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Navigation problem on Finsler manifolds

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ABSTRACT: In this article, we are going to discuss the geometry of the navigation problem on a Finsler manifold. We will give proofs for several important local and global results.

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(*Dedicated to Professor Zhongmin Shen for his warm friendship and research collaboration*)

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1. Introduction

Roughly speaking, a Finsler metric F on a manifold M is a C^∞ function F on the slit tangent bundle $TM_0 := TM\backslash\{0\}$, whose restriction to each tangent space T_xM is a Minkowski norm. The pair (M, F) is called a *Finsler manifold*. Finsler geometry is the geometry of exploring Finsler manifolds.

An important approach in studying Finsler geometry is navigation problem. For a pair (F, V) on a manifold M, where $F = F(x, y)$ is a Finsler metric and V is a vector field with $F(x, V_x) < 1$, there is a unique solution $\tilde{F} = \tilde{F}(x, y) > 0$ to the following equation

$$F\left(x, \frac{y}{\tilde{F}} + V_x\right) = 1.$$

We say $\tilde{F}$ is a *navigation Finsler metric* with respect to (F, V).

For example, if (M, h) is a Riemannian manifold, then the navigation Finsler metric F with respect to (h, V) is a Randers metric where V is a vector field on M with $h(x, V_x) < 1$.

Recall that a *Randers metric* F on a manifold M is a Finsler metric in the form $F = \alpha+\beta$, where $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a 1-form with $\|\beta\|_x < 1$ for any point x. A typical example of Randers metrics is defined on the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$:

$$F := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2} + \frac{\langle x, y\rangle}{1 - |x|^2} \tag{1.1}$$

The metric F in (1.1) is called the *Funk metric on* $\mathbb{B}^n$. This Finsler metric is produced from the Euclidean metric $h(x, y) = |y|$ and the radial field $V_x := x$ by navigation problem, and therefore it has the form $F = \alpha + \beta$, where α is the Klein metric and β is the exact form $\beta = -\frac{1}{2}d(ln(1 - |x|^2))$. It has many nice curvature properties. In particular, the Funk metric on $\mathbb{B}^n$ is of constant flag curvature $K = -\frac{1}{4}$.

The flag curvature is the most important Riemannian quantity in Finsler geometry because it is an analogue of sectional curvature in Riemannian geometry. Furthermore, Finsler metrics of constant flag curvature (or scalar curvature and dim $n \geq 3$) are the natural extension of Riemannian metric of constant sectional curvature. Professor S. S. Chern openly asked the following question on many occasions: classification Finsler metrics of constant (flag) curvature.

Recently, great progress has been made in studying Chern's question. For instance, Bao-Robles-Shen have classified Randers metrics of constant flag curvature via the navigation problem in Riemannian manifolds [2]. This class of Randers metrics contains the Funk metric on unit ball $\mathbb{B}^n$. Inspired by Bao, Robles and Shen' result, Finslerian geometers have obtained many beautiful results via navigation problem.

In this article, we are going to discuss the geometry of the navigation problem on a Finsler manifold. We will give proofs for several important local and global results.

2. Navigation problem

A *Minkowski norm* on a vector space V is a nonnegative function $F : V \to [0, \infty)$ with the following properties:

(i) F is positively y-homogeneous of degree one, i.e., for any $y \in V$ and any $\lambda > 0$, $F(\lambda y) = \lambda F(y)$,

(ii) F is C^∞ on $V\backslash\{0\}$ and for any tangent vector $y \in V\backslash\{0\}$, the following bilinear symmetric form $\mathbf{g}_y(u, v) : V \times V \to R$ is positive definite,

$$\mathbf{g}_y(u, v) := \frac{1}{2}\frac{\partial^2}{\partial s\partial t}[F^2(y + su + tv)]_{s=t=0}.$$

Let M be a differentiable manifold. Let $TM = \cup_{x\in M}T_xM$ be the tangent bundle of M, where T_xM is the tangent space at $x \in M$. We set $TM_o := TM \setminus \{0\}$ where $\{0\}$ stands for $\{(x, 0)|\, x \in M,\, 0 \in T_xM\}$.

A *Finsler metric* on M is a function $F : TM \to [0, \infty)$ with the following properties

(a) F is C^∞ on TM_o;

(b) At each point $x \in M$, the restriction $F_x := F|_{T_xM}$ is a Minkowski norm on T_xM.

Let $F = F(x, y)$ be a Finsler metric on M. Then

$$F = \sqrt{g_{ij}(x, y)y^iy^j}, \quad g_{ij} := \left(\frac{F^2}{2}\right)_{y^iy^j}.$$

Riemannian metric F is defined by:

$$F = \sqrt{g_{ij}(x)y^iy^j}, \quad g_{ij} := \left(\frac{F^2}{2}\right)_{y^iy^j}.$$

Let

$$F := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2}, \; y \in T_xB^n \cong R^n.$$

$F = F(x, y)$ is a Riemannian metric on the unit ball B^n, called the *Klein metric* on B^n.

Locally Minkowskian metric F is defined by:

$$F = \sqrt{g_{ij}(y)y^iy^j}, \quad g_{ij} := \left(\frac{F^2}{2}\right)_{y^iy^j}.$$

For instance, let $\phi = \phi(y)$ be a Minkowski norm on $\mathbb{R}^n$. Define

$$\Phi(x,\, y) := \phi(y), \quad y \in T_x\mathbb{R}^n \cong \mathbb{R}^n.$$

Then $\Phi = \Phi(x,\, y)$ is a (locally) Minkowskian metric.

Let $\alpha = \sqrt{a_{ij}(x)y^iy^j}$ be a Riemann metric on a manifold M and $\beta = b_i(x)y^i$ be a 1-form on M. Assume that

$$\|\beta_x\|_\alpha := \sup_{y \in T_xM} \frac{\beta(x, y)}{\alpha(x, y)} < 1.$$

Define $F = \alpha + \beta$. F is a Finsler metric, which is called the *Randers metric.*

Let

$$F := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1 - |x|^2}, \; y \in T_xB^n \cong R^n.$$

$F = F(x, y)$ is a Finsler metric on the unit ball B^n, called the *Funk metric* on B^n.

For an arbitrary constant vector $a \in R^n$ with $|a| < 1$, let

$$F_a := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1 - |x|^2} + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle},$$

where $y \in T_xB^n \cong R^n$. $F_a = F_a(x, y)$ is a Finsler metric on B^n. Note that $F_0 = F$ is the Funk metric on B^n. We call F_a the *generalized Funk metric* on B^n.

Randers metrics were first studied by physicist G. Randers, in 1941 from the standard point of general relativity [19].

The main technique of the navigation problem is described as follows. Given a Finsler metric F and a vector field V with $F(x,\, V_x) < 1$, define a new Finsler metric $\tilde{F}$ by

$$F\left(x,\, \frac{y}{\tilde{F}(x,\, y)} + V_x\right) = 1, \quad \forall x \in M,\; y \in T_xM. \tag{2.1}$$

Lemma 2.1 ([5]). *For any piecewise C^∞ curve C in a manifold M, the $\tilde{F}$-length of C is equal to the time for which the object travel along C.*

It follows that under the influence of V, for any two points p, q in M, the shortest time from p to q is the geodesic of the Finsler metric $\widetilde{F}$.

Given a Riemannian metric $F(x,\, y) = \sqrt{g_x(y,\, y)}$ and a vector field V on a manifold M with $F(x,\, V_x) < 1$. By (2.1), one obtains

$$\begin{aligned} 1 =& F(x,\, \frac{y}{\tilde{F}} + V)^2 \\ =& g_x\left(\frac{y}{\tilde{F}} + V,\, \frac{y}{\tilde{F}} + V\right) = \frac{1}{(\tilde{F})^2} g_x(y,\, y) + \frac{2}{\tilde{F}} g_x(V,\, y) + g_x(V,\, V). \end{aligned}$$

It follows that

$$(\tilde{F})^2 = F(x,\, y)^2 + 2g_x(V,\, y)\tilde{F} + F(x,\, V)^2(\tilde{F})^2$$

that is,

$$\left[1 - F(x,\, V)^2\right](\tilde{F})^2 - 2g_x(V,\, y)\tilde{F} - F(x,\, y)^2 = 0. \tag{2.2}$$

Note that

$$1 - F(x,\, V_x)^2 > 0.$$

Thus the solutions of (2.2) are

$$\tilde{F}_{\pm} = \frac{g_x(V,\, y) \pm \sqrt{g_x(V,\, y)^2 + [1 - F(x,\, V)^2]\, F(x,\, y)^2}}{1 - F(x,\, V)^2}.$$

Observe that

$$\tilde{F}_{-} \leq \frac{-|g_x(V,\, y)| + g_x(V,\, y)}{1 - F(x,\, V)^2} \leq 0.$$

Thus $\tilde{F}_{-}$ is not a Finsler metric, and $\tilde{F}_{+}$ is a Randers metric as follows:

$$\tilde{F}_{+} = \frac{\sqrt{g_x(V,\, y)^2 + [1 - F(x,\, V)^2]\, F(x,\, y)^2}}{1 - F(x,\, V)^2} + \frac{g_x(V,\, y)}{1 - F(x,\, V)^2}. \tag{2.3}$$

Given a Minkowski norm $\varphi : \mathbb{R}^n \to \mathbb{R}$, one can construct $\Omega := \{v \in \mathbb{R}^n \,|\, \varphi(v) < 1\}$. A domain Ω in $\mathbb{R}^n$ defined by a Minkowski norm φ is called a *strongly convex domain*. Recall that the indicatrix $\partial\Omega$ of φ is a strictly convex hypersurface enclosing the origin [11].

For each $x \in \Omega$, identify $T_x\Omega$ with $\mathbb{R}^n$. Thus $(\Omega,\, F(x,\, y))$ is a Minkowski manifold where $F(x,\, y) = \varphi(y)$ and $V_x := x$ is a radial vector field on Ω satisfying $F(x,\, V_x) = \varphi(x) < 1$. By using F and V, we produce a new Finsler metric $\tilde{F}$ in terms of the navigation problem (2.1). $\tilde{F}$ is called the *Funk metric on a strongly convex domain* Ω.

Let us take a look at a special case: when $\varphi(y) = |y|$,

$$F(x,\, y) = |y|.$$

Using (2.1) or (2.3) we get $\tilde{F}$ is the Funk metric on B^n, that is

$$\tilde{F} := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1 - |x|^2},\ y \in T_xB^n \cong R^n.$$

3. Randers metrics of isotropic S-curvature

Let F be a Finsler metric on an n-dimensional manifold M. For a non-zero vector $y \in T_xM$, F induces an *inner product* $\mathbf{g}_y$ on T_xM by

$$\mathbf{g}_y(u,\, v) = \frac{1}{2}[F^2]_{y^iy^j}(x,\, y)u^iv^j.$$

Here $(x^i,\, y^i)$ denotes the standard local coordinate system in TM, i.e. y^i's are determined by $y = y^i \frac{\partial}{\partial x^i}|_x$.

