case 2. Let  $\rho \not\subseteq \pi(C)$ . So there exists a prime  $p \in \rho$  such that  $p$  is not adjacent to 29 in  $\Gamma(G)$ . Thus by Corollary 3.4, there is a non-abelian simple group  $S_1$  such that  $S_1 \leq G/N_1 \leq \text{Aut}(S_1)$ , where  $\{p, 29\} \subseteq \pi(S_1)$  and  $\{p, 23\} \cap \pi(N_1) = \emptyset$ . By [19], we get that  $S_1$  is isomorphic to  $Fi'_{24}$  or  $A_m$  where  $29 \leq m \leq 30$ . If  $S_1 \cong Fi'_{24}$ , then  $|G|_3 < |S_1|_3$ , which is impossible.

Hence  $S \cong A_m$  where  $29 \leq m \leq 30$ . Easily we can show that  $m = n$ . Therefore, since  $|G| = |A_n|$ ,  $G$  is isomorphic  $A_n$ .  $\square$

**Lemma 4.6.** *If  $G$  is a group such that  $|G| = |A_{31}|$  and  $m_1(G) = m_1(A_{31})$ , then  $G \cong A_{31}$ .*

**Proof.** Let  $P \in \text{Syl}_{31}(G)$ ,  $C := C_G(P)/P$  and  $\rho := \{17, 19, 23, 29\}$ . We consider the following cases:

Case 1. Let  $\rho \subseteq \pi(C)$ . Since  $m_1(G) \leq m_1(A_{31}) = 2520$ ,  $m_1(G) < 31 \cdot 19 \cdot 17$ . So  $\rho$  is an independent subset of  $\Gamma(C)$  and by Corollary 3.4, there exists a non-abelian simple group  $S$  such that  $S \leq C/N \leq \text{Aut}(S)$ , where  $\rho \subseteq \pi(S)$  and  $\pi(N) \cap \rho = \emptyset$ . By [19],  $S$  is isomorphic to  $A_m$  where  $29 \leq m \leq 30$ . On the other hand  $m_1(A_{29}) = 1540$ . This implies that  $m_1(G) \geq m_1(C) \geq 31 \cdot 1540$ , which is impossible since  $m_1(G) = 2520$ .

Case 2. Let  $\rho \not\subseteq \pi(C)$ . So there exists a prime  $p \in \rho$  such that  $p$  is not adjacent to 31 in  $\Gamma(G)$ . Thus by Corollary 3.4, there is a non-abelian simple group  $S_1$  such that  $S_1 \leq G/N_1 \leq \text{Aut}(S_1)$ , where  $\{p, 31\} \subseteq \pi(S_1)$  and  $\{p, 31\} \cap \pi(N_1) = \emptyset$ . Since  $|G|_2 = |A_{31}|_2 = 2^{25}$ , by [19], we get that  $S_1$  is isomorphic to  $O_{10}^+(2), L_5(4), ON, Th$  or  $A_{31}$ .

Let  $S_1 \cong O_{10}^+(2), L_5(4), ON$  or  $Th$ . Thus by the order of  $S_1$ ,  $\{29, 23\} \subseteq \pi(N_1)$ . By Lemma 2.4,  $\{29, 23\}$  is an independent subset of  $\Gamma(N_1)$ . So by Corollary 3.4, there is a non-abelian simple group  $S_2$  such that  $S_2 \leq N_1/N_2 \leq \text{Aut}(S_2)$ , where  $\{29, 23\} \subseteq \pi(S_2)$ . Then by [19],  $|S_2|_2 \geq 2^{21}$ . So  $2^{25} = |G|_2 \geq |S_1|_2 |S_2|_2 \geq 2^9 \cdot 2^{21}$ , which is a contradiction. Therefore  $S \cong A_{31}$  and so  $G \cong A_{31}$ .  $\square$

*Proof of Theorem 1.1.* It comes from the previous Lemmas.

