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Characterization of some alternating groups by order and largest element order

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ABSTRACT: The prime graph (or Gruenberg-Kegel graph) of a finite group is a well-known graph. In this paper, first, we investigate the structure of the finite groups with a non-complete prime graph. Then as an application, we prove that every alternating group A_n, where $n \leq 31$ is determined by its order and its largest element order. Also, we show that A_{32} is not characterizable by order and the largest element order.

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1. Introduction

Throughout this paper, n and G denote a natural number and a finite group, respectively. For a given prime number p, we let n_p denote the p-part of n; i.e., $n_p = p^k$ if $p^k \mid n$ but $p^{k+1} \nmid n$. The set of all prime divisors of $|G|$ is denoted by $\pi(G)$. Also, the set of all element orders of G is denoted by $\pi_e(G)$. The *prime graph* (or *Gruenberg-Kegel graph*) of G, which is denoted by $\Gamma(G)$ is a simple graph whose vertex set is $\pi(G)$ and two distinct primes p and q are adjacent in $\Gamma(G)$ if and only if $pq \in \pi_e(G)$. A subset ρ of vertices of $\Gamma(G)$ is called an independent subset (or an isolated point set) of $\Gamma(G)$, whenever every two distinct primes in ρ are non-adjacent in $\Gamma(G)$.

Let $m_1(G)$ be the largest element order of G, in the other word, $m_1(G)$ is the maximum of $\pi_e(G)$. In general, if $k = |\pi_e(G)|$, then for $2 \leq i \leq k$, we define $m_i(G)$ as follows:

$$m_i(G) = \max\{a \mid a \in \pi_e(G) \setminus \{m_1(G), \ldots, m_{i-1}(G)\}\}$$

For a finite simple group S there are a lot of results about the numbers $m_1(S)$, $m_2(S)$ and $m_3(S)$ (see [7, 12]). Also, the characterization of finite simple groups by their arithmetical properties has been researched widely. For instance, Mazurov et al. in [17], show that every finite simple group S can be determined by $|S|$ and $\pi_e(S)$. Then some authors tried to investigate the characterization of finite simple groups by using fewer conditions. In [8, 18], it is proved that there are some finite simple groups S, which are determined by $|S|$ and $m_1(S)$. For more results see [1, 3, 11, 13, 9]. However, the main result in [4], is not true in general (it is enough to consider the classical simple groups $B_4(3^4)$ and $C_4(3^4)$).

In this paper, first, we consider the finite groups whose prime graphs are not complete. Then as an application we prove the following theorem:

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Theorem 1.1. *Let G be a finite group and A_n be an alternating group such that $n \leq 31$. Then G is isomorphic to A_n if and only if $|G| = |A_n|$ and $m_1(G) = m_1(A_n)$.*

By the above theorem, one may ask is any alternating group characterizable by the order and the largest element order? The following proposition gives a negative answer to this question.

Proposition 1.2. *Let Z_2 be the cyclic group of order 2. If $G = S_{31} \times Z_2 \times Z_2 \times Z_2 \times Z_2$, then $|G| = |A_{32}|$ and $m_1(G) = m_1(A_{32})$. In particular, A_{32} is not characterizable by the order and the largest element order.*

Proof. We know that $|G| = |A_{32}|$ and $\pi_e(G) \subseteq \{\mathrm{lcm}(a, 2) \mid a \in \pi_e(S_{31})\}$. In Appendix, it is shown that $m_1(A_{32}) = m_1(S_{31}) = 4620$ and also if $a \in \pi_e(S_{31})$ is an odd number, then $a \leq 1365$. Thus $m_1(G) \geq 2a$ and this implies that $m_1(G) = m_1(S_{31}) = m_1(A_{32})$.

□

We note that our main tool for considering Theorem 1.1 is the fact that when $n \leq 20$ or $n \in \{23, 24\}$, since $m_1(G) = m_1(A_n)$, $\rho := \{p \mid n/2 \leq p \leq n\} \cap \pi(G)$ would be an independent subset of $\Gamma(G)$ and so G has a non-complete prime graph. Also, if $n \in \{21, 22\}$ or $25 \leq n \leq 31$, then we use a method, which is inspired by [5, Page 8]. We note that in the appendix, there are two procedures by Maple software for computing $\pi_e(A_n)$ and $\pi_e(S_n)$.

Recall that $\mathrm{Soc}(G)$ denotes the socle of G (the subgroup generated by all the minimal nontrivial normal subgroups of G). The other notation and terminologies in this paper are standard and the reader is referred to [2, 10] if necessary.

2. Preliminary Results

Lemma 2.1. *[20, Lemma 4] In S_m (resp. in A_m) there is an element of order $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s}$, where $p_1, p_2, \ldots, p_s$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_s$ are naturals, if and only if $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq m$ (resp. $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq m$ for odd n and $p_1^{\alpha_1} + p_2^{\alpha_2} + \cdots + p_s^{\alpha_s} \leq m - 2$ for even n).*

Lemma 2.2. *[16, Lemma 1] Let a finite group G have a normal series of subgroups $1 \leq K \leq M \leq G$, and the primes p, q and r are such that p divides $|K|$, q divides $|M/K|$, and r divides $|G/M|$. Then p, q, and r cannot be pairwise nonadjacent in $\Gamma(G)$.*

Lemma 2.3. *(See, for example, [10]) Let $G = F \rtimes H$ be a Frobenius group with kernel F and complement H. Then $|H|$ divides $|F| - 1$.*

Corollary 2.4. *Let G be a finite group and N be a normal subgroup of G. Then the following assertions hold:*

1) *Let p and q be two distinct primes in $\pi(G)$. If $p \in \pi(N)$, $q \in \pi(G/N)$ and $\{p, q\}$ is an independent subset of $\Gamma(G)$, then $q \mid (|N|_p - 1)$.*

2) *Let p, q and r be three pariwise distinct primes in $\pi(G)$. If $p \in \pi(N)$ and $\{q, r\} \subseteq \pi(G/N)$ and G/N is solvable, then p, q and r cannot be pairwise nonadjacent in $\Gamma(G)$.*

Proof. 1) Let P be a Sylow p-subgroup of N. By Frattini's argument, $G/N \cong N_G(P)/N_N(P)$. In view of the hypothesis, we conclude that $N_G(P)$ contains an element of order q. So $N_G(P)$ contains a subgroup isomorphic to the semidirect product $P \rtimes Q$ where Q is a cyclic subgroup of order q. On the other hand, by the assumption, G does not contain any element of order pq. Hence, Q acts fixed point freely on P. Thus, $P \rtimes Q$ is a Frobenius group and so by Lemma 2.3, $q \mid (|P| - 1)$, which implies that $q \mid (|N|_p - 1)$.