The *Riemannian curvature* of F is a family of endomorphisms $\mathbf{R}_y = R^i{}_k dx^k \otimes \frac{\partial}{\partial x^i} : T_xM \to T_xM$, defined by

$$R^i{}_k := 2\frac{\partial G^i}{\partial x^k} - y^j\frac{\partial^2 G^i}{\partial x^j \partial y^k} + 2G^j\frac{\partial^2 G^i}{\partial y^j \partial y^k} - \frac{\partial G^i}{\partial y^j}\frac{\partial G^j}{\partial y^k} \tag{3.1}$$

where

$$G^i := \frac{1}{4}g^{il}\left\{[F^2]_{x^ky^l}y^k - [F^2]_{x^l}\right\}$$

are the *geodesic coefficients* of F.

For a two-dimensional plane $P \subset T_xM$ and a non-zero vector $y \in T_xM$, the *flag curvature* $\mathbf{K}(y,\, P)$ is defined by

$$\mathbf{K}(y,\, P) := \frac{\mathbf{g}_y(u,\, \mathbf{R}_y(u))}{\mathbf{g}_y(y,\, y)\mathbf{g}_y(u,\, u) - \mathbf{g}_y(y,\, u)^2}$$

where $P = y \wedge u$. The flag curvature in Finsler geometry is an analogue of sectional curvature in Riemannian geometry which was first introduced by L. Berwald. A Finsler metric F is said to be of *constant flag curvature* if the flag curvature $K(y,\, P) =$ *constant*. In the local coordinate, F has constant flag curvature if and only if

$$R^i{}_k = \lambda(F^2\delta^i{}_k - F\frac{\partial F}{\partial y^k}y^i).$$

Finsler metrics of constant flag curvature are the natural extension of Riemannian metrics of constant sectional curvature. Below are three interesting examples:

The Klein metric on B^n

$$F = \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 - |x|^2}, \; y \in T_xB^n \cong R^n$$

is of constant flag curvature -1.

The generalized Funk metric on B^n

$$F_a := \frac{\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle}{1 - |x|^2} + \frac{\langle a, y\rangle}{1 + \langle a, x\rangle},$$

is of constant flag curvature $-\frac{1}{4}$ where $y \in T_xB^n \cong R^n$.

Consider the following Randers metric defined nearby the origin in $\mathbb{R}^n$

$$F := \frac{\sqrt{|y|^2 - (|xQ|^2|y|^2 - \langle y, xQ\rangle^2)}}{1 - |xQ|^2} - \frac{\langle y, xQ\rangle}{1 - |xQ|^2}$$

where $Q = (q^i_j)$ is an anti-symmetric matrix. F is of constant flag curvature zero.

Let $F = \alpha + \beta$ be a Randers metric on M. Then F is the solution of the following equation

$$h\left(x, \frac{y}{F} + V_x\right) = 1$$

where h is a Riemannian metric and V is a vector field where $h(x, V_x) < 1$. We call (h, V) the *navigation representation (or navigation data) of* F.

Professor S. S. Chern openly asked the following question on many occasions: Classification Finsler metrics of constant (flag) curvature. The complete classification of Randers metrics of constant flag curvature due to Bao-Robles-Shen is motivated by the following result:

A Randers metric F is of constant flag curvature $K = \lambda$ if and only if (i) *h has constant sectional curvature $\mu = \lambda + c^2$; and* (ii) *V is a homothetic field of h with dilation c, where (h, V) is the navigation representation of F.*

The condition (ii) is equivalent that F has constant S-curvature [26]. Lately, Cheng-Shen established relation between the flag curvature of F and h for a Randers metric F of isotropic S-curvature [7] generalizing Bao-Robles-Shen's the flag curvature non-increasing equation [2].

Let $\alpha_\mu := \frac{\sqrt{\omega^2|y|^2 - \mu\langle x,y\rangle^2}}{\omega^2}$, $\omega := \sqrt{1 + \mu|x|^2}$. By studying the homothetic fields in Riemannian manifolds of constant sectional curvature, Bao-Robles-Shen obtain the following important classification result of Ransders metrics of constant flag curvature [2].

Theorem 3.1. *Let F be a Randers metric on a manifold M, and (h, V) be the navigation representation. Then F has constant (flag) curvaure if and only if h is a Riemannian metric and V is a vector field on M with the following property: at any point in M, there is a local coordinate system (x^i) with $(x^i(p)) = 0$ such that h is locally expressed by α_μ and V is given by*

$$V :\equiv \begin{cases} -2cx + xQ + b, & \mu = 0 \\ xQ + b + \mu\langle b, x\rangle x, & \mu \neq 0 \end{cases}$$

where Q is a skew-symmetric matrix and b is a constant vector field with $|b| < 1$.

Recall that the S-curvature is one of most important non-Riemannian quantities in Finsler geometry [6]. In the rest of section, we are going to discuss Randers metrics of isotropic S-curvature.

Define the *(mean) distortion* $\tau : TM\backslash\{0\} \longrightarrow \mathbb{R}$ by

$$\tau(x, y) := ln\frac{\sqrt{det(g_{ij}(y))}}{\sigma(x)}$$

where

$$\sigma(x) = \frac{Vol(B^n)}{Vol\{(y^i) \in \mathbb{R}^n | F(x, y^i\frac{\partial}{\partial x^i}) < 1\}}$$

where B^n denotes the unit ball in $\mathbb{R}^n$ and Vol denotes the Euclidean measure on $\mathbb{R}^n$.

To measure the rate of change of distortion along geodesics, we define

$$S(x,y) := \frac{d}{dt}[\tau(\dot{c}(t))]_{t=0}$$

where $c(t)$ is the geodesic with $\dot{c}(0) = y$. We call the scalar function S the *S-curvature*. The S-curvature S is said to be *isotropic* if there is a scalar function $c(x)$ on M such that

$$S(x,y) = (n+1)c(x)F(x,y)$$

In particular, S is said to be of *constant* c if $c =$ *constant*.

Example 3.1. *If Randers metric $F = \alpha + \beta$ has constant (flag) curvature, then the S-curvature of F is constant, i.e.*

$$S = (n+1)cF$$

where $c = constant$ and $n = dimM$.

Example 3.2. *Let F be the Finsler metric on an open ball $B^n(1/\sqrt{|a|})$ in $\mathbb{R}^n$ defined by*

$$F = \frac{\sqrt{(|x|^2\langle a,\, y\rangle - 2\langle a,\, x\rangle\langle x,\, y\rangle)^2 + |y|^2(1-|a|^2|x|^4)}}{1-|a|^2|x|^4} - \frac{|x|^2\langle a,\, y\rangle - 2\langle a,\, x\rangle\langle x,\, y\rangle}{1-|a|^2|x|^4}$$

where $a \in \mathbb{R}^n$ is a constant vector. We can show that F is of isotropic S-curvature

$$S = (n+1)\langle a,\, x\rangle F.$$

Example 3.3. *Let ζ be an arbitrary constant and*

$$\Omega = \begin{cases} \mathbb{R}^n, & \zeta \geqslant 0 \\ B(\sqrt{-\frac{1}{\zeta}}), & \zeta < 0 \end{cases}$$

Define $F : T\Omega \to [0,\infty)$ by

$$\alpha(x,y) := \frac{\sqrt{\kappa^2\langle x,y\rangle^2 + \epsilon|y|^2(1+\zeta|x|^2)}}{1+\zeta|x|^2}, \quad \beta(x,y) := \frac{\langle x,y\rangle}{1+\zeta|x|^2}$$

where ϵ is an arbitrary positive constant, and κ is an arbitrary constant. Then F has isotropic S-curvature, i.e. $S = (n+1)cF$ where

$$c = \frac{\kappa}{2[\epsilon + (\epsilon\zeta + \kappa^2)|x|^2]}$$

Note that when $\zeta = -1, \kappa = \pm 1$ and $\epsilon = 1$, these metrics reduce the famous Funk metrics on the unit ball.

Proposition 3.2. *[26] Let $F = \alpha + \beta$ be an n-dimensional Randers metric expressed in terms of a Riemannian metric h and a vector V by*

$$\alpha = \frac{\sqrt{\lambda h^2 + V_0}}{\lambda}, \quad \beta = -\frac{V_0}{\lambda} \tag{3.2}$$

respectively, where $\lambda = 1 - h(x, -V_x)$ and $V_0 = v^i y^j h_{ij}$. Then F has isotropic S-curvature,

$$S(x,y) = (n+1)c(x)F(x,y),$$

if and only if V is a conformal vector field, $\mathscr{L}_v h = -4c(x)h$.

Let ${R^i}_k$ be the Riemannian curvature of F given in (3.1). Let

$$Ric := \sum {R^i}_i$$

Ric is a well-defined scalar function on $TM\backslash\{0\}$. We call *Ric* the *Ricci scalar* (or *Ricci curvature*).

Proposition 3.3. *[16] Let M be an n-dimensional compact manifold and $F = \alpha + \beta$ be an n-dimensional Randers metric expressed in terms of a Riemannian metric h and a vector field V by (3.2). Suppose that F has isotropic S-curvature. Then*

$$\int_M \{[1 + h(x, V)]^2 \left[Ric(x, V) - (n-1)\left(\frac{3c_{x^i}V^i}{F(x, V)} - c^2 - 2c_{x^i}V^i\right) F(x, V)^2\right] - |\nabla V|^2 - \frac{n-2}{n}(divV)^2\} \, {}^*1 = 0 \tag{3.3}$$

where $h(x, y) := \sqrt{h_{ij}(x)y^i y^j}$, $V = V^j \frac{\partial}{\partial x^j}$, $c_{x^j} = \frac{\partial c}{\partial x^j}$.

We say $Ric < -(n-1)\lambda$ for some $\lambda \in \mathbb{R}$ if for any $y \in T_xM$ [22]

$$Ric(x, y) < -(n-1)\lambda F(x, y)^2$$

Theorem 3.4. *[16] Let M be an n-dimensional compact manifold and $F = \alpha + \beta$ be an n-dimensional Randers metric expressed in terms of a Riemannian metric h and a vector V by (3.2). Suppose that F has constant S-curvature. Then*

$$\int_M [1 + h(x, V)]^2 [Ric(x, V) + (n-1)c^2 F(x, V)^2] \, {}^*1 \geqslant 0 \tag{3.4}$$

Furthemore, if $Ric < -(n-1)c^2$, then $F = \alpha$ is a Riemannian metric.