## 5. Appendix

Here, we have listed two simple Maple procedures that compute  $\pi_e(A_n)$  and  $\pi_e(S_n)$ . The electronic versions of these procedures can be obtained by contacting the author.

```
with(NumberTheory): with(ArrayTools):
#Procedure mA computes the set of element orders of the alternating group A_n:
mA := proc(n) local l, T_o, T_e, i, T;
l := proc(m) local S, A, B, k, r;
S := 0; A := ifactors(m); B := A[2]; k := Size(B);
```

Table 1: The conditions of Corollary 3.4 for  $A_n$  when  $7 \leq n \leq 24$

$n$	$ A_n $	$m_1(A_n)$	$\rho$	$S$
7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	7	{5, 7}	$A_7$
8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	15	{5, 7}	$A_8, L_3(4), A_7$
9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	15	{5, 7}	$A_9, A_8, L_3(4), A_7$
10	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	21	{5, 7}	$A_{10}, J_2$
11	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	21	{7, 11}	$A_{11}, M_{22}$
12	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	35	{7, 11}	$A_{12}, A_{11}, M_{22}$
13	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	35	{7, 11, 13}	$A_{13}$
14	$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$	60	{11, 13}	$A_{13}, A_{14}$
15	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	105	{11, 13}	$A_{13}, A_{14}, A_{15}$
16	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	105	{11, 13}	$A_{13}, A_{14}, A_{15}, A_{16}$
17	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	105	{11, 13, 17}	$A_{17}$
18	$2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	140	{11, 13, 17}	$A_{18}, A_{17}$
19	$2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	210	{13, 17, 19}	$A_{19}$
20	$2^{17} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	210	{13, 17, 19}	$A_{19}, A_{20}$
21	$2^{17} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	420		
22	$2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19$	420		
23	$2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23$	420	{19, 23}	$A_{23}$
24	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23$	420	{19, 23}	$A_{23}, A_{24}$

```

for r to 1/2*k[2] do S := S + B[r][1]B[r][2]; end do; S; end proc;
T_o := ; T_e := ;
for i to Landau(n) do if i::even and l(i) = n - 2 then T_e := T_e union i; end if; end do;
for i to Landau(n) do if i::odd and l(i) = n then T_o := T_o union i; end if; end do;
T := T_o union T_e; T;
end proc;

```

```

#Procedure mS computes the set of element orders of the symmetric group S_n:
mS := proc(n) local l, i, T;
l := proc(m) local S, A, B, k, r;
S := 0; A := ifactors(m); B := A[2]; k := Size(B);
for r to 1/2*k[2] do S := S + B[r][1]B[r][2]; end do;
S; end proc;
T := ; for i to Landau(n) do if l(i) = n then T := T union i; end if; end do; T;
end proc;

```

For example by using these procedures, we compute the set of all element orders of the alternating group  $A_{32}$  and the symmetric group  $S_{31}$ :

```

mA(32);
{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35,
36, 38, 39, 40, 42, 44, 45, 46, 48, 50, 51, 52, 54, 55, 56, 57, 60, 63, 65, 66, 68, 69, 70, 72, 75, 76, 77, 78, 80, 84, 85,
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765, 770, 780, 792, 819, 840, 858, 910, 924, 936, 990, 1001, 1020, 1092, 1155, 1170, 1260, 1320, 1365, 1386, 1540,
1560, 1785, 1820, 1848, 1980, 2145, 2310, 2520, 2730, 3465, 4620}

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336, 340, 342, 345, 357, 360, 364, 374, 380, 385, 390, 396, 399, 408, 420, 429, 440, 455, 456, 462, 468, 476, 495, 504, 510, 520, 528, 532, 546, 560, 561, 570, 572, 585, 595, 612, 616, 630, 660, 665, 680, 693, 714, 715, 720, 728, 765, 770, 780, 792, 798, 819, 840, 858, 910, 924, 936, 990, 1001, 1020, 1092, 1140, 1155, 1170, 1190, 1260, 1320, 1365, 1386, 1428, 1430, 1540, 1560, 1638, 1680, 1716, 1820, 1848, 1980, 2184, 2310, 2340, 2520, 2730, 2772, 3080, 4620}

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