2) Put $\bar{G} = G/N$ and $\rho = \{q, r\}$. Recall that $\bar{G}$ is a solvable group and $\rho \subseteq \pi(\bar{G})$. Take a Hall ρ- subgroup $\bar{H}$ of $\bar{G}$. We know that $O_q(\bar{H}) \neq 1$ or $O_r(\bar{H}) \neq 1$. So without loss of generality, we may assume that $\bar{G}$ contains a subgroup isomorphic to the semidirect product $\bar{H}_1 \rtimes \bar{H}_2$ in which $\pi(\bar{H}_1) = \{q\}$ and $\pi(\bar{H}_2) = \{r\}$.

Now let P be a Sylow p-subgroup of N. Similar to the previous case, it follows that $\bar{G} \cong N_G(P)/N_N(P)$. Recall that $\bar{H}_1 \rtimes \bar{H}_2$ is a subgroup of $\bar{G}$. Consequently, $N_G(P)/N_N(P)$ contains a subgroup isomorphic to $\bar{H}_1 \rtimes \bar{H}_2$. Hence, there is a normal series $1 < N_N(P) < T_1 < T_2$ in $N_G(P)$ such that $T_1/N_N(P) \cong \bar{H}_1$ and $T_2/N_N(P) \cong \bar{H}_1 \rtimes \bar{H}_2$. Also, by the above argument, $p \in \pi(N_N(P))$, $\pi(T_1/N_N(P)) = \{q\}$ and $\pi(T_2/T_1) = \{r\}$. Therefore, by Lemma 2.2, we get that the subset $\{p, q, r\}$ can not be an independent subset of $\Gamma(G)$, which completes the proof.

□

Lemma 2.5. *Let G be a finite group, M be a normal subgroup of G and G/M contain a subgroup S, which is isomorphic to a simple group. If R is a Sylow r-subgroup of M, then one of the following assertions holds:*

1) $|S| \mid |\mathrm{Aut}(R)|$,

2) *If $a \in \pi_e(S)$ and $r^\alpha \in \pi_e(R)$, then $\mathrm{lcm}(r^\alpha, a) \in \pi_e(G)$.*

Proof. Put $N = N_G(R)$, $L = N_M(R)$ and $C = C_G(R)$. By Frattini's argument, $G/M \cong N/L$. Hence by the assumption we get that N/L contains a subgroup isomorphic to the simple group S. Let K be a subgroup of N such that $K/L \cong S$ is a simple group. Since K/L is a simple subgroup of N/L and CL/L is a normal subgroup of N/L, it follows that either $K/L \cap CL/L = 1$ or $K/L \leq CL/L$. We consider each possibilities:

1) Let $K/L \cap CL/L = 1$. Then we obtain the following relation:

$$K/L \cong \frac{(K/L)(CL/L)}{CL/L} \leq \frac{N/L}{CL/L} \cong N/CL.$$

So $|K/L| \mid |N/CL|$. On the other hand:

$$|N/CL| = |\frac{N/C}{CL/C}| \mid |\mathrm{Aut}(R)|.$$

Therefore, $|K/L| \mid |\mathrm{Aut}(R)|$ and consequently, $|S| \mid |\mathrm{Aut}(R)|$.

2) Let $K/L \leq CL/L$. Since $CL/L \cong C/C_L(R)$, it follows that $C/C_L(R)$ contains a subgroup isomorphic to K/L. Recall that $C = C_G(R)$. Hence if $a \in \pi_e(S) = \pi_e(K/L)$ and $r^\alpha \in \pi_e(R)$, then $a \in \pi_e(C_G(R))$ and so $\mathrm{lcm}(r^\alpha, a) \in \pi_e(G)$, which completes the proof.

□

Lemma 2.6. *Let G be a finite group, $N_2 \leq N_1$ be some characteristic subgroups of G such that $S_1 \leq G/N_1 \leq \mathrm{Aut}(S_1)$ and $S_2 \leq N_1/N_2 \leq \mathrm{Aut}(S_2)$ where S_1 and S_2 are some non-abelian simple groups. Then G/N_2 has a subgroup isomorphic to $S_1 \times S_2$.*

Proof. Let K_1 and K_2 be the subgroups of G such that $K_1/N_1 \cong S_1$ and $K_2/N_2 \cong S_2$. Put $C/N_2 := C_{G/N}(K_2/N_2)$. Note that since N_1 and N_2 are characteristic subgroups in G, K_1 and K_2 are some normal subgroups of G and so C/N_2 is a normal subgroup of G/N_2.

By the hypothesis, K_2/N_2 is not abelian and $K_2/N_2 \leq N_1/N_2 \leq \mathrm{Aut}(K_2/N_2)$. This shows that $C_{N_1/N_2}(K_2/N_2) = 1$. Hence because of $C/N_2 \cap N_1/N_2 \leq C_{N_1/N_2}(K_2/N_2)$, we have:

$$\frac{C}{N_2} \cap \frac{N_1}{N_2} = 1. \tag{1}$$

This yields that $(C/N_2) \times (K_2/N_2) \leq G/N_2$. So in the sequel, we show that C/N_2 has a subgroup isomorphic to K_1/N_1, which implies that G/N_2 has a subgroup isomorphic to $S_1 \times S_2$.