Proof. By $c = constant$ and (3.3), we have (3.4). Assume that V is a non-zero vector field. Note that

$$[1 + h(x, V)]^2 > 0.$$

If $Ric < -(n-1)c^2$, then we have

$$Ric(x, V) + (n-1)c^2 F(x, V)^2 < 0.$$

It follows that the left side of (3.4) is negative. This is a contradiction. It follows that

$$V \equiv 0.$$

Plugging this into (3.2) yields $\beta = 0$. Hence $F = \alpha$ is a Riemannian metric.

□

4. Global classification result for Randers metrics

Let F be a Finsler metric on a manifold. F is said to be of *scalar flag curvature* if the flag curvature $K = K(y)$ is a scalar function on the slit tangent bundle $TM\backslash\{0\}$, or equivalently if

$$R^i{}_k = K(F^2 \delta^i{}_k - F F_{y^k} y^i).$$

Proposition 4.1. *[14] Let (M, g) be an n-dimensional Riemannian manifold and V a conformal vector field on (M, g), that is,*

$$\mathscr{L}_V g = 2\lambda(x) g \tag{4.1}$$

for some scalar funtion $\lambda(x)$ on M. If the scalar curvature r of (M, g) is constant, then

$$\Delta\lambda = \frac{\lambda r}{n-1} \tag{4.2}$$

where Δ denotes the Laplacian of g. In particular,
(i) if M is compact, then r is positive unless λ is constant;
(ii) if g is an Einstein metric, and $dimM \geqslant 3$, then

$$Dd\lambda = -\frac{r\lambda}{n(n-1)} g, \tag{4.3}$$

where Dd denotes the Hessian of g.

Proof. (4.2) is an immediate consequence of the following:

$$\mathscr{L}_V r = 2(n-1)\Delta\lambda - 2\lambda r,\ r = constant.$$

(i) Assume that M is compact and λ is not constant. Note that

$$\begin{aligned}\frac{1}{2}\Delta\lambda^2 &= -\frac{1}{2}\sum_i(\lambda^2)_{ii}\\ &= -2\sum_i(\lambda)_{ii} - \sum\lambda_i^2 = \lambda\Delta\lambda - |\nabla\lambda|^2.\end{aligned} \tag{4.4}$$

We denote the global inner product of tensor fields by (,). Then

$$\begin{aligned}0 &= \frac{1}{2}\int_M \Delta(\lambda^2)\ {}^*1\\ &= \int_M \lambda\Delta\lambda\ {}^*1 - \int_M |\nabla\lambda|^2\ {}^*1\\ &= \frac{r}{n-1}\int_M \lambda^2\ {}^*1 - \int_M |\nabla\lambda|^2\ {}^*1 = \frac{r}{n-1}(\lambda,\lambda) - (\nabla\lambda,\nabla\lambda)\end{aligned}$$

from (4.2) and (4.4). It follows that

$$r = (n-1)\frac{(\nabla\lambda,\nabla\lambda)}{(\lambda,\lambda)} > 0.$$

(ii) Assume that g is an Einstein metric and $dimM \geqslant 3$. Then

$$Ric = \frac{r}{n}g, \quad r = constant. \tag{4.5}$$

Using (4.1), we have

$$\mathscr{L}_V Ric = (\Delta\lambda)g - (n-2)Dd\lambda. \tag{4.6}$$

Taking (4.5) together with (4.1), (4.6) and (4.2) we obtain

$$\begin{aligned}(n-2)Dd\lambda &= (\Delta\lambda)g - \mathscr{L}_V Ric\\ &= \frac{r\lambda}{n-1}g - \mathscr{L}_V(\frac{r}{n}g) = \frac{r\lambda}{n-1}g - \frac{r}{n}\mathscr{L}_V g\\ &= \frac{r\lambda}{n-1}g - \frac{r}{n}\cdot 2\lambda g = -\frac{r\lambda(n-2)}{n(n-1)}g.\end{aligned}$$

Note that $dimM \geqslant 3$, so we obtain (4.3).

□

This proposition is a natural generalization of Theorem 1.1 of [26] in the case of g with constant sectional curvature.

Lemma 4.2. *[14] Let (M, F) be an n-dimensional compact Finsler manifold with constant S-curvature c. Then* $c = 0$.

Proof. Proposition 3.1 in [13] tell us that

$$div_G(\tau\omega) = \tau_{|n} = \frac{S}{F}$$

where

$$\tau : SM \to \mathbb{R}$$

is the distortion of F.

Note that M is compact and $c = constant$. It follows that

$$\begin{aligned}0 &= \int_{SM} div_G(\tau\omega)\mathit{\Pi}\\ &= \int_{SM}\frac{S}{F}\mathit{\Pi}\\ &= \int_{SM}(n+1)c\mathit{\Pi} = (n+1)cVol(SM)\end{aligned}$$

where $\omega := F_{y^j}dx^j$ is the Hilbert form of F.

□

In 2009, Cheng-Shen classified locally Randers metrics of scalar flag curvature whose S-curvature are isotropic [7]. Mo gave the following global classification for these metrics on compact manifolds.

Theorem 4.3. *Let $F=\alpha+\beta$ be a Randers metric on a compact manifold of dimensional $n \geqslant 3$, which is expressed in terms of a Riemannian metric h and a vector field V. Suppose that F is of isotropic S-curvature $S=(n+1)cF$ and of scalar flag curvature. Let μ denote the constant sectional curvature of h.*

(a) If $\mu=-1$, then $F=\alpha$ is Riemannian.

(b) If $\mu=0$, then F is locally Minkowskian.

(c) If $\mu=1$, then c is an eigenfunction of Laplace operator corresponding to the first eigenvalue $\lambda_1=n$. In this case, (M,h) is isometric to a unit sphere, or V is a killing vector field on (M,h).

Proof. Denote the navigation representation of F by (h,V). By Proposition 3.5, the condition isotropic S-curvature implies that [26]

$$\mathscr{L}_V h=-4ch.$$

Furthermore, Theorem 5.1 of [7] tells us that h has constant sectional curvature μ.

Case 1. $\mu \leqslant 0$. Assume that c is not constant. It follows from (i) of Proposition that

$$\mu=\frac{r}{n(n-1)}>0$$

That is a contradiction. Thus $c=constant$, that is, F has constant S-curvature. By Lemma 4.2, $c=0$. According to (8) in [7], the scalar flag curvature K of F satisfies that

$$K=\frac{3c_{x^j}y^j}{F}+\sigma,\ \sigma=\mu-c^2-2c_{x^j}V^j$$

where $c_{x^i}=\frac{\partial c}{\partial x^i}, V=V^i\frac{\partial}{\partial x^i}$. It follows that

$$K=\mu=constant$$

Thus F has constant flag curvature. By Akbar-Zadeh's rigidity theorem [1], F must be Riemannian if $\mu=-1$ and F must be locally Minkowskian if $\mu=0$.

Case 2. $\mu=1$. Then

$$r=n(n-1) \tag{4.7}$$

Plugging (4.7) into (4.2) yields that $c(x)$ is eigenfunction corresponding to the first eigenvalue $\lambda_1=n$. If $c\equiv 0$, then $\mathscr{L}_V h=0$ and V is a Killing field. Otherwise $c(x_0)\neq 0$ for some $x_0\in M$, then c is not constant from Lemma 4.2. By Obata's theorem A [18], (M,h) must be isometric to a sphere.

□

5. Homothetic navigation problem

A vector field V on a Finsler manifold (M,F) is called *homothetic with dilation c* if its flow ϕ_t satisfies that

$$F(\phi_t(x),(\phi_t)_*(y))=e^{2ct}F(x,y),\ \forall x\in M,\ \forall y\in T_xM.$$

In particular, V is called *Killing* if $c=0$.

The lift of a flow ϕ_t on a manifold M is again a flow $\breve{\phi}_t^*$ on TM

$$\breve{\phi}_t^* := (\phi_t(x),\ \phi_{t*}(y)) \tag{5.1}$$

By the relationship of vector fields and flows, (5.1) induces a natural way to lift a vector field V to a vector field X_V on TM. Let $V=v^i\frac{\partial}{\partial x^i}$ be a vector field on M. In natural coordinates, we have

$$X_V=v^i\frac{\partial}{\partial x^i}+y^j\frac{\partial v^i}{\partial x^j}\frac{\partial}{\partial y^i}.$$

Lemma 5.1. *V is a homothetic field of F with dilation λ if and only if $X_V(F)=2\lambda F$. In particular, V is a killing field of F if and only if $X_V(F)=0$.*

For non-collinear $u,\, v \in T_xM$, we denote the tangent plane Span$\{u,\, v\}$ by $u\wedge v$. In 2004 International Conference on Riemannian-Finsler Geometry at Nankai University, P.Foulon announced that if a Finsler metric and V is a Killing field, then F and $\tilde{F}$ have the same flag curvature where $\tilde{F}$ is the navigation Finsler metric with respect to (F, V). Mo-Huang studied the navigation problem for any Finsler metric F and any homothetic field (for instance, the Funk metric on a strongly convex domain) in the spirit of Bao-Robles-Shen's the flag curvature non-increasing formula above and the announcement of P. Foulon and obtained the following:

Theorem 5.2. *Let $F = F(x,\, y)$ be a Finsler metric on a manifold M and V a vector field on M with $F(x,\, V_x) < 1$. Let $\tilde{F} = \tilde{F}(x,\, y)$ denote the Finsler metric on M defined by (2.1). Suppose that V is homothetic with dilation c. Then the flag curvature of $\tilde{F}$ and F is related by*

$$\mathbf{K}_{\tilde{F}}(y,\, y\wedge u) = \mathbf{K}_F(\tilde{y},\, \tilde{y}\wedge u) - c^2 \tag{5.2}$$

where $\tilde{y} = y - F(x,\, y)V$.

Theorem 5.2 tells us that for a homothetic field, the navigation representation satisfies the flag curvature non-increasing equation. In particular, the navigation problem has the flag curvature preserving property for a Killing field.

Corollary 5.3. *Let $F = F(x,\, y)$ be a Finsler metric on a manifold M and V a vector field on M with $F(x,\, V_x) < 1$. Let $\tilde{F} = \tilde{F}(x,\, y)$ denote the Finsler metric on M defined by (2.1). Suppose that V is homothetic with dilation c. If F is of scalar curvature, then $\tilde{F}$ is also of scalar curvature. Moreover, if F has constant curvature, then so does $\tilde{F}$.*

Let $\Omega \subset \mathbb{R}^n$ be a strongly convex domain defined by a Minkowski norm $\phi = \phi(y)$. For each $x \in \Omega$, identify $T_x\Omega$ with $\mathbb{R}^N$. This $(\Omega,\, F(x,\, y))$ is a Minkowski manifold where $F(x,\, y) = \phi(y)$ and $V_x := x$ is a radial vector field on Ω satisfying $F(x,\, V_x) = \phi(x) < 1$. Moreover, we have

$$\begin{aligned} X_V(F) &= v^i\frac{\partial F}{\partial x^i} + y^j\frac{\partial v^i}{\partial x^j}\frac{\partial F}{\partial y^i} \\ &= y^j\delta^i_j\frac{\partial \phi(y)}{\partial y^i} \\ &= y^i\frac{\partial \phi}{\partial y^i} \\ &= \phi = F. \end{aligned}$$

Hence V is a homothetic field of F with dilation $\frac{1}{2}$.

On the other hand, all Minkowski metrics have zero flag curvature. Combining this with (5.2) we have

$$\mathbf{K}_{\tilde{F}} = \mathbf{K}_F - \left(\frac{1}{2}\right)^2 = -\frac{1}{4}$$

where $\mathbf{K}_{\tilde{F}}$ denotes the flag curvature of the Funk metric on the strongly convex domain Ω.

A smooth curve in a Finsler manifold is called a *geodesic* if it is locally the shorted path connecting any two neaby points on this curve. Now we are going to give a geodesic description of the geodesics of Finsler metric $\tilde{F}$ obtained from arbitrary Finsler metric F and arbitrary homothetic field V of F in terms of the navigation representation. Precisely we show the following:

Theorem 5.4. *Let $F = F(x,y)$ be a Finsler metric on a manifold M and V a vector field on M with $F(x,V_x) < 1$. Suppose that V is homothetic with dilation c. Let $\tilde{F} = \tilde{F}(x,y)$ denote the Finsler metric on M defined by $F(x, y + \tilde{F}(x,y)V) = \tilde{F}(x,y)$. Then the geodesics of $\tilde{F}$ are given by $\psi_t(\gamma(a(t)))$, where ψ_t is the flow of $-V$, $\gamma(t)$ is a geodesic of $\tilde{F}$ are defined by*

$$a(t) := \begin{cases} \frac{e^{2ct}-1}{2c}, & if\ c \neq 0, \\ t, & if\ c = 0. \end{cases}$$

Our result generalizes a theorem previously only known in the case of Randers metrics with constant S-curvature [20]. As its application, we represent explicitly the geodesics of the Funk metric on a strongly convex domain (see Theorem 5.5 below)

Theorem 5.5. *Let $\varphi : \mathbb{E} \longrightarrow \mathbb{R}$ be a Minkowski norm and Ω its strongly convex domain. Assume that $\tilde{F}$ is the Funk metric on Ω defined by*

$$\varphi(\frac{y}{\tilde{F}(x,y)} + x) < 1.$$

Then the geodesics of $\tilde{F}$ are given by

$$e^{-t}\left[x + \frac{e^t - 1}{\varphi(y)}y\right].$$

6. Navigation Finsler metrics of constant (or scalar) flag curvature

Let M^n be an n-dimensional manifold and V a vector field of M^n. Let F and $\tilde{F}$ be two Finsler metrics on M^n. We say $\tilde{F}$ is a *navigation Finsler metric* with respect to (F, V) if $\tilde{F}$ is produced from the Finsler metric F and the vector field V in terms of the navigation problem.

Example 6.1. *Let F be a Riemannian metric. Then the navigation Finsler metric $\tilde{F}$ with respect to (F, V) is a Randers metric.*

Example 6.2. *For each $x \in \Omega := \{v \in \mathbb{R}^N | \varphi(v) < 1\}$, identify $T_x\Omega$ with $\mathbb{R}^N$ with $\varphi : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a Minkowski norm. Thus $(\Omega, F(x, y))$ is a Minkowski manifold where*

$$F(x, y) = \varphi(y)$$

and $V_x := x$ is a radical vector field on Ω. Then the navigation Finsler metric $\tilde{F}$ with respect to (F, V) is the Funk metric on a strongly convex domain Ω.

Consider the following function

$$F := \alpha\phi(s),\ s = \frac{\beta}{\alpha} \tag{6.1}$$

where $\phi = \phi(s)$ is a positive C^∞ function on $(-b_0, b_0)$ satisfying

$$\phi(s) - s\phi^{'}(s) + (b^2 - s^2)\phi^{''}(s) > 0,\ |s| \leqslant b < b_0$$

Then by Lemma 1.2 in [5], F is a Finsler metric if $\|\beta_x\|_\alpha < b_0$ for any $x \in M$. A Finsler metric in the form (6.1) is called an (α, β)*-metric*.

Let $\Phi = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric on an open subset $\mathcal{U} \subset \mathbb{R}^N$. Define

$$\alpha := \rho(h)\bar{\alpha},\ \beta := B\rho(h)^{r+1}dh \tag{6.2}$$

where

$$\bar{\alpha} = \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1 + \mu|x|^2}$$

where

$$\rho(t) = \left[-\frac{2rB^2}{p}\left(C + \eta t - \frac{1}{2}\mu t^2\right)\right]^{-\frac{1}{2r}} \tag{6.3}$$

$$h := \frac{1}{\sqrt{1 + \mu|x|^2}}\left(A + \langle a, x\rangle + \frac{\eta|x|^2}{1 + \sqrt{1 + \mu|x|^2}}\right) \tag{6.4}$$

where A, B, C, p, r and η are constants $(B > 0)$ and $a \in \mathbb{R}^N$ is a constant vector.

Proposition 6.1. *Let $\Phi = \alpha\phi(\beta/\alpha)$ be an (α, β)-metric on an open subset $\mathcal{U} \subset \mathbb{R}^N$. Assume that α and β satisfying (6.2). Let V denote a vector field on $\mathcal{U}$ defined by*

$$V_x = xQ \text{ at } x \in \mathcal{U} \tag{6.5}$$

where Q is skew-symmetric and satisfies that

$$Qa^T = 0. \tag{6.6}$$

where $a \in \mathbb{R}^N$ is a constant vector given in (6.4). Then V is of Killing type with respect to Φ.

Now we take a look at the special case of (6.2)∼(6.4): when $A = B = 1$, $C = \eta = 0$, $\mu = -1$, $r = -\frac{1}{2n}$,

$$\alpha = \frac{(1 + \langle a, x\rangle)^{2n}}{(1 - |x|^2)^{n+1}}\sqrt{(1 - |x|^2)|y|^2 + \langle x, y\rangle^2}$$

$$\beta = \frac{(1 + \langle a, x\rangle)^{2n-1}}{(1 - |x|^2)^{n+1}}[(1 - |x|^2)\langle a, y\rangle + (1 + \langle a, x\rangle)\langle x, y\rangle]$$

In [17], authors given an explicit constriction of polynomial of arbitrary degree (α, β)-metrics with scalar flag curvature and determine their scalar flag scalar. From Mo-Yu's Finsler metric of (α, β) type, we produce new Finsler metrics with scalar curvature by using Theorem 5.2 and Corollary 5.3.

Theorem 6.2. *Let*

$$\Phi := \frac{(1+\langle a,x\rangle)^{2n}}{(1-|x|^2)^{n+1}}\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2} \times \phi\left(\frac{(1-|x|^2)\langle a,y\rangle+(1+\langle a,x\rangle)\langle x,y\rangle}{(1+\langle a,x\rangle)\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}\right)$$

be an (α,β)-metric on an open subset $\mathcal{U}$ at origin in $\mathbb{R}^N$ where

$$\phi(s) = 1 + 2^n s + 2n\sum_{k=0}^{n-1}\frac{(-1)^k C_k^{n-1} s^{2k+2}}{(2k+1)(2k+2)}.$$

Assume that V is a vector field on $\mathcal{U}$ defined by (6.5) where Q is skew-symmetric and satisfies that (6.6) and $\Phi(x,V_x)<1$. Then Finsler metric F given by

$$\Phi(x, \frac{y}{F(x,y)} + V_x) = 1, \forall x\in\mathcal{U},\ y\in T_x\mathcal{U}$$

is of scalar curvature with flag curvature

$$\begin{aligned} K_F = &-\frac{(n+1)|y-\Phi(x,y)V_x|^2}{\Phi^2\omega^2} + \frac{(n^2-1)\langle x, y-\Phi(x,y)V_x\rangle^2}{\Phi^2\omega^4} \\ &- \frac{\zeta^{2n-4}\psi^2\phi^{''}}{2\theta\Phi^3\omega^{2n+2}} + \frac{\zeta^{2n-2}}{4\Phi^4\omega^{4n+4}}(2n\langle a, y-\Phi(x,y)V_x\rangle\theta\phi\zeta + \phi^{'}\psi) \\ &\times [4(n+1)\Phi\langle x, y-\Phi(x,y)V_x\rangle\omega^{2n} + 3\zeta^{2n-2}] \\ &- (n\langle a, y-\Phi(x,y)V_x\rangle\theta\phi\zeta + \phi^{'}\psi)\frac{(2n-1)\langle a, y-\Phi(x,y)V_x\rangle\zeta^{2n-3}}{\Phi^3\omega^{2n+2}} \end{aligned}$$

where

$$\begin{aligned} \theta &:= \sqrt{(1-|x|^2)|y-\Phi(x,y)V_x|^2 + \langle x, y-\Phi(x,y)V_x\rangle^2} \\ \psi &:= \zeta^2|y-\Phi(x,y)V_x|^2 - 2\langle a, y-\Phi(x,y)V_x\rangle\langle x, y-\Phi(x,y)V_x\rangle\zeta \\ &\quad - \omega^2\langle a, y-\Phi(x,y)V_x\rangle^2 \\ \phi^{(i)} &:= \phi^{(i)}\left(\frac{\omega^2\langle a, y-\Phi(x,y)V_x\rangle + \zeta\langle x, y-\Phi(x,y)V_x\rangle}{\zeta\theta}\right) \end{aligned}$$

where

$$\omega := \sqrt{1+\mu|x|^2},\ \zeta = A + \langle a,x\rangle.$$

where $\phi^{(i)}$ denotes i-order derivative for $\phi(s)$.

Note that when $n=0$, $a=0$, $N=2$, $Q=\begin{pmatrix}0 & \epsilon\\ -\epsilon & 0\end{pmatrix}$, we recover Shen's construction [23].