First, let $K_1/N_1 \cap CN_1/N_1 = 1$. Similar to the above argument, since K_1/N_1 is a non-ableian simple and $C_{G/N_1}(K_1/N_1) = 1$, $CN_1/N_1 = 1$. Hence $C \leq N_1$ and so by Relation (1), $C/N_2 = 1$. This implies that:

$$S_2 \cong \frac{K_2}{N_2} \cong \frac{(K_2/N_2)(C/N_2)}{C/N_2} \leq \frac{G/N_2}{C/N_2} \cong \frac{G}{C} \leq \mathrm{Aut}(S_2). \tag{2}$$

On the other hand, because of $C/N_2 = 1$, by Relation (2), we get that:

$$\frac{G}{K_2} \cong \frac{G/N_2}{K_2/N_2} \hookrightarrow \frac{\mathrm{Aut}(S_2)}{S_2}. \tag{3}$$

This implies that G/K_2 is a solvable group. Also, since G/N_1 is a quotient of G/K_2, we deduce that G/N_1 is solvable, too. On the other hand, K_1/N_1 is a subgroup of the solvable group G/N_1, which is impossible since $K_1/N_1 \cong S_1$ is a non-abelian simple group.

Therefore, $K_1/N_1 \cap CN_1/N_1 \neq 1$ and this implues that $K_1/N_1 \leq CN_1/N_1$. We remark that by Realtion (1), $C/N_2 \cap N_1/N_2 = 1$. So we have:

$$\frac{CN_1}{N_1} \cong \frac{CN_1/N_2}{N_1/N_2} \cong \frac{(C/N_2)(N_1/N_2)}{N_1/N_2} \cong \frac{C}{N_2} \tag{4}$$

Thus, since $K_1/N_1 \leq CN_1/N_1$, by the above relation we conclude that C/N_2 has a subgroup isomorphic to K_1/N_1, which completes the proof.

□

3. Finite groups with the non-complete prime graph

Lemma 3.1. *Let G be a finite group, K_1 and K_2 two normal subgroups of G and ρ an independent subset of $\Gamma(G)$. Then either $\pi(K_1) \cap \rho \subseteq \pi(K_2) \cap \rho$ or $\pi(K_2) \cap \rho \subseteq \pi(K_1) \cap \rho$. Moreover, if N is the product of all normal subgroups K of G such that $|\pi(K) \cap \rho| \leq 1$, then $|\pi(N) \cap \rho| \leq 1$.*

Proof. For $1 \leq i \leq 2$, put $\pi_i = \pi(K_i) \cap \rho$. If $\pi_1 \nsubseteq \pi_2$ and $\pi_2 \nsubseteq \pi_1$, then there exist two primes p_1 and p_2 such that $p_1 \in \pi_1 \setminus \pi_2$ and $p_2 \in \pi_2 \setminus \pi_1$. This implies that $p_1 \in \pi(K_1/(K_1 \cap K_2))$ and $p_2 \in \pi(K_2/(K_1 \cap K_2))$. By the following relation:

$$\frac{K_1K_2}{K_1 \cap K_2} \cong \frac{K_1}{K_1 \cap K_2} \times \frac{K_2}{K_1 \cap K_2}$$

it follows that K_1K_2 contains an element of order p_1p_2, which contradicts to the assumption. Therefore, $\pi_1 \subseteq \pi_2$ or $\pi_2 \subseteq \pi_1$ and consequently, there is $i \in \{1, 2\}$, such that $\pi(K_1K_2) \cap \rho \subseteq \pi_i$. Also this implies that if $|\pi_1| \leq 1$ and $|\pi_2| \leq 1$, then $|\pi(K_1K_2) \cap \rho| \leq 1$.

Finally, let N be the product of all normal subgroups K of G such that $|\pi(K) \cap \rho| \leq 1$. Then by the above discussion, $|\pi(N) \cap \rho| \leq 1$, which completes the proof.

□

We note that by the previous lemma, if ρ is an independent subset of $\Gamma(G)$ such that $|\rho| \geq 2$, then G contains a normal subgroup N, which is the largest normal subgroup of G among the normal subgroups of G with the property $|\pi(N) \cap \rho| \leq 1$.

Theorem 3.2. *Let G be a finite group and ρ be an independent subset of $\Gamma(G)$ such that $|\rho| \geq 2$. Then one of the following assertions holds:*

1) G has a normal series $1 \leq N \leq L \leq G$, where $L/N = \mathrm{Soc}(G/N)$ is the socle of G/N. Moreover, in this case $\pi(N) \cap \rho = \{p\}$, $\pi(L/N) \cap \rho = \{q\}$ and $\rho = \{p, q\}$.

2) There exists a normal subgroup N of G and a non-abelian simple group S such that

$$S \leq \frac{G}{N} \leq \mathrm{Aut}(S),$$

where $|\pi(N) \cap \rho| \leq 1$ and $|\pi(S) \cap \rho| \geq 2$. Moreover, if $|\rho| \geq 3$, then $|\pi(S) \cap \rho| \geq |\rho| - 1$.

Proof. Let G be a finite group, ρ be an independent subset of $\Gamma(G)$ such that $|\rho| \geq 2$ and N be the product of all normal subgroups K of G such that $|\pi(K) \cap \rho| \leq 1$. Also let L/N be the socle of G/N. By Lemma 3.1, $|\pi(N) \cap \rho| \leq 1$. Let $M_1/N, \ldots, M_t/N$ be the minimal normal subgroups of G/N such that $L/N \cong M_1/N \times \cdots \times M_t/N$. We know that for each $1 \leq i \leq t$, M_i/N is a direct product of some isomorphic simple groups. Also since N is a pure subgroup of M_i, $|\pi(M_i) \cap \rho| > 1$ and so $|\pi(M_i/N) \cap \rho| \geq 1$. In the sequel, we consider the following cases, seperaitly:

1) Let for every $1 \leq i \leq t$, $|\pi(M_i/N) \cap \rho| = 1$. In view of the definition of N, we conclude that there exist two distinct primes p and q such that $\pi(N) \cap \rho = \{p\}$ and for every $1 \leq i \leq t$, $\pi(M_i/N) \cap \rho = \{q\}$. This implies that $\pi(L/N) \cap \rho = \{q\}$.