When $n=1$, we obtain the following:

Theorem 6.3. *Let*

$$\Phi := \frac{[(1+\langle a,x\rangle)(\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}+\langle x,y\rangle)+(1-|x|^2)\langle a,y\rangle]^2}{(1-|x|^2)^2\sqrt{(1-|x|^2)|y|^2+\langle x,y\rangle^2}}$$

be an (α,β)-metric on the open subset $\mathcal{U}$ at origin in $\mathbb{R}^N$. Assume that V is a vector field on $\mathcal{U}$ defined by (6.5) where Q is skew-symmetric and satisfies that (6.6) and $\Phi(x,V_x)<1$. Then Finsler metric F given by

$$\Phi(x, \frac{y}{F(x,y)} + V_x) = 1, \forall x\in\mathcal{U},\ y\in T_x\mathcal{U}$$

is of zero flag curvature.

Now we are going to give a lot of new Finsler metrics of constant (or scalar) curvature and determine their scalar flag curvature by fining Killing fields of general Bryant's metric.

Finsler metric on a manifold M in the following form is said to be *general* (α, β) *type*

$$F = \alpha\phi(b^2, \frac{\beta}{\alpha})$$

where α is a Riemannian metric, β is a 1-form, $b = \|\beta\|_\alpha$ and $\phi(\rho, s)$ is a C^∞ function [28]. A Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is said to be *projectively flat* if all geodesics are straight in $\mathcal{U}$. All projectively flat Finsler metrics are of scalar curvature (Proposition 6.1.3 of [5]).

Theorem 6.4. *Let* $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$ *and* $\Omega = B^n(r)$, *where* $r = \frac{1}{\sqrt{-\mu}}$ *if* $\mu < 0$ *and* $r = +\infty$ *if* $\mu \geqslant 0$. *Define*

$$F = \sqrt{\frac{\sqrt{A}+B}{2E} + \left(\frac{U}{E}\right)^2} + \frac{U}{E}$$

where

$$\begin{aligned} A &:= (\alpha^2 cos\varphi + b^2\alpha^2 - \beta^2)^2 + (\alpha^2 sin\varphi)^2 \\ B &:= \alpha^2 cos\varphi + b^2\alpha^2 - \beta^2 \\ U &:= \beta sin\varphi \\ E &:= b^4 + 2b^2 cos\varphi + 1 \end{aligned}$$

Then F *is a Finsler metric on* Ω *with scalar curvature*

$$K = \frac{(\lambda - \mu\langle a, x\rangle)^2}{1+\mu|x|^2} + \frac{\mu}{F^2}\left[\alpha^2 + \beta\left(\epsilon\sqrt{\frac{\sqrt{A}-B}{2E} - \left(\frac{U}{E}\right)^2} - \frac{V}{E}\right)\right] \tag{6.7}$$

where

$$\begin{aligned} V &:= \beta(cos\varphi + b^2) \\ \epsilon &:= sgn\left(\frac{\alpha^2 sin\varphi}{2} - \frac{UV}{E}\right) \end{aligned}$$

where

$$\alpha := \frac{\sqrt{|y|^2 + \mu(|x|^2|y|^2 - \langle x, y\rangle^2)}}{1+\mu|x|^2} \tag{6.8}$$

$$\beta := \frac{\langle a, y\rangle}{\sqrt{1+\mu|x|^2}} + \frac{\lambda - \mu\langle a, x\rangle}{(\sqrt{1+\mu|x|})^3}\langle x, y\rangle, \quad b := \|\beta\|_\alpha \tag{6.9}$$

$a \in \mathbb{R}^n$ *is an arbitrary constant vector and* λ *an arbitrary non-zero constant.*

Proof. In 2011, Yu and Zhu proved that F is projectively flat [28], equivalently, its geodesic coefficients G^i satisfies the following [25]

$$G^i = Py^i$$

where P is its projective factor. Hence P and its scalar flag curvature K are given by

$$P = \frac{F_0}{2F}, \quad K = \frac{P^2 - P_0}{F^2} \tag{6.10}$$

A function f defined on $T\Omega$ can be expressed as $f(x^1, \cdots, x^n; y^1, \cdots, y^n)$. We use the following notation:

$$f_0 = \frac{\partial f}{\partial x^i}y^i$$

Then we have

$$F_0 = -\frac{2\mu\langle x, y\rangle}{\omega^2}F + \frac{2\zeta}{\omega}\left(\epsilon Q - \frac{V}{E}\right)F$$

where

$$\omega := \sqrt{1+\mu|x|^2}, \quad \zeta := \lambda - \mu\langle a, x\rangle.$$

Together with the first equation of (6.10) we obtain

$$P = -\frac{\mu\langle x, y\rangle}{\omega^2} + \frac{\zeta}{\omega}\left(\epsilon Q - \frac{V}{E}\right)$$

It follows that

$$P^2 - P_0 = \frac{\zeta^2}{\omega^2}F^2 + \mu\left[\alpha^2 + \beta\left(\epsilon Q - \frac{V}{E}\right)\right]$$

Together with the second equation of (6.10) yields (6.7).

□

Let us take a look at the special case: when $\mu = 0$,

$$\alpha = |y|, \, \beta = \langle \lambda x + a, \, y\rangle$$

Then we have the following:

Corollary 6.5. *Let $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$, $a \in \mathbb{R}^n$ be an arbitrary constant vector and λ an arbitrary constant. The following Finsler metric is projectively flat with non-negative constant flag curvature $K = \lambda^2$,*

$$F = Im\frac{-\langle \lambda x + a, y\rangle + i\sqrt{(e^{i\varphi} + |\lambda x + a|^2)|y|^2 - \langle \lambda x + a, y\rangle^2}}{e^{i\varphi} + |\lambda x + a|^2}.$$

Remark 6.6. *When $\lambda = 1$ and $a = 0$, them F was constructed by Bryant with constant flag curvature $K = 1$ [3].*

Proposition 6.7. *Let $\Phi = \alpha\phi(b^2, \frac{\beta}{\alpha})$ be a general (α, β)-metric on an open subset $\mathcal{U} \subseteq \mathbb{R}^n$. Define α and β by (6.8) and (6.9) respectively. Let V denote a vector field on $\mathcal{U}$ defined by*

$$V_x = xQ \tag{6.11}$$

where Q is skew-symmetric and satisfies that

$$Qa^T = 0 \tag{6.12}$$

Then V is of Killing type with respect to Φ.

Proof. By a straightforward computation one obtains

$$X_V(\Phi) = (\phi - \frac{\beta}{\alpha}\phi_2)X_V(\alpha) + \phi_2 X_V(\beta) + \alpha\phi_1 X_V(b^2) \tag{6.13}$$

where

$$X_V = V^i\frac{\partial}{\partial x^i} + y^j\frac{\partial V^i}{\partial x^j}\frac{\partial}{\partial y^i} = V + y^j\frac{\partial V^i}{\partial x^j}\frac{\partial}{\partial y^i}$$

By using (6.8) and (6.9), we have

$$X_V(\alpha) = X_V(\beta) = X_V(b^2) = 0$$

Plugging these into (6.13) yields

$$X_V(\Phi) = 0.$$

Therefore V is a Killing field of Φ.

□

Let $\Phi = \alpha\phi(b^2, \frac{\beta}{\alpha})$ be a general (α, β)-metric on an open subset $\mathcal{U} \subset \mathbb{R}^n$. Assume that α and β satisfies (6.8) and (6.9) respectively. Proposition 6.10 tells us $V := xQ$ is a Killing field of Φ, where Q satisfies $Q^T = -Q$ and $Qa^T = 0$. Define

$$\phi(\rho, \, s) = Im\frac{-s + i\sqrt{e^{i\varphi} + \rho - s^2}}{e^{i\varphi} + \rho}.$$

Then the elementary function expression of Φ is given by

$$F=\sqrt{\frac{\sqrt{A}+B}{2E}+\left(\frac{U}{E}\right)^2}+\frac{U}{E}.$$

Theorem 6.7 implies that Φ is of scalar curvature. Let V be a Killing field of Φ on $\mathcal{U}$ with $\Phi(x,\,V_x)<1$. Define a new Finsler metric F by

$$\Phi\left(x,\,\frac{y}{F(x,\,y)}+V_x\right)=1,\,\forall x\in\mathcal{U},\,y\in T_x\mathcal{U}. \tag{6.14}$$

By using Theorem 5.2 and Corollary 5.3, we obtain F is also of scalar curvature. Moreover, its scalar flag curvature K_F is given by

$$K_F(x,\,y)=K_\Phi(x,\,y-\Phi(x,\,y)V_x) \tag{6.15}$$

Combining this with Theorem above we have the following:

Theorem 6.8. *Let*

$$\Phi=Im\frac{-\beta+i\sqrt{\alpha^2e^{i\varphi}+b^2\alpha^2-\beta^2}}{e^{i\varphi}+b^2}$$

be a general $(\alpha,\,\beta)$-metric on an open subset $\mathcal{U}$ at origin in $\mathbb{R}^n$, where α and β are defined by (6.8) and (6.9). Assume that V is a vector field on $\mathcal{U}$ defined by (6.11) where Q is skew-symmetric and satisfies that (6.12) and $\Phi(x,\,V_x)<1$. Then Finsler metric F given by (6.14) is of scalar curvature with the flag curvature

$$K_F=\frac{(\lambda-\mu\langle a,\,x\rangle)^2}{1+\mu|x|^2}+\frac{\mu}{F^2}\left[\alpha^2+\beta\left(\epsilon\sqrt{\frac{\sqrt{A}-B}{2E}-\left(\frac{U}{E}\right)^2}-\frac{V}{E}\right)\right]$$

where A,B,U,V,E and ϵ are defined in Theorem 6.7, where

$$\alpha=\alpha(x,\,y-\Phi(x,\,y)V_x),\,\beta=\beta(x,\,y-\Phi(x,\,y)V_x).$$

Taking $\mu=0$ in Theorem 6.11 and then using (6.15) and Corollary 6.7 we have the following:

Theorem 6.9. *Let $\varphi\in(-\frac{\pi}{2},\,\frac{\pi}{2})$ and*

$$F=Im\frac{-\langle\lambda x+a,y\rangle+i\sqrt{(e^{i\varphi}+|\lambda x+a|^2)|y|^2-\langle\lambda x+a,y\rangle^2}}{e^{i\varphi}+|\lambda x+a|^2}$$

be a general $(\alpha,\,\beta)$-metric on an open subset $\mathcal{U}$ at origin in $\mathbb{R}^n$ where $a\in\mathbb{R}^n$ is an arbitrary constant vector and λ an arbitrary non-zero constant. Assume that V is a vector field on $\mathcal{U}$ defined by (6.11)

$$V_x=xQ$$

where Q is skew-symmetric and satisfies

$$Qa^T=0$$

and $\Phi(x,\,V_x)<1$. Then Finsler metric F is given by

$$\Phi(x,\,\frac{y}{F(x,\,y)}+V_x)=1,\,\forall x\in\mathcal{U},\,y\in T_x\mathcal{U}$$

which is of positive constant flag curvature λ^2.

7. Explicit construction of all dual flat Randers metrics

Recently, a signification progress has been made in studying Randers metrics. For instance, Bao-Robles-Shen have classified Randers metrics of constant flag curvature via the navigation problem on Riemannian manifolds. Huang-Mo showed that there exists no non-homothetic conformal field on a Randers manifold of isotopic S-curvature in terms of the shortest time problem. Recently C. Yu have given a direct characterization of the dually flat Randers metrics in terms of their navigation data [27]. He showed that a Randers metric on an open subset $\mathcal{U}\subseteq\mathbb{R}^n$ is dually flat if and only if its navigation data (h,W) satisfies

$$G^i_h=2\theta y^i+h^2\theta^i, \tag{7.1}$$

and $W^\flat$ is dually related to h where G^i_h denotes the geodesic coefficients of h, $^\flat : T\mathcal{U} \to T^*\mathcal{U}$ is the musical isomorphism, $\theta = \theta_i(x)y^i$ is a 1-form on $\mathcal{U}$ and $\theta^i = h^{ij}\theta_j$.

Recall that a Finsler metric $F = F(x, y)$ on an open subset $\mathcal{U} \subseteq \mathbb{R}^n$ is *dually flat* if it satisfies the following dually flat equations

$$F^2_{x^iy^j}y^i - 2F^2_{x^j} = 0,$$

where $x = (x^1, \cdots, x^n) \in \mathcal{U}$ and $y = y^i\frac{\partial}{\partial x^i} \in T\mathcal{U}$. Such Finsler metrics arise from α-flat information structures on Riemann-Finsler manifolds.

In the Riemannian case, let $h = \sqrt{h_{ij}(x)y^iy^j}$ be a dually flat Riemannian metric on $\mathcal{U} \subset \mathbb{R}^n$. Then h is the Hessian of some locally scalar function ψ, i.e.,

$$h_{ij}(x) = \frac{\partial^2\psi}{\partial x^i\partial x^j}(x).$$

Therefore h is a Hessian metric on $(\mathcal{U}, D)$ where D is the standard flat connection on $\mathcal{U}$.

In this section, we completely determine Riemannian metrics which satisfy condition (7.1) and 1-forms which are dually related to such Riemannian metrics. We have the following:

Theorem 7.1. *A Randers metric $F = \alpha + \beta$ on an open subset $\mathcal{U} \subseteq \mathbb{R}^n$ is dually flat if and only if its navigation data (h, W) satisfies*

$$(h, W^\flat) = \omega^{\frac{1}{2}}(\widetilde{h}, \widetilde{W}^\flat) \tag{7.2}$$

where $\widetilde{h}$ is the Riemannian metric of constant sectional curvature given in (7.5), $\widetilde{W}^\flat$ is closed and conformal with respect to $\widetilde{h}$ and $\omega := \sqrt{1 + \mu|x|^2}$.

By using Theorem 7.1 we construct explicitly all dually flat Randers metrics by using the bijection between Randers metrics and their navigation representation. More precisely, we show the following result.

Theorem 7.2. *Let F be a Randers metric on an open subset $\mathcal{U} \subseteq \mathbb{R}^n$. Then F is dually flat if and only if F can be expressed in the following form*

$$\begin{aligned} F &= \frac{\omega^{\frac{1}{2}}}{\Psi}\sqrt{\Psi|y|^2 - (\mu - \lambda^2 - \mu|v|^2)\langle x, y\rangle^2 + 2\Phi\langle v, y\rangle + \omega^2\langle v, y\rangle^2} \\ &\quad + \frac{\Phi + \omega^2\langle v, y\rangle}{\omega^{\frac{1}{2}}\Psi}, \qquad y \in T_x\mathcal{U} \cong \mathbb{R}^n \end{aligned} \tag{7.3}$$

where $\Phi := (\lambda - \mu\langle v, x\rangle)\langle x, y\rangle$

$$\Psi := 1 + (\mu - \lambda^2)|x|^2 - |v|^2\omega^2 - \langle v, x\rangle(2\lambda - \mu\langle v, x\rangle),$$

$\omega := \sqrt{1 + \mu|x|^2}$, *$v \in \mathbb{R}^n$ is a constant vector, and λ, μ are constants*

Let us take a look at a special case. When $v = 0$,

$$\Psi = \omega^2 - \lambda^2|x|^2.$$

Then F has obtained in [27]. In particular, when $\mu = 0$ and $\lambda = 1$, (7.3) is reduced to the famous Funk metric on the unit ball $\mathbb{B}^n$.

It is worth mentioning our recent result that dually flat Riemannian metrics form a broader class than Riemannian metrics which satisfy (7.1).

A Riemannian metric h on an open subset $\mathcal{U} \subset \mathbb{R}^n$ is called to be *strongly dually flat* if its geodesic coefficients satisfies (7.1). Note that every strongly dually flat Riemannian metric must be dually flat.

Let h be a strongly dually flat Riemannian metric on an open subset $\mathcal{U} \subseteq \mathbb{R}^n$. A 1-form β on $\mathcal{U}$ is said to be *dually related to* h if

$$b_{i;j} = c(x)h_{ij} + 2\theta_ib_j, \tag{7.4}$$

where $c(x)$ is a scalar function, θ_i is given in (7.1) and $b_{i;j}$ denotes the covariant derivative of b_i with respect to the Levi-Civita connection of h.

We can show that each strongly dually flat Riemannian metric on an open subset $\mathcal{U}$ is conformal to a projectively flat Riemannian metric.

Lemma 7.3. *Let $h = \sqrt{h_{ij}(x)y^iy^j}$ be a strongly dually flat Riemannian metric on an open subset $\mathcal{U} \subseteq \mathbb{R}^n$. Then*

$$\widetilde{h} = [\det(h_{ij})]^{\frac{1}{n+2}}\, h.$$

is projectively flat.

Combining this with Beltrami's theorem, $\widetilde{h}$ has constant sectional curvature. Hence we have the following:

Proposition 7.4. *Let $h = \sqrt{h_{ij}(x)y^iy^j}$ be a strongly dually flat Riemannian metric on an open subset $\mathcal{U} \subseteq \mathbb{R}^n$. Then*

$$\widetilde{h} = [\det(h_{ij})]^{\frac{1}{n+2}}\, h.$$

is of constant sectional curvature.

We assume that $\mathcal{U} \subseteq \mathbb{R}^n$ is complete, connected and simply connected. As we know, every Riemannian metric of constant sectional curvature on $\mathcal{U}$ is isometric to the following metric on $\mathbb{B}^n(r_\mu) \subseteq \mathbb{R}^n$,

$$\widetilde{h} = \frac{\sqrt{\omega^2|y|^2 - \mu\langle x, y\rangle^2}}{\omega^2}, \quad \omega := \sqrt{1+\mu|x|^2}, \tag{7.5}$$

where $r_\mu := +\infty$ if $\mu \geq 0$ and $r_\mu := 1/\sqrt{-\mu}$ if $\mu < 0$.

Let h be a strongly dually flat Riemannian metric on $\mathcal{U}$. By Proposition 7.4, we have

$$h(x, y) = e^{\lambda(x)}\widetilde{h}(x, y), \tag{7.6}$$

for some scalar function λ on $\mathcal{U} \subseteq \mathbb{R}^n$. Then we have, up to a positive constant,

$$e^{\lambda(x)} = (1+\mu|x|^2)^{\frac{1}{4}}.$$

Thus we obtain the following theorem.

Theorem 7.5. *Let h be a strongly dually flat Riemannian metric on $\mathcal{U} \subseteq \mathbb{R}^n$. Then, up to a scaling,*

$$h = \frac{\sqrt{(1+\mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}}{(1+\mu|x|^2)^{\frac{3}{4}}}. \tag{7.7}$$

Let h be the strongly dually flat Riemannian metric given in (7.7). Then

$$G^i_h = 2\theta y^i + h^2\theta^i, \tag{7.8}$$

where $\theta = \theta_i(x)y^i$ satisfies [27]

$$\theta_i(x) = -\frac{\mu}{4\omega^2}x^i. \tag{7.9}$$

Assume that $\beta = b_j(x)y^j$ is a 1-form and it is dually related to h. Then

$$b_{i;j} = c(x)h_{ij} + 2\theta_i b_j, \tag{7.10}$$

where θ_i is given in (7.9) and $b_{i;j}$ denotes the covariant derivative of b_i with respect to the Levi-Civita connection of h.

Let $\widetilde{h}$ be the Riemannian metric of constant sectional curvature given in (7.5). Then

$$\widetilde{c}(x) := \omega[c(x) + 2b_k\theta^k], \tag{7.11}$$

satisfies

$$\widetilde{c}_{|k} + 2\widetilde{c}\theta_k + \mu b_k = 0. \tag{7.12}$$

$\widetilde{c}_{|k}$ denotes the covariant derivative of $\widetilde{c}$ with respect to the Levi-Civita connection of $\widetilde{h}$.

Case 1. $\mu \neq 0$. We obtain a general formula for $\widetilde{c}$,

$$\widetilde{c} = \frac{\lambda + \langle a, x\rangle}{(1+\mu|x|^2)^{\frac{1}{4}}}.$$

It follows that

$$\beta = b_k y^k = \frac{\lambda - \mu\langle v, x\rangle}{\omega^{\frac{5}{2}}}\langle x, y\rangle + \frac{\langle v, y\rangle}{\omega^{\frac{1}{2}}}, \tag{7.13}$$

where $v := -\frac{1}{\mu}a$.

Case 2. $\mu = 0$.

By (7.9), we obtain

$$\theta_i = 0, \quad \theta^i = 0. \tag{7.14}$$

It follows that $c(x) = \widetilde{c}(x)$ is a constant. Let $\lambda = c(x)$. Then we obtain a general formula for b_i,

$$b_i = \lambda x^i + v^i,$$

where $v = (v^1, \cdots, v^n)$ is a constant vector. Thus, (7.13) also holds for the case $\mu = 0$.