By the above discussion, $\{p, q\} \subseteq \rho$. Let there exist $r \in \rho \setminus \{p, q\}$. Recall that, $\pi(N) \cap \rho = \{p\}$ and $\pi(L/N) \cap \rho = \{q\}$. This shows that $r \in \pi(G/L)$. On the other hand, $\{p, q, r\}$ is an independent subset of $\Gamma(G)$, which contradicts to Lemma 2.2. Therefore, $\rho = \{p, q\}$, which get the assertion (1) in the theroem.

2) Let there exist $1 \leq i \leq t$, such that $|\pi(M_i/N) \cap \rho| \geq 2$. Without lose of generality, suppose that $|\pi(M_1/N) \cap \rho| \geq 2$. In this case, if $t \geq 2$, then $M_1/N \times M_2/N$ contains an element of order pq where $p \in \pi(M_1/N) \cap \rho$ and $q \in \pi(M_2/N) \cap \rho$, which is a contradiction. Thus, $t = 1$. Also since M_1/N is a direct product of some isomorphic simple groups, by a similar argument, we conclude that $L/N = M_1/N$ is isomorphic to a non-abelian simple group. Then in this case, $C_{G/N}(L/N) = 1$ since L/N is the socle of G/N. Let L/N be isomorphic to a non-abelian simple group S. So the following relation holds:

$$S \leq \bar{G} := \frac{G}{N} \leq \mathrm{Aut}(S).$$

We recall that in this case, $L/N = M_1/N \cong S$ and by the assumption $|\pi(M_1/N) \cap \rho| \geq 2$. So $|\pi(S) \cap \rho| \geq 2$.

Finally, we prove that if $|\rho| \geq 3$, then $|\pi(S) \cap \rho| \geq |\rho| - 1$. On the contrary, let $|\rho| \geq 3$ and $|\pi(S) \cap \rho| \leq |\rho| - 2$. This implies that there are two distinct primes p and q in ρ such that $\{p, q\} \subseteq \pi(N) \cup \pi(\bar{G}/S)$ and $\{p, q\} \cap \pi(S) = \emptyset$. Since $|\rho| \geq 3$, if $\{p, q\} \subseteq \pi(\bar{G}/S)$, then by Corollary 2.4 (Assertion 2), we get a contradiction since $\bar{G}/S$ is solvable. Similarly, if $p \in \pi(N)$ and $q \in \pi(\bar{G}/S)$, then by Lemma 2.2, we arrive a contradiction. Therefore, when $|\rho| \geq 3$, we deduce that $|\pi(S) \cap \rho| \geq |\rho| - 1$, which completes the proof.

□

Example 3.1. *Let $G = 11^2 : \mathrm{SL}_2(5)$, which is a Frobenius group with kernel 11^2 and complement $\mathrm{SL}_2(5)$. In the prime graph of G, the subsets $\rho_1 = \{2, 11\}$ and $\rho_2 = \{11, 3, 5\}$ are two independent subsets. If we choose ρ_1 as the independent subset said in Theorem 3.2, then we have $N = 11^2$ and $L = 11^2 : 2$, which shows that Case (1) of Theorem 3.2 holds. Also if we choose ρ_2 as the independent subset ρ in Theorem 3.2, then $N = 11^2 : 2$ and we have*

$$\mathrm{PSL}_2(5) \leq G/N \leq \mathrm{Aut}(\mathrm{PSL}_2(5)),$$

which satisfies Case (2) of Theorem 3.2.

Now by Theorem 3.2, we can easily get the following two corollaries which modify [6, Lemma 10] and [13, Lemma 2.3]:

Corollary 3.3. *If G is a finite group and ρ an independent subset of $\Gamma(G)$ such that $|\rho| \geq 3$, then there exists a nonabelian simple group S and a normal subgroup N of G such that*

$$S \leq \frac{G}{N} \leq \mathrm{Aut}(S),$$

and also we have $|\pi(S) \cap \rho| \geq |\rho| - 1$ and $|\pi(N) \cap \rho| \leq 1$.

Corollary 3.4. *Let G be a finite group, ρ be an independent subset of $\Gamma(G)$ such that $|\rho| \geq 2$. Also let for every pair of distinct prime numbers p and q belong to ρ we have $p \nmid (q^j - 1)$ and $q \nmid (p^i - 1)$ where $1 < p^i \leq |G|_p$ and $1 < q^j \leq |G|_q$. Then there exists a non-abelian simple group S such that*

$$S \leq \frac{G}{O_{\rho'}(G)} \leq \mathrm{Aut}(S),$$

and also we have $\rho \subseteq \pi(S)$ and $\rho \cap \pi(\mathrm{Out}(S)) = \emptyset$.

Proof. It immediately comes from Theorem 3.2 and Corollary 2.4.

□

4. Proof of Theorem 1.1

Recall that in number theory $Landau(n)$ is a familar notation for $m_1(S_n)$.

Lemma 4.1. *Let A_n be an alternating group. If $n \geq 25$ or $n \in \{21, 22\}$, then $m_1(A_n) \geq pq$ for all distinct primes p and q in $\pi(A_n)$.*

Proof. Let p and q be two distincet primes in $\pi(A_n)$. By the definition of $m_1(A_n)$ and Lemma 2.1, $m_1(A_n) \geq m_1(S_{n-2}) = Landau(n-2)$. In view of [14], if $n \geq 906$, then

$$Landau(n) \geq e^{\sqrt{n\, ln(n)}}.$$

Hence,

$$m_1(A_n) \geq e^{\sqrt{(n-2)\, ln(n-2)}}.$$

On the other hand, by the hypothesis, $(n-2)^3 > n\,(n-2) \geq p\,q$. Using an easy computation, we can show that if $n \geq 906$, then

$$e^{\sqrt{(n-2)\, ln(n-2)}} \geq (n-2)^3,$$

Thus, by the above argument if $n \geq 906$, then $m_1(A_n) \geq (n-2)^3$ and consequently, $m_1(A_n) > pq$. Finally, by the program in the appendix, and an easy compution we deduce that if $25 \leq n \leq 905$ or $n \in \{21, 22\}$, then $m_1(A_n) \geq p\,q$, which completes the proof.