Proof of Theorem 7.1 The conformal field of $\widetilde{h}$ is given by [5, 26]

$$\widetilde{W} = (\lambda\omega + \langle a, x\rangle)x - \frac{|x|^2}{1+\omega}a + Qx + v + \mu\langle v, x\rangle x,$$

where λ is a constant, Q is an antisymmetric matrix independent of x and a, $v \in \mathbb{R}^n$ are constant vectors. It follows that

$$\widetilde{W}^\flat = \frac{1}{\omega^3}\left[(\lambda + \langle a, x\rangle)\langle x, y\rangle + \omega\left(x^T Q y + \langle v, y\rangle - \frac{|x|^2}{1+\omega}\langle a, y\rangle\right)\right]. \tag{7.15}$$

By a direct calculation we have $d\widetilde{W}^\flat = 0$ if and only if

$$Q = 0, \quad a = -\mu v.$$

Plugging this into (7.15) yields

$$\widetilde{W}^\flat = \frac{1}{\omega^3}\left[(\lambda - \mu\langle v, x\rangle)\langle x, y\rangle + \omega^2\langle v, y\rangle\right]. \tag{7.16}$$

Combining this with (7.8) and (7.13) we obtain (7.2).

Proof of Theorem 7.2 Denote the navigation data of F by (h, W). Then F can be expressed as

$$F = \frac{\sqrt{(1 - \|W\|_h^2)h^2 + (W^\flat)^2}}{1 - \|W\|_h^2} - \frac{W^\flat}{1 - \|W\|_h^2}. \tag{7.17}$$

By a direct calculation, we obtain (7.4).

8. Flag curvature in conformal navigation problem

In this section, we determine the flag curvature of Finsler metric produced from any Finsler metric and any conformal field in terms of the navigation problem. To characterize Riemannian metrics among Finsler metric, we define the *Cartan torsion* $\mathbf{A} = \{\mathbf{A_y}\}_{y \in T_xM\setminus\{0\}}$ by

$$\mathbf{A}_y(u, v) = A^i_{jk}u^j v^k \frac{\partial}{\partial x^i}, \quad A^i_{jk} := \frac{F}{4}g^{il}\frac{\partial^2 F^2}{\partial y^j \partial y^k \partial y^l}$$

where $u = u^j\frac{\partial}{\partial x^j}$, $v = v^k\frac{\partial}{\partial x^k} \in T_xM$.

Let (M, F) be a Finsler manifold with its Hilbert form ω. Let SM be the projective sphere bundle of M, obtained from TM by identifying nonzero vectors which differ from each other by a positive multiplicative factor. It is easy to verify that [4]

$$\omega \wedge (d\omega)^{n-1} \neq 0, \qquad n = \dim M$$

i.e., ω defines a contact structure on SM. Hence there is a unique vector field X on SM that satisfies $\omega(X) = 1$ and $X \lrcorner (d\omega) = 0$. This vector field X is known as the *Reeb vector field*.

Every vector $y \in T_xM\setminus\{0\}$ uniquely determines a covector $p \in T_x^*M\setminus\{0\}$ by

$$p(u) := \frac{1}{2}\frac{d}{dt}\left(F^2(x,\, y+tu)\right)|_{t=0}, \quad u \in T_xM.$$

The resulting map $L_x^F : y \in T_xM \to p \in T_x^*M$ is called the *Legendre transformation* at x.

Define a non-negative scalar function $H = H(x,\, p)$ by

$$H(x,\, p) := \max_{y\in T_xM\setminus\{0\}} \frac{p(y)}{F(x,\, y)}. \tag{8.1}$$

The function H is C^∞ on $T^*M\setminus\{0\}$ and $H_x := H|_{T_x^*M}$ is a Minkowski norm on T_x^*M for $x \in M$. Such a function is called a *Cartan metric* [12] (co-Finsler metric in an alternative terminology [23]). The pair $(M,\, H)$ is called a *Cartan manifold*.

Every covector $p \in T_x^*M\setminus\{0\}$ uniquely determines a vector $y \in T_xM\setminus\{0\}$ by

$$q(y) := \frac{1}{2}\frac{d}{dt}\left(H^2(x,\, p+tq)\right)|_{t=0}, \quad q \in T_x^*M.$$

The resulting map $L_x^{F*} : p \in T_x^*M \to y \in T_xM$ is called the *inverse Legendre transformation* at x.

Indeed L_x^F and L_x^{F*} are inverses of each other. Moreover, they preserve the Minkowski norms $H(x,\, p) = F\left(x,\, L_x^{F*}p\right)$.

A *(local) flow* (a *local one-parameter group* in an alternative terminology) on a manifold M is a map $\phi : (-\epsilon,\, \epsilon)\times M \to M$, also denoted by $\phi_t := \phi(t,\, \cdot)$, satisfying

- $\phi_0 = \mathrm{id} : M \to M$;
- $\phi_s \circ \phi_t = \phi_{s+t}$ for any $s, t \in (-\epsilon, \epsilon)$ with $s + t \in (-\epsilon, \epsilon)$.

Hence, the lift of a flow ϕ_t on M is a flow $\hat{\phi}_t$ on T^*M,

$$\hat{\phi}_t(x,\, p) := (\phi_t(x),\, (\phi_t^*)^{-1}(p)). \tag{8.2}$$

By the relationship between vector fields and flows, (8.2) induces a natural way a lift of a vector field V on M to a vector field X_V^* on T^*M.

A vector field V on a Finsler manifold $(M,\, F)$ is called *conformal with dilation* $c(x)$ if its flow ϕ_t satisfies

$$F(\phi_t(x),\, \phi_{t*}(y)) = e^{2\sigma_t(x)}F(x,\, y), \quad \forall x \in M,\ y \in T_xM \tag{8.3}$$

where $c(x) = \left[\frac{d\sigma_t(x)}{dt}\right]_{t=0}$. In particular, V is called a *homothetic* field if $c =$ constant.

Similarly, a vector field V on a Cartan manifold $(M,\, H)$ is called *conformal with dilation* $c(x)$ if its flow ϕ_t is a conformal transformation on $(M,\, H)$, i.e.,

$$H(\phi_t(x),\, (\phi_t^*)^{-1}(p)) = e^{-2\sigma_t(x)}H(x,\, p), \quad \forall x \in M,\ p \in T_x^*M \tag{8.4}$$

where $c(x) = \left[\frac{d\sigma_t(x)}{dt}\right]_{t=0}$.

Lemma 8.1. *[10] Let V be a conformal field on a Finsler manifold $(M,\, F)$ with dilation $c(x)$ and H its Cartan metric defined by (8.1). Then V is a conformal field of H with dilation $c(x)$.*

Proposition 8.2. *[10] Let φ be a conformal transformation on a Cartan manifold $(M,\, H)$, i.e. $\hat{\varphi}^*H = e^{-2\sigma(x)}H$. Then*

$$\hat{\varphi}_*X^\flat = e^{2\sigma(x)}\left(X^\flat + 2Y_{\sigma(x)}\right)$$

where $Y_{\sigma(x)}$ is defined by

$$Y_f := X_f - fX^\flat = -\phi Df \in VS^*M \tag{8.5}$$

for $f \in C^\infty(M)$.

Let F be a Finsler metric and $\tilde{F}$ denote the Finsler metric defined in (2.1). With the help of the inverse Legendre transformation at x, we obtain co-Finsler metric $H(x,\, p)$ and $\tilde{H}(x,\, p)$ respectively. Then H and $\tilde{H}$ are related by [15]

$$\tilde{H}(x,\, p) = H(x,\, p) - p(V). \tag{8.6}$$

Proposition 8.3. *[10] For a conformal transformation φ on a Cartan manifold (M, H), we have*

$$\hat{\varphi}_* \mathcal{H}^\flat(v) = e^{2\sigma(x)} \left[\mathcal{H}^\flat(\hat{\varphi}_* v) + 2\dot{\sigma}\hat{\varphi}_* v - 2A^\flat(Y_\sigma, \hat{\varphi}_* v)\right] \tag{8.7}$$

where $\mathcal{H}^\flat$ is the horizontal endomorphism.

Proposition 8.4. *[10] Let V be a conformal field of H with dilation $c(x)$. Then*

$$\left[X_V^*, \mathcal{H}^\flat(v)\right] = -2c\mathcal{H}^\flat(v) + \mathcal{H}^\flat\left[X_V^*, v\right] - 2\dot{c}v + 2A^\flat(Y_c, v) \tag{8.8}$$

where

$$X_V^* = v^i \frac{\partial}{\partial x^i} - p_j \frac{\partial v^j}{\partial x^i}\frac{\partial}{\partial p_i}$$

where $V = v^i \frac{\partial}{\partial x^i}$.

Proposition 8.5. *[10] Let V be a conformal field on a Cartan manifold (M, H) with dilation $c(x)$. Let $\tilde{H}$ be the Cartan metric given in (8.6). Then*

$$\tilde{\mathcal{R}}^\flat(v) = \mathcal{R}^\flat(v) + \left[3\tilde{X}^\flat(c) - c^2 + 2X_V^*(c)\right] v - 2A^\flat(Y_c, v) \tag{8.9}$$

where $\mathcal{R}^\flat$ (resp. $\tilde{\mathcal{R}}^\flat$) is the Riemann tensor of H (resp. $\tilde{H}$)

Proof.

$$\begin{aligned}\tilde{\mathcal{R}}^\flat(v) =& P^\flat_{\tilde{\mathcal{V}}}\left[\tilde{X}^\flat, \tilde{\mathcal{H}}^\flat(v)\right] \\ =& \mathcal{R}^\flat(v) - P^\flat_{\tilde{\mathcal{V}}}\left[X_V^*, \mathcal{H}^\flat(v)\right] - c\left[X_V^*, v\right] + \left[\tilde{X}^\flat(c) + c^2\right] v \\ =& \mathcal{R}^\flat(v) + \left[3\tilde{X}^\flat(c) - c^2 + 2X_V^*(c)\right] v - 2A^\flat(Y_c, v)\end{aligned}$$

□

Proposition 8.6. *[10] Let V be a conformal field on a Cartan manifold (M, H) with dilation $c(x)$. Let $\tilde{H}$ be the Cartan metric given in (8.6). Then*

$$\tilde{K}^\flat(v) - \left[3\tilde{X}^\flat(c) - c^2 + 2V(c)\right] = K^\flat(v) - 2\frac{A^\flat(v, Y_c, v)}{h^\flat(v, v)} \tag{8.10}$$

where $K^\flat$ (resp. $\tilde{K}^\flat$) is the flag curvature of H (resp. $\tilde{H}$)

Proof. The flag curvature $K^\flat$ is given by

$$K^\flat(v) = \frac{h^\flat(\mathcal{R}^\flat(v), v)}{h^\flat(v, v)}, \qquad v \in VS^*M \setminus \{0\} \tag{8.11}$$

where $h^\flat$ is the angular metric on VS^*M. Combining with Proposition we have (8.10).