□

Lemma 4.2. *If G is a finite group such that $|G| = |A_n|$ and $m_1(G) = m_1(A_n)$, where $n \leq 20$ or $n \in \{23, 24\}$, then $G \cong A_n$.*

Proof. If $n \leq 4$, then by using GAP we can see that $G \cong A_n$. Also, if $n \in \{5, 6\}$, then by [18, Theorem 1], $G \cong A_n$. So let $7 \leq n \leq 20$ or $n \in \{23, 24\}$. By Table 1, there exists an independent subset ρ of $\Gamma(G)$ such that ρ satisfies the conditions of Corollay 3.4, which implies that there is a non-abelian simple group S such that

$$S \leq \bar{G} := \frac{G}{M} \leq \mathrm{Aut}(S)$$

where $M = O_{\rho'}(G)$, $\rho \subseteq \pi(S)$ and $\pi(\bar{G}/S) \cap \rho = \emptyset$. Moreover, by the assumption $|S| \mid |A_n|$. In view of [19, Table 1], the possible cases for S are indicated in Table 1. Hence, if $n \in \{7, 13, 14, 17, 19, 23\}$, then by Table 1, $S \cong A_n$ and so $G \cong A_n$ since $|G| = |A_n|$. In the sequel, for the other cases, we suppose that S is not isomorphic to A_n.

Let $n = 8$. By Table l, $S \cong A_7$ or $L_3(4)$. If $S \cong A_7$, then G/M is isomorphic to either A_7 or S_7 and $|M| \mid 8$. On the other hand, A_7 and S_7 do not contain any element of order 15, in while $m_1(G) = m_1(A_8) = 15$, which is a contradiction. If $S \cong L_3(4)$, then $|S| = |A_8|$ and so $G \cong L_3(4)$, which is impossible since by [2], $m_1(L_3(4)) = 7$.

Let $n = 9$. By Table l, $S \cong A_8, A_7$ or $L_3(4)$. If $S \cong A_8, A_7$ or $L_3(4)$, then $7 \mid |G/M|$ and $|M|_3 = 3$ or 9. By Corollary 2.4, we get that G contains an element of order 21, which is a contradiction since $m_1(G) = m_1(A_9) = 15$.

Let $n = 10$. By Table l, $S \cong J_2$. Then by [2], we deduce that $|M| = 9$ and S contains an element of order 10. Hence by Lemma 2.5, we get that G contains an element of order 30, which is a contradiction since $m_1(G) = m_1(A_{10}) = 21$.

Let $n = 11$. By Table l, $S \cong M_{22}$. Then $11 \mid |S|$ and $|M|_3 = 3^2$. So by Corollary 2.4, we get that G contains an element of order 33, which is a contradiction since $m_1(G) = m_1(A_{11}) = 21$.

Let $n = 12$. By Table l, $S \cong A_{11}$ or M_{22}. Let $S \cong M_{22}$. Then $11 \mid |S|$ and $|M|_5 = 5$. So by Lemma 2.5, we get that $55 \in \pi_e(G)$, which is impossible since $m_1(G) = m_1(A_{12}) = 35$. Let $S \cong A_{11}$. Then $|M|_3 = 3$ and S contains an element of order 20. So by Lemma 2.5, we get that $60 \in \pi_e(G)$, which is a contradiction.

Let $n = 14$. By Table l, $S \cong A_{13}$. Then $|M|_7 = 7$. So $|M|_7 = 7$ and S contains an element of order 30. Hence by Lemma 2.5, we get that $210 \in \pi_e(G)$, which is a contradiction since $m_1(G) = m_1(A_{14}) = 60$.

Let $n = 15$. By Table l, $S \cong A_{14}$ or A_{13}. Let $S \cong A_{13}$ or A_{14}. Then $|M|_5 = 5$ and S contains an element of order 28. Hence by Lemma 2.5, we get that $140 \in \pi_e(G)$, which is a contradiction since $m_1(G) = m_1(A_{15}) = 105$.

Let $n = 16$. By Table l, $S \cong A_{15}, A_{14}$ or A_{13}. We note that $m_1(G) = m_1(A_{16}) = 105$. So if $S \cong A_{13}$ or A_{14}, then similar to the case $n = 15$, we get that $140 \in \pi_e(G)$, which is a contradiction. Let $S \cong A_{15}$. In this case, we have S contains an element of order 105 and also $|M| = 8$ or 16. Thus, by Lemma 2.5, we get that $210 \in \pi_e(G)$, which is impossible.

Let $n = 18$. By Table l, $S \cong A_{17}$ and $m_1(G) = m_1(A_{18}) = 140$. If $S \cong A_{17}$, then $|M|_3 = 9$ and $70 \in \pi_e(S)$. So by Lemma 2.5, $210 \in \pi_e(G)$, which is impossible.

Let $n = 20$. By Table l, $S \cong A_{19}$ and $m_1(G) = m_1(A_{20}) = 210$. If $S \cong A_{19}$, then $|M|_5 = 5$ and $77 \in \pi_e(S)$, and so by Lemma 2.5, $5 \cdot 77 \in \pi_e(G)$, which is impossible.

Let $n = 24$. By Table l, $S \cong A_{23}$ and $m_1(G) = m_1(A_{24}) = 420$. If $S \cong A_{23}$, then $|M|_3 = 3$ and $385 \in \pi_e(A_{23})$ and so by Lemma 2.5, $3 \cdot 385 \in \pi_e(G)$, which is impossible.

Finally, by the above discussions we conclude that if $|G| = |A_n|$ and $m_1(G) = m_1(A_n)$, then $S \cong A_n$ and consequently, $G \cong A_n$, which completes the proof.

□

Lemma 4.3. *Let G be a finite group such that $|G| = |A_n|$ and $m_1(G) = m_1(A_n)$, where $n \in \{21, 22\}$, then $G \cong A_n$.*

Proof. Let $P \in \mathrm{Syl}_{19}(G)$, $C := C_G(P)/P$ and $\rho := \{11, 13, 17\}$. We consider the following cases:

Case 1. Let $\rho \subseteq \pi(C)$. Since $m_1(G) = m_1(A_{21}) = m_1(A_{22}) = 420$ and $19 \cdot 13 \cdot 11 > 420$, we get that ρ is an independent subset of $\Gamma(C)$. So by Corollary 3.4, there exists a non-abelian simple group S such that $S \leq C/N \leq \mathrm{Aut}(S)$, where $\rho \subseteq \pi(S)$ and $\pi(N) \cap \rho = \emptyset$. Hence by [19], S is isomorphic to either A_{17} or A_{18}. On the other hand, $m_1(A_{17}) = 105$. This implies that $m_1(G) \geq m_1(C) \geq 19 \cdot 105$, which is impossible since $m_1(G) = 405$.

Case 2. Let $\rho \nsubseteq \pi(C)$. So there exists a prime $p \in \rho$ such that p is not adjacent to 19 in $\Gamma(G)$. Thus by Corollary 3.4, there is a non-abelian simple group S_1 such that $S_1 \leq G/N_1 \leq \mathrm{Aut}(S_1)$, where $\{p, 19\} \subseteq \pi(S_1)$ and $\{p, 19\} \cap \pi(N_1) = \emptyset$. By [19], we get that $S_1 \cong J_3$, J_1, HN, $U_4(8)$ or A_m where $19 \leq m \leq 22$.

Let $S_1 \cong J_3$. In this case, $\{17, 19\} \subseteq \pi(S_1)$ and $\{11, 13\} \subseteq \pi(N_1)$. By Lemma 2.4, it follows that G contains some elements of orders $19 \cdot 11$ and $19 \cdot 13$. However, $\{11, 13\}$ is an independent subset of $\Gamma(N_1)$, since $m_1(G) < 19 \cdot 13 \cdot 11$. So again by Corollary 3.4, there is a non-abelian simple group S_2 such that $S_2 \leq N_1/N_2 \leq \mathrm{Aut}(S_2)$, where $\{11, 13\} \subseteq \pi(S_2)$. Then by [19], S_2 is isomorphic to A_{13}, A_{14}, A_{15}, A_{16}, Suz or Fi_{22}. By using GAP, we can see that in every possible cases for S_2, $m_1(S_2) \geq 24$. Thus, by Lemma 2.6, $m_1(G) \geq 19 \times m_1(S_2) \geq 19 \cdot 24 = 456$, which is a contradiction.

Let $S_1 \cong J_1$ or HN. In this case, $\{13, 17\} \subseteq \pi(N_1)$. Similar to the above discussion, we cocnlude that $\{13, 17\}$ is an independent subset of $\Gamma(N_1)$ and so there is a non-abelian simple group S_2 such that $S_2 \leq N_1/N_2 \leq \mathrm{Aut}(S_2)$, where $\{13,17\} \subseteq \pi(S_2)$. Then by [19], S_2 is isomorphic to $U_4(4)$, $L_3(16)$, A_{17} or A_{18}. By [12, Tables 1 and 2] and using GAP, $m_1(S_2) \geq 65$. So by Lemma 2.6, $m_1(G) \geq 19 \times m_1(S_2) \geq 19 \cdot 65$, which is a contradiction.

Let $S_1 \cong U_4(8)$. In this case, $\{11,17\} \subseteq \pi(N_1)$. Thus $\{11,17\}$ is an independent subset of $\Gamma(N_1)$ and so there is a non-abelian simple group S_2 such that $S_2 \leq N_1/N_2 \leq \mathrm{Aut}(S_2)$, where $\{11,17\} \subseteq \pi(S_2)$. Similar to the previous case, we get a contradiction.

Hence $S \cong A_m$ where $m \in \{19,20,21,22\}$. Let $m = 19$ or $m = 20$. Then $|N|_7 = 7$ and $165 \in \pi_e(S)$. So by Lemma 2.5, $7 \cdot 165 \in \pi_e(G)$, which is impossible. If $n = 22$ and $m = 21$, then $|N|_{11} = 11$ and by Lemma 2.5, $11 \cdot 165 \in \pi_e(G)$, which is impossible. Therefore, if $n \in \{21,22\}$, then G is isomorphic A_n.

□

Lemma 4.4. *Let n be an integer such that $25 \leq n \leq 28$. If G is a group such that $|G| = |A_n|$ and $m_1(G) = m_1(A_n)$, then $G \cong A_n$.*

Proof. Let $P \in \mathrm{Syl}_{23}(G)$, $C := C_G(P)/P$ and $\rho := \{13,17,19\}$. We consider the following cases:

Case 1. Let $\rho \subseteq \pi(C)$. Since $m_1(G) \leq m_1(A_{28}) = 1365$, $m_1(G) < 23 \cdot 17 \cdot 13$. So ρ is an independent subset of $\Gamma(C)$. So by Corollary 3.4, there exists a non-abelian simple group S such that $S \leq C/N \leq \mathrm{Aut}(S)$, where $\rho \subseteq \pi(S)$ and $\pi(N) \cap \rho = \emptyset$. Since $|S|_2 \leq |G|_2 \leq |A_{28}|_2 = 2^{24}$ by [19], S is isomorphic to either A_m where $19 \leq m \leq 22$. On the other hand $m_1(A_{19}) = 210$. This implies that $m_1(G) \geq m_1(C) \geq 23 \cdot 210$, which is impossible since $m_1(G) \leq 1365$.

Case 2. Let $\rho \not\subseteq \pi(C)$. So there exists a prime $p \in \rho$ such that p is not adjacent to 23 in $\Gamma(G)$. Thus by Corollary 3.4, there is a non-abelian simple group S_1 such that $S_1 \leq G/N_1 \leq \mathrm{Aut}(S_1)$, where $\{p,23\} \subseteq \pi(S_1)$ and $\{p,23\} \cap \pi(N_1) = \emptyset$. By [19], we get that $S_1 \cong Co_1$, Fi_{23} or A_m where $23 \leq m \leq 28$.