□

Theorem 8.7. *[10] Let $F = F(x, y)$ be a Finsler metric on a manifold M with its Cartan torsion A and V be a vector field on M with $F(x, V_x) < 1$. Let $\tilde{F} = \tilde{F}(x, y)$ denote the Finsler metric on M defined in (2.2). Suppose that V is conformal with dilation $c(x)$. Then the flag curvature of $\tilde{F}$ and F is related by*

$$\begin{aligned}K_{\tilde{F}}(y, y \wedge u) - \left[3\frac{y^i c_{x^i}}{\tilde{F}(x, y)} - c^2 + 2V(c)\right] =& K_F(\tilde{y}, \tilde{y} \wedge u) \\ & - 2\frac{A_{(x, [\tilde{y}])}(u, \nabla c, u)}{h_{(x, [\tilde{y}])}(u, u)}\end{aligned}$$

where $\tilde{y} = y + F(x, \tilde{y})V$ and h is the angular metric of F.

Remark 8.8. *We have two special case of Theorem:*

- *When V is homothetic, i.e. its dilation $c(x) = constant$, then $\nabla c = 0$ and our formula is reduced to Mo-Huang's formula in [15].*
- *When F is Riemannian and it has sectional curvature $K = K(x)$, then our formula is reduced to Cheng-Shen's formula in [7].*

Consider the case $\dim M = 2$, so $x = (x^1, x^2)$ and $y = (y^1, y^2)$. In order to avoid the excessive use of parentheses, we shall abbreviate x^1, x^2 as s, t and y^1, y^2 as p, q respectively. Let

$$M := \left\{(s, t) \in \mathbb{R}^2 | \, t > 1\right\}.$$

Define $F : TM \to \mathbb{R}$ by

$$F(s, t; p, q) := \frac{1}{t}\phi(p, q) \tag{8.12}$$

where

$$\phi(p, q) := (p^4 + 2\epsilon p^2 q^2 + q^4)^{\frac{1}{4}}, \quad \epsilon \in (0, 3) \tag{8.13}$$

is a Minkowski norm on $\mathbb{R}^2$. F is a Finsler metric on M.

For the Finsler surface (M, F), its Gaussian curvature K takes the place of the flag curvature in general case. A direct calculation shows that the Gauss curvature of F is given by

$$K_F(s, t; p, q) = \frac{[\phi(p, q)]^2 Q(p, q)}{[\Delta(p, q)]^4} \tag{8.14}$$

where

$$\begin{aligned} Q(p, q) :=& \epsilon(2\epsilon^2 - 3)p^{14} + (17\epsilon^4 - 42\epsilon^3 + 18)p^{12}q^2 \\ &+ \epsilon(8\epsilon^4 - 50\epsilon^2 + 21)p^{10}q^4 \\ &+ (9\epsilon^6 - 89\epsilon^4 + 81\epsilon^2 - 36)p^8q^6 - 5\epsilon(5\epsilon^4 - 4\epsilon^2 + 6)p^6q^8 \\ &+ \epsilon^2(5\epsilon^4 - 5\epsilon^2 - 21)p^4q^{10} + \epsilon^3(5\epsilon^2 - 12)p^2q^{12} - \epsilon^4q^{14} \end{aligned} \tag{8.15}$$

and

$$\Delta(p, q) := \epsilon p^4 + (3 - \epsilon^2)p^2q^2 + \epsilon q^4. \tag{8.16}$$

Let V denote a vector field on M defined by

$$V := \frac{\partial}{\partial t}. \tag{8.17}$$

By using the isomorphism $T_xM \simeq \mathbb{R}^2$ we have $F(x, V_x) < 1$ on M. Denoted the lift of V by X_V. Then [8]

$$X_V(F) = \frac{\partial F}{\partial t} = -\frac{1}{t}F$$

where we have made use of (8.12). Thus V is conformal with dilation $c = -\frac{1}{2t}$ (see [9]). In particular, V is not homothetic.

Using Theorem 8.7, we obtain that the Gauss curvature $K_{\tilde{F}}$ is given by

$$K_{\tilde{F}}(x, y) = K_F(x, \tilde{y}) + \frac{3q}{2t^2 F(x, \tilde{y})} + \frac{3}{4t^2} - 2\lambda(x, \tilde{y})I(x, \tilde{y})$$

where

$$\tilde{y} = y + F(x, y)V = \left(p, \; q + \frac{(p^4 + 2\epsilon p^2q^2 + q^4)^{\frac{1}{4}}}{t}\right)$$

$$I(x, y) = \frac{3(1-\epsilon^2)pq}{[\Delta(p, q)]^{\frac{3}{2}}}(p^4 - q^4), \quad \lambda(x, y) = \frac{p(p^2 + \epsilon q^2)}{2F\sqrt{\Delta(p, q)}t^2}.$$

Let us take a look at the special case: when $\epsilon = 1$,

$$F(s, t; p, q) := \frac{(p^2 + q^2)^{\frac{1}{2}}}{t}.$$

F is the famous Poincaré metric of constant sectional curvature $K_F = -1$. In this case, $\tilde{F}$ is of Randers type and its Gauss curvature is given by

$$K_{\tilde{F}}(x, y) = \frac{3}{4t^2}\left(\frac{2q}{\tilde{F}(x, y)} + 1\right) - 1.$$

9. Landsberg curvature in conformal navigation problem

The following Landsberg curvature, it gives the rate of change of the Cartan torsion along geodesics. For a vector $y \in T_xM\backslash\{0\}$, define

$$\mathbf{L}_y(u,\, v) = \dot{A}^i_{jk} u^j v^k \frac{\partial}{\partial x^i}$$

where $u = u^j \frac{\partial}{\partial x^j}$, $v = v^k \frac{\partial}{\partial x^k} \in T_xM$ and "$\cdot$" denotes the covariant derivative along geodesics. $\mathbf{L} = \{\mathbf{L}_y\}_{y\in T_xM\backslash\{0\}}$ is called the *Landsberg curvature*. We say that F has *relatively isotropic Landsberg curvature* if $\mathbf{L} + c\dot{\mathbf{A}} = 0$ where $c = c(x)$ is a scalar function on M [5]. We say that F is a *Landsberg metric* if $\mathbf{L} = 0$.

Let (M, F) be a Finsler manifold. A vector field V on (M, F) is said to be *closed* if $dV^\flat \equiv 0 \pmod{\delta y^i}$, where

$$V^\flat = V^j g_{ij} dx^i, \qquad V = V^j \frac{\partial}{\partial x^j}$$

and δy^i are defined by [5]

$$\delta y^i := dy^i + N^i_j dx^j. \tag{9.1}$$

In the case of a vector field V on a Riemannian manifold, our notion is reduced to $dV^\flat = 0$ where $\flat : TM \to T^*M$ denotes the musical isomorphism.

In this section, we discuss the Landsberg curvature of a Finsler metric via conformal navigation problem. We have the following:

Theorem 9.1. *Let $F = F(x,\, y)$ be a Finsler metric on a manifold M with its Landsberg curvature $\mathbf{L}$ and V a closed vector field on $(M,\, F)$ with $F(x,\, V_x) < 1$. Let $\widetilde{F} = \widetilde{F}(x,\, y)$ denote the Finsler metric on M defined by (2.1) Suppose that V is conformal with dilation $c(x)$. Then the Landsberg curvature $\widetilde{\mathbf{L}}$ and the Cartan torsion $\widetilde{\mathbf{A}}$ of $\widetilde{F}$ satisfy*

$$\widetilde{\mathbf{L}}_y + c(x)\widetilde{\mathbf{A}}_y = \mathbf{L}_{\widetilde{y}}$$

where $\widetilde{y} = y + F(x,\, \widetilde{y})V$.

Theorem 9.1 tells us that the Killing navigation problem (i.e. $c(x) \equiv 0$) has the Landsberg curvature preserving property for a closed vector field.

Our method to proving Theorem 9.1 is partially in the contact geometry. It follows that our method is quite different from Shen in [24].

A Randers metric can be expressed in the navigation form

$$F = \frac{\sqrt{(1-b^2)\alpha^2 + \beta^2}}{1-b^2} + \frac{\beta}{1-b^2}$$

where $(\alpha,\, \beta^{\flat^{-1}})$ is the navigation data of F and $b := \|\beta\|_\alpha$ is the length of β. Suppose that $\beta^{\flat^{-1}}$ is closed and conformal with dilation $c(x)$. Then F satisfies $\mathbf{L} + c(x)\mathbf{A} = 0$ [24].

Theorem 9.2. *Let $F = F(x,\, y)$ be a Landsberg metric on a manifold M and V a closed vector field on $(M,\, F)$ with $F(x,\, V_x) < 1$. Let $\widetilde{F} = \widetilde{F}(x,\, y)$ denote the Finsler metric on M defined in (2.1). Suppose that V is a conformal field of F. Then $\widetilde{F}$ has relatively isotropic Landsberg curvature, i.e. the Landsberg curvature of $\widetilde{F}$ is proportional to its Cartan torsion.*

If c = constant, we have non-trivial example satisfying the conditions and conclusions in Theorem. Given a Minkowski norm $\varphi : \mathbb{R}^n \to \mathbb{R}$, a constant vector b and a constant c, one can construct a domain $\Omega := \{v \in \mathbb{R}^n \,|\, \varphi(2cv + b) < 1\}$. For each $x \in \Omega$, identify $T_x\Omega$ with $\mathbb{R}^n$. This $F(x,\, y))$ is a Minkowski metric on the domain Ω where $F(x,\, y) = \varphi(y)$ and $V_x := 2cx + b$ is a vector field on Ω satisfying $F(x,\, V_x) = \varphi(2cx + b) < 1$. It can be shown that V is conformal with constant dilation c and V is a closed vector on $(F,\, \Omega)$. Define a new Finsler metric $\widetilde{F}$ by (2.1). Note that any Minkowski metric must be Landsberg type. By Theorem 9.1, we have

$$\widetilde{\mathbf{L}} + c\widetilde{\mathbf{A}} = 0. \tag{9.2}$$

Moreover, the geodesics of $\tilde{F}$ are given by $e^{-2ct}\left[x + \frac{e^{2ct}-1}{2c\varphi(y)}y\right] - tb$ (resp. $x + \frac{t}{\varphi(y)}y - tb$) for $c \neq 0$ (resp. $c = 0$). When $c = \frac{1}{2}$ and $b = 0$, $\widetilde{F}$ is the Funk metric on a strongly convex domain. Our result (9.2) has been obtained in [21]. Again, the technique and method used in this paper is quite different from Shen in [21].

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