Let $S_1 \cong Co_1$. Thus by the order of S_1, $\{19,17\} \subseteq \pi(N_1)$. By Lemma 2.4, $\{19,17\}$ is an independent subset of $\Gamma(N_1)$. So by Corollary 3.4, there is a non-abelian simple group S_2 such that $S_2 \leq N_1/N_2 \leq \mathrm{Aut}(S_2)$, where $\{19,17\} \subseteq \pi(S_2)$. Then by [19], S_2 is isomorphic to J_3, A_{19}, A_{20}, A_{21} or A_{23}. In each case, $|S_2|_2 \geq 2^7$. So $|G|_2 \geq |S_1|_2|S_2|_2 \geq 2^{21} \cdot 2^7$, which is a contradiction.

Let $S_1 \cong Fi_{23}$. In this case, $\{11,19\} \subseteq \pi(N_1)$. Similar to the above discussion, we cocnlude that $\{11,19\}$ is an independent subset of $\Gamma(N_1)$ and so there is a non-abelian simple group S_2 such that $S_2 \leq N_1/N_2 \leq \mathrm{Aut}(S_2)$, where $\{11,19\} \subseteq \pi(S_2)$. Then by [19], S_2 is isomorphic to J_1. So $|G|_3 \geq |S_1|_3|S_2|_3 \geq 3^{13} \cdot 3$, which is a contradiction since $|G|_3 \leq |A_{28}|_3$.

Hence $S \cong A_m$ where $23 \leq m \leq 28$. Easily we can show that $m = n$. Therefore, since $|G| = |A_n|$, where $25 \leq n \leq 28$, G is isomorphic A_n.

□

Lemma 4.5. *Let n be an integer such that $n \in \{29,30\}$. If G is a group such that $|G| = |A_n|$ and $m_1(G) = m_1(A_n)$, then $G \cong A_n$.*

Proof. Let $P \in \mathrm{Syl}_{29}(G)$, $C := C_G(P)/P$ and $\rho := \{17,19,23\}$. We consider the following cases:

Case 1. Let $\rho \subseteq \pi(C)$. Since $m_1(G) \leq m_1(A_{30}) = 2310$, $m_1(G) < 29 \cdot 19 \cdot 17$. So ρ is an independent subset of $\Gamma(C)$. By Corollary 3.4, there exists a non-abelian simple group S such that $S \leq C/N \leq \mathrm{Aut}(S)$, where $\rho \subseteq \pi(S)$ and $\pi(N) \cap \rho = \emptyset$. By [19], S is isomorphic to either A_m where $23 \leq m \leq 28$. On the other hand $m_1(A_{23}) = 420$. This implies that $m_1(G) \geq m_1(C) \geq 29 \cdot 420$, which is impossible since $m_1(G) \leq 2310$.

Case 2. Let $\rho \not\subseteq \pi(C)$. So there exists a prime $p \in \rho$ such that p is not adjacent to 29 in $\Gamma(G)$. Thus by Corollary 3.4, there is a non-abelian simple group S_1 such that $S_1 \leq G/N_1 \leq \mathrm{Aut}(S_1)$, where $\{p,29\} \subseteq \pi(S_1)$ and $\{p,23\} \cap \pi(N_1) = \emptyset$. By [19], we get that S_1 is isomorphic to Fi'_{24} or A_m where $29 \leq m \leq 30$. If $S_1 \cong Fi'_{24}$, then $|G|_3 < |S_1|_3$, which is impossible.

Hence $S \cong A_m$ where $29 \leq m \leq 30$. Easily we can show that $m = n$. Therefore, since $|G| = |A_n|$, G is isomorphic A_n.

□

Lemma 4.6. *If G is a group such that $|G| = |A_{31}|$ and $m_1(G) = m_1(A_{31})$, then $G \cong A_{31}$.*

Proof. Let $P \in \mathrm{Syl}_{31}(G)$, $C := C_G(P)/P$ and $\rho := \{17,19,23,29\}$. We consider the following cases:

Case 1. Let $\rho \subseteq \pi(C)$. Since $m_1(G) \leq m_1(A_{31}) = 2520$, $m_1(G) < 31 \cdot 19 \cdot 17$. So ρ is an independent subset of $\Gamma(C)$ and by Corollary 3.4, there exists a non-abelian simple group S such that $S \leq C/N \leq \mathrm{Aut}(S)$, where $\rho \subseteq \pi(S)$ and $\pi(N) \cap \rho = \emptyset$. By [19], S is isomorphic to A_m where $29 \leq m \leq 30$. On the other hand $m_1(A_{29}) = 1540$. This implies that $m_1(G) \geq m_1(C) \geq 31 \cdot 1540$, which is impossible since $m_1(G) = 2520$.

Table 1: The conditions of Corollary 3.4 for A_n when $7 \leq n \leq 24$

n	$\|A_n\|$	$m_1(A_n)$	ρ	S
7	$2^3 \cdot 3^2 \cdot 5 \cdot 7$	7	$\{5, 7\}$	A_7
8	$2^6 \cdot 3^2 \cdot 5 \cdot 7$	15	$\{5, 7\}$	$A_8, L_3(4), A_7$
9	$2^6 \cdot 3^4 \cdot 5 \cdot 7$	15	$\{5, 7\}$	$A_9, A_8, L_3(4), A_7$
10	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7$	21	$\{5, 7\}$	A_{10}, J_2
11	$2^7 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11$	21	$\{7, 11\}$	A_{11}, M_{22}
12	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$	35	$\{7, 11\}$	A_{12}, A_{11}, M_{22}
13	$2^9 \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$	35	$\{7, 11, 13\}$	A_{13}
14	$2^{10} \cdot 3^5 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13$	60	$\{11, 13\}$	A_{13}, A_{14}
15	$2^{10} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	105	$\{11, 13\}$	A_{13}, A_{14}, A_{15}
16	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13$	105	$\{11, 13\}$	$A_{13}, A_{14}, A_{15}, A_{16}$
17	$2^{14} \cdot 3^6 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	105	$\{11, 13, 17\}$	A_{17}
18	$2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17$	140	$\{11, 13, 17\}$	A_{18}, A_{17}
19	$2^{15} \cdot 3^8 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	210	$\{13, 17, 19\}$	A_{19}
20	$2^{17} \cdot 3^8 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	210	$\{13, 17, 19\}$	A_{19}, A_{20}
21	$2^{17} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19$	420		
22	$2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19$	420		
23	$2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23$	420	$\{19, 23\}$	A_{23}
24	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23$	420	$\{19, 23\}$	A_{23}, A_{24}

Case 2. Let $\rho \nsubseteq \pi(C)$. So there exists a prime $p \in \rho$ such that p is not adjacent to 31 in $\Gamma(G)$. Thus by Corollary 3.4, there is a non-abelian simple group S_1 such that $S_1 \leq G/N_1 \leq \mathrm{Aut}(S_1)$, where $\{p, 31\} \subseteq \pi(S_1)$ and $\{p, 31\} \cap \pi(N_1) = \emptyset$. Since $|G|_2 = |A_{31}|_2 = 2^{25}$, by [19], we get that S_1 is isomorphic to $O^+_{10}(2)$, $L_5(4)$, ON, Th or A_{31}.

Let $S_1 \cong O^+_{10}(2)$, $L_5(4)$, ON or Th. Thus by the order of S_1, $\{29, 23\} \subseteq \pi(N_1)$. By Lemma 2.4, $\{29, 23\}$ is an independent subset of $\Gamma(N_1)$. So by Corollary 3.4, there is a non-abelian simple group S_2 such that $S_2 \leq N_1/N_2 \leq \mathrm{Aut}(S_2)$, where $\{29, 23\} \subseteq \pi(S_2)$. Then by [19], $|S_2|_2 \geq 2^{21}$. So $2^{25} = |G|_2 \geq |S_1|_2|S_2|_2 \geq 2^9 \cdot 2^{21}$, which is a contradiction. Therefore $S \cong A_{31}$ and so $G \cong A_{31}$.

□

Proof of Theorem 1.1. It comes from the previous Lemmas.

5. Appendix

Here, we have listed two simple Maple procedures that compute $\pi_e(A_n)$ and $\pi_e(S_n)$. The electronic versions of these procedures can be obtained by contacting the author.

```
with(NumberTheory): with(ArrayTools):
#Procedure mA computes the set of element orders of the alternating group A_n:
mA := proc(n) local l, T_o, T_e, i, T;
l := proc(m) local S, A, B, k, r;
S := 0; A := ifactors(m); B := A[2]; k := Size(B);
for r to 1/2*k[2] do S := S + B[r][1]B̂[r][2]; end do; S; end proc;
T_o := ; T_e := ;
for i to Landau(n) do if i::even and l(i) ¡= n - 2 then T_e := T_e union i; end if; end do;
for i to Landau(n) do if i::odd and l(i) ¡= n then T_o := T_o union i; end if; end do;
T := T_o union T_e; T;
end proc;

#Procedure mS computes the set of element orders of the symmetric group S_n:
mS := proc(n) local l, i, T;
l := proc(m) local S, A, B, k, r;
S := 0; A := ifactors(m); B := A[2]; k := Size(B);
for r to 1/2*k[2] do S := S + B[r][1]B̂[r][2]; end do;
```

S; end proc;
T := ; for i to Landau(n) do if l(i) ¡= n then T := T union i; end if; end do; T;
end proc;

For example by using these procedures, we compute the set of all element orders of the alternaging group A_{32} and the symmetric group S_{31}:

mA(32);
{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 46, 48, 50, 51, 52, 54, 55, 56, 57, 60, 63, 65, 66, 68, 69, 70, 72, 75, 76, 77, 78, 80, 84, 85, 87, 88, 90, 91, 92, 95, 99, 100, 102, 104, 105, 110, 112, 114, 115, 117, 119, 120, 126, 130, 132, 133, 135, 136, 138, 140, 143, 144, 150, 152, 153, 154, 156, 161, 165, 168, 170, 171, 175, 176, 180, 182, 187, 190, 195, 198, 204, 207, 208, 209, 210, 220, 221, 228, 230, 231, 234, 238, 240, 247, 252, 255, 260, 264, 266, 273, 276, 280, 285, 286, 306, 308, 312, 315, 330, 336, 340, 342, 345, 357, 360, 364, 374, 380, 385, 390, 396, 399, 408, 420, 429, 440, 455, 456, 462, 468, 476, 495, 504, 510, 520, 528, 532, 546, 560, 561, 570, 572, 585, 595, 612, 616, 630, 660, 665, 680, 693, 714, 715, 720, 728, 765, 770, 780, 792, 819, 840, 858, 910, 924, 936, 990, 1001, 1020, 1092, 1155, 1170, 1260, 1320, 1365, 1386, 1540, 1560, 1785, 1820, 1848, 1980, 2145, 2310, 2520, 2730, 3465, 4620}

mS(31);
{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 33, 34, 35, 36, 38, 39, 40, 42, 44, 45, 46, 48, 50, 51, 52, 54, 55, 56, 57, 58, 60, 63, 65, 66, 68, 69, 70, 72, 75, 76, 77, 78, 80, 84, 85, 88, 90, 91, 92, 95, 99, 100, 102, 104, 105, 108, 110, 112, 114, 115, 117, 119, 120, 126, 130, 132, 133, 136, 138, 140, 143, 144, 150, 152, 153, 154, 156, 161, 165, 168, 170, 171, 176, 180, 182, 184, 187, 190, 195, 198, 204, 208, 209, 210, 220, 221, 228, 230, 231, 234, 238, 240, 252, 255, 260, 264, 266, 273, 276, 280, 285, 286, 306, 308, 312, 315, 330, 336, 340, 342, 345, 357, 360, 364, 374, 380, 385, 390, 396, 399, 408, 420, 429, 440, 455, 456, 462, 468, 476, 495, 504, 510, 520, 528, 532, 546, 560, 561, 570, 572, 585, 595, 612, 616, 630, 660, 665, 680, 693, 714, 715, 720, 728, 765, 770, 780, 792, 798, 819, 840, 858, 910, 924, 936, 990, 1001, 1020, 1092, 1140, 1155, 1170, 1190, 1260, 1320, 1365, 1386, 1428, 1430, 1540, 1560, 1638, 1680, 1716, 1820, 1848, 1980, 2184, 2310, 2340, 2520, 2730, 2772, 3080, 4620}

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